EXERCISES

1. Find the linear fractional transformation that maps the points $z_1 = 2, z_2 = i, z_3 = -2$ onto the points $w_1 = 1, w_2 = i, w_3 = -1$.

Ans. $w = \dfrac{3z + 2i}{iz + 6}$.

2. Find the linear fractional transformation that maps the points $z_1 = -i, z_2 = 0, z_3 = i$ onto the points $w_1 = -1, w_2 = i, w_3 = 1$. Into what curve is the imaginary axis $x = 0$ transformed?

3. Find the bilinear transformation that maps the points $z_1 = \infty, z_2 = i, z_3 = 0$ onto the points $w_1 = 0, w_2 = i, w_3 = \infty$.

Ans. $w = -1/z$.

4. Find the bilinear transformation that maps distinct points z_1, z_2, z_3 onto the points $w_1 = 0, w_2 = 1, w_3 = \infty$.

Ans. $w = \dfrac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$.

5. Show that a composition of two linear fractional transformations is again a linear fractional transformation, as stated in Sec. 93. To do this, consider two such transformations

$$T(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \quad (a_1 d_1 - b_1 c_1 \neq 0)$$

and

$$S(z) = \frac{a_2 z + b_2}{c_2 z + d_2} \quad (a_2 d_2 - b_2 c_2 \neq 0).$$

Then show that the composition $S[T(z)]$ has the form

$$S[T(z)] = \frac{a_3 z + b_3}{c_3 z + d_3},$$

where

$$a_3 d_3 - b_3 c_3 = (a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2) \neq 0.$$

6. A *fixed point* of a transformation $w = f(z)$ is a point z_0 such that $f(z_0) = z_0$. Show that every linear fractional transformation, with the exception of the identity transformation $w = z$, has at most two fixed points in the extended plane.

7. Find the fixed points (see Exercise 6) of the transformation

(a) $w = \dfrac{z - 1}{z + 1}$; *(b)* $w = \dfrac{6z - 9}{z}$.

Ans. (a) $z = \pm i$; *(b)* $z = 3$.

8. Modify equation (1), Sec. 94, for the case in which both z_2 and w_2 are the point at infinity. Then show that any linear fractional transformation must be of the form $w = az \ (a \neq 0)$ when its fixed points (Exercise 6) are 0 and ∞.

9. Prove that if the origin is a fixed point (Exercise 6) of a linear fractional transformation, then the transformation can be written in the form

$$w = \frac{z}{cz + d} \qquad (d \neq 0).$$

10. Show that there is only one linear fractional transformation which maps three given distinct points z_1, z_2, and z_3 in the extended z plane onto three specified distinct points w_1, w_2, and w_3 in the extended w plane.

Suggestion: Let T and S be two such linear fractional transformations. Then, after pointing out why $S^{-1}[T(z_k)] = z_k$ $(k = 1, 2, 3)$, use the results in Exercises 5 and 6 to show that $S^{-1}[T(z)] = z$ for all z. Thus show that $T(z) = S(z)$ for all z.

11. With the aid of equation (1), Sec. 94, prove that if a linear fractional transformation maps the points of the x axis onto points of the u axis, then the coefficients in the transformation are all real, except possibly for a common complex factor. The converse statement is evident.

12. Let

$$T(z) = \frac{az + b}{cz + d} \quad (ad - bc \neq 0)$$

be any linear fractional transformation other than $T(z) = z$. Show that

$$T^{-1} = T \quad \text{if and only if} \quad d = -a.$$

Suggestion: Write the equation $T^{-1}(z) = T(z)$ as

$$(a + d)[cz^2 + (d - a)z - b] = 0.$$

95. MAPPINGS OF THE UPPER HALF PLANE

Let us determine all linear fractional transformations that map the upper half plane $\operatorname{Im} z > 0$ onto the open disk $|w| < 1$ and the boundary $\operatorname{Im} z = 0$ of the half plane onto the boundary $|w| = 1$ of the disk (Fig. 113).

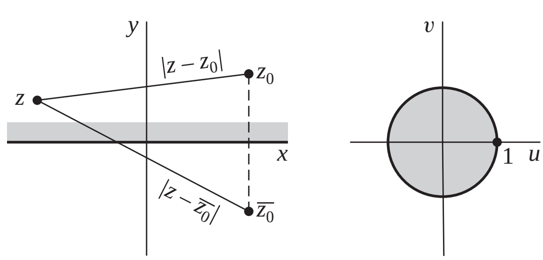

FIGURE 113

$$w = e^{i\alpha}\left(\frac{z - z_0}{z - \overline{z_0}}\right) \quad (\operatorname{Im} z_0 > 0).$$

Keeping in mind that points on the line $\operatorname{Im} z = 0$ are to be transformed into points on the circle $|w| = 1$, we start by selecting the points $z = 0$, $z = 1$, and $z = \infty$ on the line and determining conditions on a linear fractional transformation

$$(1) \qquad w = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0)$$

which are necessary in order for the images of those points to have unit modulus.