the choice of Θ_0 varies between $\Theta_0 = 0$ to $\Theta_0 = \pi$, the mapping of the half plane $Y > 0$ onto the strip is, in fact, one to one.

This shows that the composition (9) of the mappings (10) transforms the plane $y > 0$ onto the strip $0 < v < \pi$. Corresponding boundary points are shown in Fig. 19 of Appendix 2.

EXERCISES

1. Recall from Example 1 in Sec. 95 that the transformation

$$w = \frac{i - z}{i + z}$$

maps the half plane $\operatorname{Im} z > 0$ onto the disk $|w| < 1$ and the boundary of the half plane onto the boundary of the disk. Show that a point $z = x$ is mapped onto the point

$$w = \frac{1 - x^2}{1 + x^2} + i\frac{2x}{1 + x^2},$$

and then complete the verification of the mapping illustrated in Fig. 13, Appendix 2, by showing that segments of the x axis are mapped as indicated there.

2. Verify the mapping shown in Fig. 12, Appendix 2, where

$$w = \frac{z - 1}{z + 1}.$$

Suggestion: Write the given transformation as a composition of the mappings

$$Z = iz, \quad W = \frac{i - Z}{i + Z}, \quad w = -W.$$

Then refer to the mapping whose verification was completed in Exercise 1.

3. *(a)* By finding the inverse of the transformation

$$w = \frac{i - z}{i + z}$$

and appealing to Fig. 13, Appendix 2, whose verification was completed in Exercise 1, show that the transformation

$$w = i\frac{1 - z}{1 + z}$$

maps the disk $|z| \leq 1$ onto the half plane $\operatorname{Im} w \geq 0$.

(b) Show that the linear fractional transformation

$$w = \frac{z - 2}{z}$$

can be written

$$Z = z - 1, \quad W = i\frac{1 - Z}{1 + Z}, \quad w = iW.$$

Then, with the aid of the result in part *(a)*, verify that it maps the disk $|z - 1| \leq 1$ onto the left half plane $\operatorname{Re} w \leq 0$.

4. Transformation (6), Sec. 95, maps the point $z = \infty$ onto the point $w = \exp(i\alpha)$, which lies on the boundary of the disk $|w| \le 1$. Show that if $0 < \alpha < 2\pi$ and the points $z = 0$ and $z = 1$ are to be mapped onto the points $w = 1$ and $w = \exp(i\alpha/2)$, respectively, the transformation can be written

$$w = e^{i\alpha}\left[\frac{z + \exp(-i\alpha/2)}{z + \exp(i\alpha/2)}\right].$$

5. Note that when $\alpha = \pi/2$, the transformation in Exercise 4 becomes

$$w = \frac{iz + \exp(i\pi/4)}{z + \exp(i\pi/4)}.$$

Verify that this special case maps points on the x axis as indicated in Fig. 115.

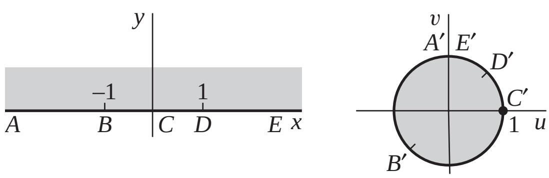

FIGURE 115
$$w = \frac{iz + \exp(i\pi/4)}{z + \exp(i\pi/4)}.$$

6. Show that if $\operatorname{Im} z_0 < 0$, transformation (6), Sec. 95, maps the lower half plane $\operatorname{Im} z \le 0$ onto the unit disk $|w| \le 1$.

7. The equation $w = \log(z - 1)$ can be written

$$Z = z - 1, \quad w = \log Z.$$

Find a branch of $\log Z$ such that the cut z plane consisting of all points except those on the segment $x \ge 1$ of the real axis is mapped by $w = \log(z - 1)$ onto the strip $0 < v < 2\pi$ in the w plane.

96. THE TRANSFORMATION $w = \sin z$

Since (Sec. 34)

$$\sin z = \sin x \cosh y + i \cos x \sinh y,$$

the transformation $w = \sin z$ can be written

$$u = \sin x \cosh y, \quad v = \cos x \sinh y. \tag{1}$$

One method that is often useful in finding images of regions under this transformation is to examine images of vertical lines $x = c_1$. If $0 < c_1 < \pi/2$, points on the line $x = c_1$ are transformed into points on the curve

$$u = \sin c_1 \cosh y, \quad v = \cos c_1 \sinh y \qquad (-\infty < y < \infty), \tag{2}$$

APPENDIX

2

TABLE OF TRANSFORMATIONS OF REGIONS (See Chap. 8)

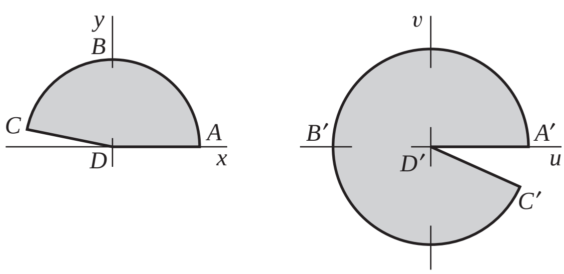

FIGURE 1
$w = z^2$.

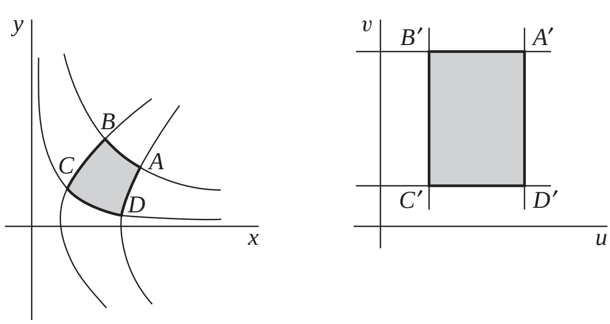

FIGURE 2
$w = z^2$.

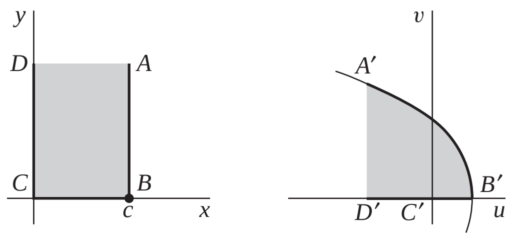

FIGURE 3
$w = z^2$;
$A'B'$ on parabola $v^2 = -4c^2(u - c^2)$.

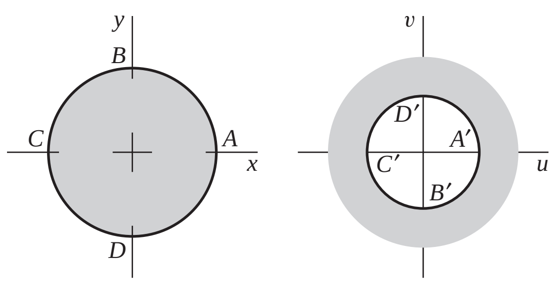

FIGURE 4
$w = 1/z$.

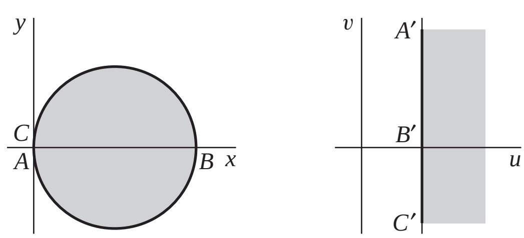

FIGURE 5
$w = 1/z$.

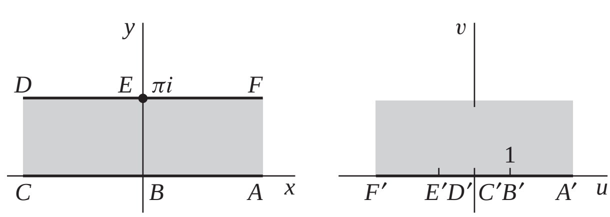

FIGURE 6
$w = \exp z$.

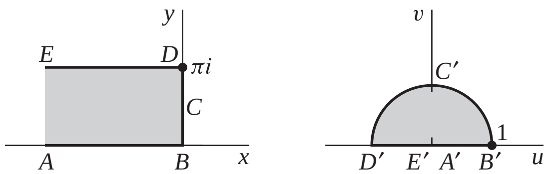

FIGURE 7
$w = \exp z$.

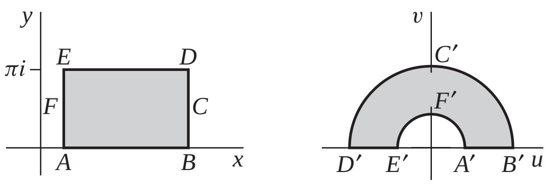

FIGURE 8
$w = \exp z$.

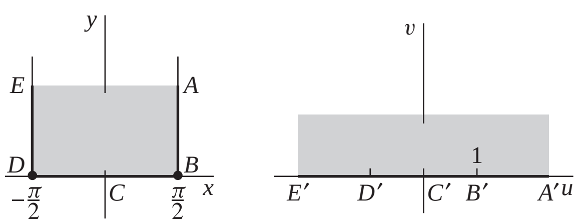

FIGURE 9
$w = \sin z$.

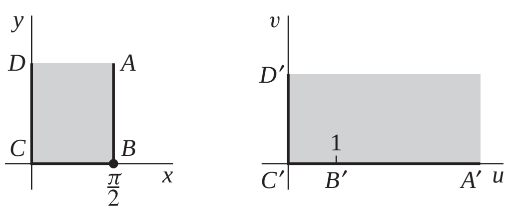

FIGURE 10
$w = \sin z$.

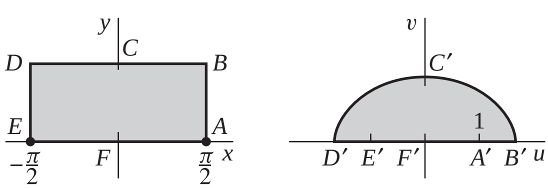

FIGURE 11
$w = \sin z$; BCD on line $y = b$ $(b > 0)$,

$B'C'D'$ on ellipse $\dfrac{u^2}{\cosh^2 b} + \dfrac{v^2}{\sinh^2 b} = 1$.

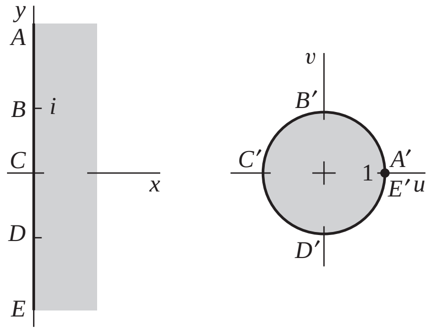

FIGURE 12

$$w = \frac{z-1}{z+1}.$$

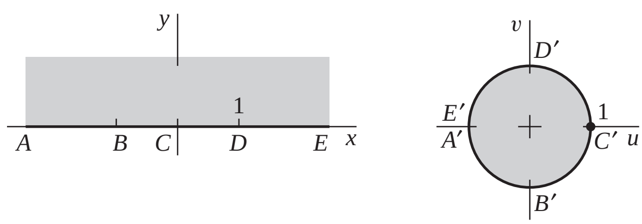

FIGURE 13

$$w = \frac{i-z}{i+z}.$$

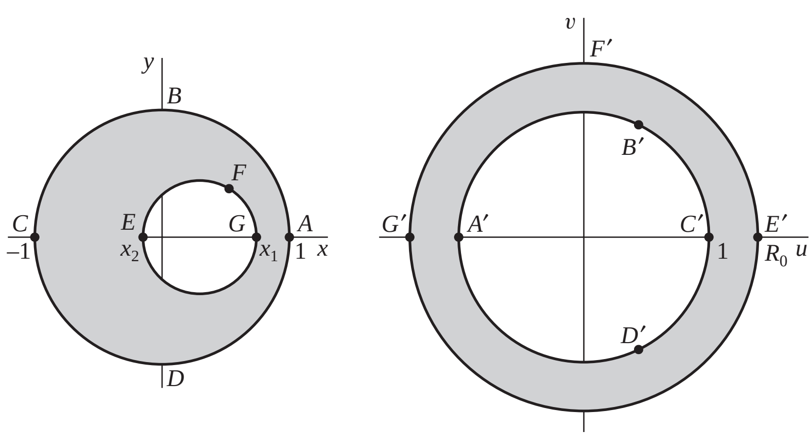

FIGURE 14

$$w = \frac{z-a}{az-1};\ a = \frac{1 + x_1 x_2 + \sqrt{(1-x_1^2)(1-x_2^2)}}{x_1 + x_2},$$

$$R_0 = \frac{1 - x_1 x_2 + \sqrt{(1-x_1^2)(1-x_2^2)}}{x_1 - x_2} \quad (a > 1 \text{ and } R_0 > 1 \text{ when } -1 < x_2 < x_1 < 1).$$

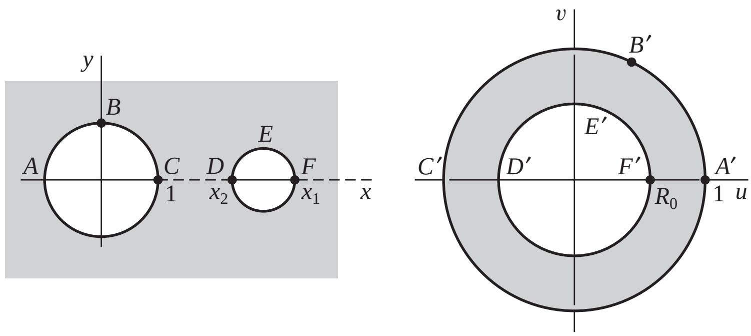

FIGURE 15

$$w = \frac{z-a}{az-1};\ a = \frac{1 + x_1x_2 + \sqrt{(x_1^2-1)(x_2^2-1)}}{x_1+x_2},$$

$$R_0 = \frac{x_1x_2 - 1 - \sqrt{(x_1^2-1)(x_2^2-1)}}{x_1-x_2}\ \ (x_2 < a < x_1 \text{ and } 0 < R_0 < 1 \text{ when } 1 < x_2 < x_1).$$

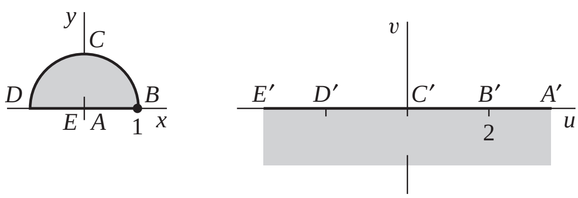

FIGURE 16

$$w = z + \frac{1}{z}.$$

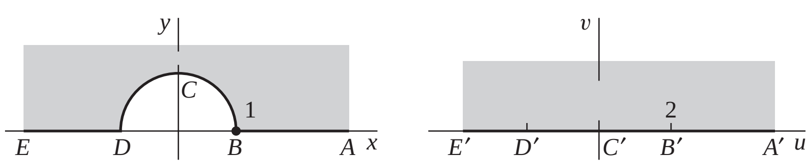

FIGURE 17

$$w = z + \frac{1}{z}.$$

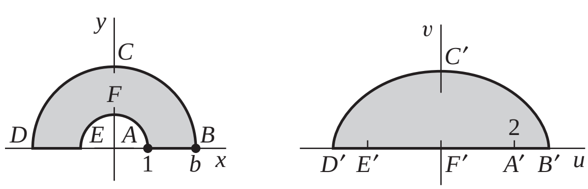

FIGURE 18

$$w = z + \frac{1}{z};\ B'C'D' \text{ on ellipse } \frac{u^2}{(b+1/b)^2} + \frac{v^2}{(b-1/b)^2} = 1.$$

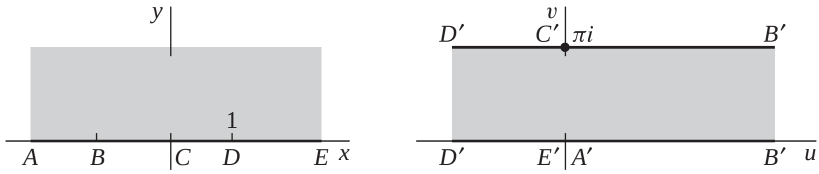

FIGURE 19

$w = \text{Log}\,\dfrac{z-1}{z+1}$; $z = -\coth\dfrac{w}{2}$.

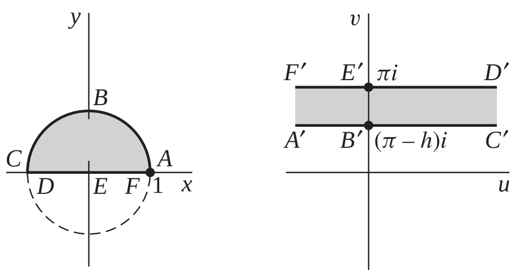

FIGURE 20

$w = \text{Log}\,\dfrac{z-1}{z+1}$;

ABC on circle $x^2 + (y + \cot h)^2 = \csc^2 h \quad (0 < h < \pi)$.

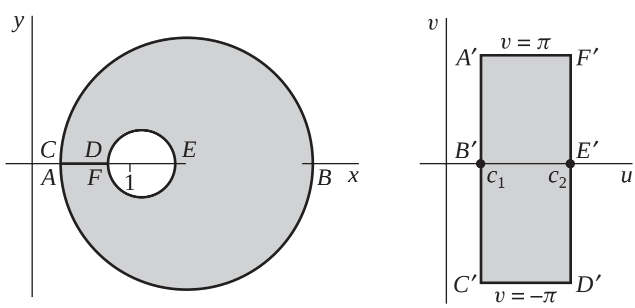

FIGURE 21

$w = \text{Log}\,\dfrac{z+1}{z-1}$; centers of circles at $z = \coth c_n$, radii: $\text{csch}\, c_n \quad (n = 1, 2)$.

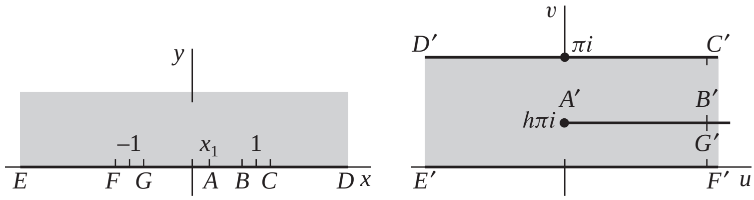

FIGURE 22

$$w = h \ln \frac{h}{1-h} + \ln 2(1-h) + i\pi - h\ \mathrm{Log}(z+1) - (1-h)\ \mathrm{Log}(z-1);\ x_1 = 2h - 1\,.$$

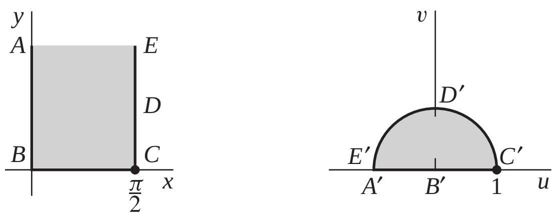

FIGURE 23

$$w = \left(\tan \frac{z}{2}\right)^2 = \frac{1-\cos z}{1+\cos z}.$$

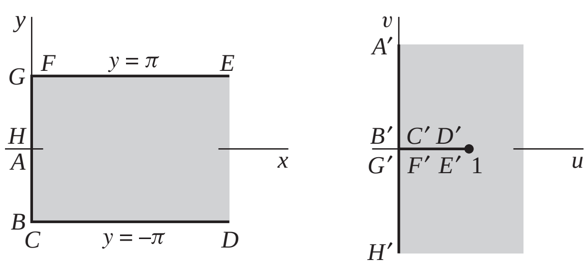

FIGURE 24

$$w = \coth \frac{z}{2} = \frac{e^z + 1}{e^z - 1}.$$

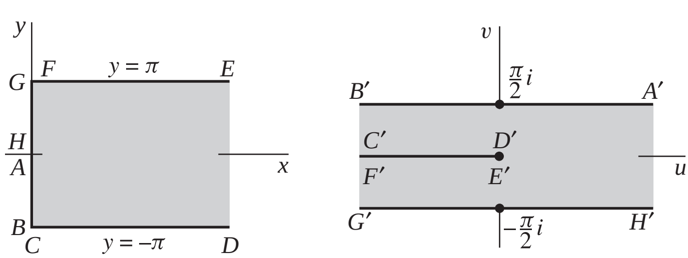

FIGURE 25

$$w = \mathrm{Log}\left(\coth \frac{z}{2}\right).$$

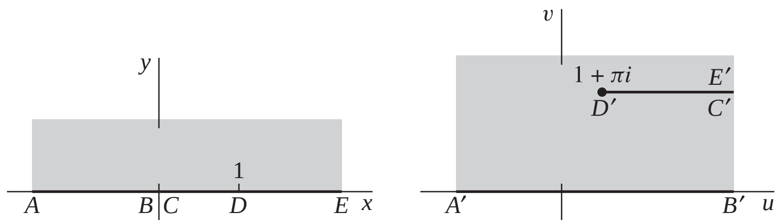

FIGURE 26

$w = \pi i + z - \operatorname{Log} z.$

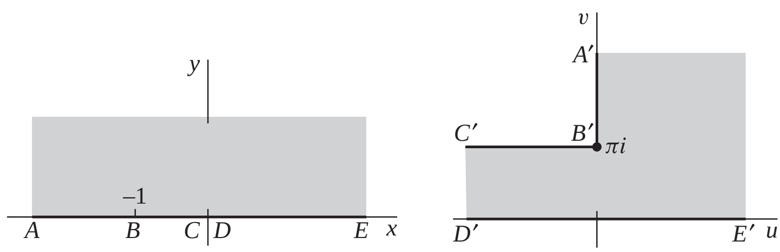

FIGURE 27

$$w = 2(z+1)^{1/2} + \operatorname{Log}\frac{(z+1)^{1/2}-1}{(z+1)^{1/2}+1}.$$

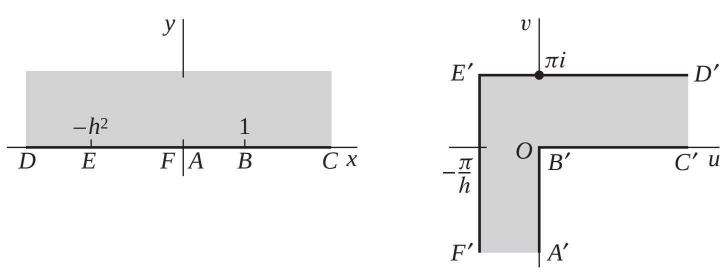

FIGURE 28

$$w = \frac{i}{h}\operatorname{Log}\frac{1+iht}{1-iht} + \operatorname{Log}\frac{1+t}{1-t};\ t = \left(\frac{z-1}{z+h^2}\right)^{1/2}.$$

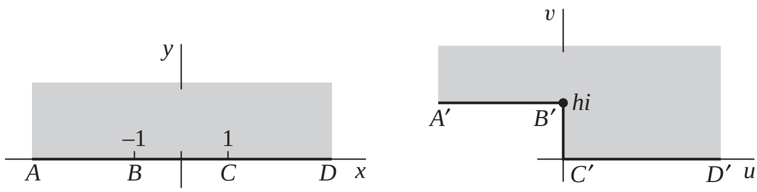

FIGURE 29

$$w = \frac{h}{\pi}[(z^2 - 1)^{1/2} + \cosh^{-1} z].^*$$

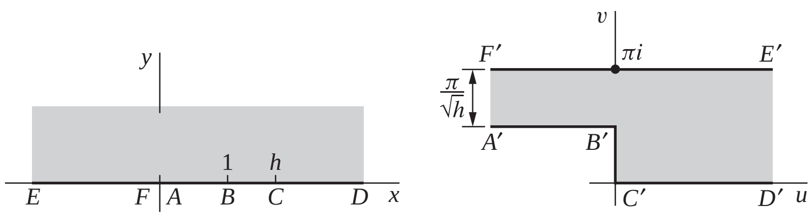

FIGURE 30

$$w = \cosh^{-1}\left(\frac{2z - h - 1}{h - 1}\right) - \frac{1}{\sqrt{h}} \cosh^{-1}\left[\frac{(h+1)z - 2h}{(h-1)z}\right].$$

*See Exercise 3, Sec. 122.