4. Transformation (6), Sec. 95, maps the point $z = \infty$ onto the point $w = \exp(i\alpha)$, which lies on the boundary of the disk $|w| \le 1$. Show that if $0 < \alpha < 2\pi$ and the points $z = 0$ and $z = 1$ are to be mapped onto the points $w = 1$ and $w = \exp(i\alpha/2)$, respectively, the transformation can be written

$$w = e^{i\alpha}\left[\frac{z + \exp(-i\alpha/2)}{z + \exp(i\alpha/2)}\right].$$

5. Note that when $\alpha = \pi/2$, the transformation in Exercise 4 becomes

$$w = \frac{iz + \exp(i\pi/4)}{z + \exp(i\pi/4)}.$$

Verify that this special case maps points on the x axis as indicated in Fig. 115.

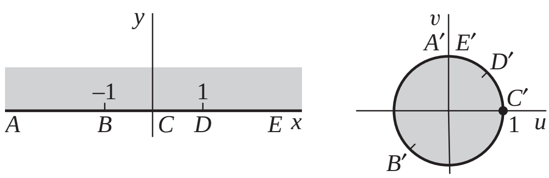

FIGURE 115
$$w = \frac{iz + \exp(i\pi/4)}{z + \exp(i\pi/4)}.$$

6. Show that if $\operatorname{Im} z_0 < 0$, transformation (6), Sec. 95, maps the lower half plane $\operatorname{Im} z \le 0$ onto the unit disk $|w| \le 1$.

7. The equation $w = \log(z - 1)$ can be written

$$Z = z - 1, \quad w = \log Z.$$

Find a branch of $\log Z$ such that the cut z plane consisting of all points except those on the segment $x \ge 1$ of the real axis is mapped by $w = \log(z - 1)$ onto the strip $0 < v < 2\pi$ in the w plane.

96. THE TRANSFORMATION $w = \sin z$

Since (Sec. 34)

$$\sin z = \sin x \cosh y + i \cos x \sinh y,$$

the transformation $w = \sin z$ can be written

$$u = \sin x \cosh y, \quad v = \cos x \sinh y. \tag{1}$$

One method that is often useful in finding images of regions under this transformation is to examine images of vertical lines $x = c_1$. If $0 < c_1 < \pi/2$, points on the line $x = c_1$ are transformed into points on the curve

$$u = \sin c_1 \cosh y, \quad v = \cos c_1 \sinh y \qquad (-\infty < y < \infty), \tag{2}$$

which is the right-hand branch of the hyperbola

$$\frac{u^2}{\sin^2 c_1} - \frac{v^2}{\cos^2 c_1} = 1 \tag{3}$$

with foci at the points

$$w = \pm\sqrt{\sin^2 c_1 + \cos^2 c_1} = \pm 1.$$

The second of equations (2) shows that as a point (c_1, y) moves upward along the entire length of the line, its image moves upward along the entire length of the hyperbola's branch. Such a line and its image are shown in Fig. 116, where corresponding points are labeled. Note that, in particular, there is a one to one mapping of the top half $(y > 0)$ of the line onto the top half $(v > 0)$ of the hyperbola's branch. If $-\pi/2 < c_1 < 0$, the line $x = c_1$ is mapped onto the left-hand branch of the same hyperbola. As before, corresponding points are indicated in Fig. 116.

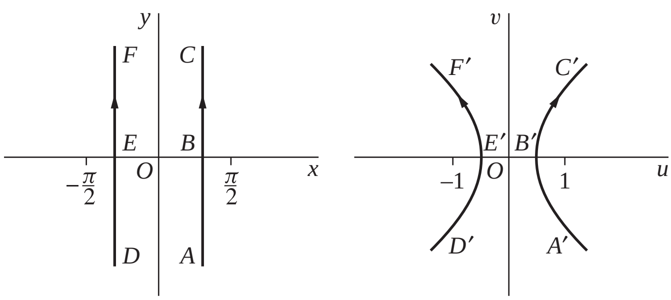

FIGURE 116
$w = \sin z$.

The line $x = 0$, or the y axis, needs to be considered separately. According to equations (1), the image of each point $(0, y)$ is $(0, \sinh y)$. Hence the y axis is mapped onto the v axis in a one to one manner, the positive y axis corresponding to the positive v axis.

We now illustrate how these observations can be used to establish the images of certain regions.

EXAMPLE 1. Here we show that the transformation $w = \sin z$ is a one to one mapping of the semi-infinite strip $-\pi/2 \le x \le \pi/2$, $y \ge 0$ in the z plane onto the upper half $v \ge 0$ of the w plane.

To do this, we first show that the boundary of the strip is mapped in a one to one manner onto the real axis in the w plane, as indicated in Fig. 117. The image of the line segment BA there is found by writing $x = \pi/2$ in equations (1) and restricting y to be nonnegative. Since $u = \cosh y$ and $v = 0$ when $x = \pi/2$, a typical point $(\pi/2, y)$ on BA is mapped onto the point $(\cosh y, 0)$ in the w plane; and that image must move to the right from B' along the u axis as $(\pi/2, y)$ moves upward from B. A point $(x, 0)$ on the horizontal segment DB has image $(\sin x, 0)$, which moves to the right from D' to B' as x increases from $x = -\pi/2$ to $x = \pi/2$, or as $(x, 0)$ goes from D to B. Finally, as a point $(-\pi/2, y)$ on the line segment DE moves upward from D, its image $(-\cosh y, 0)$ moves to the left from D'.

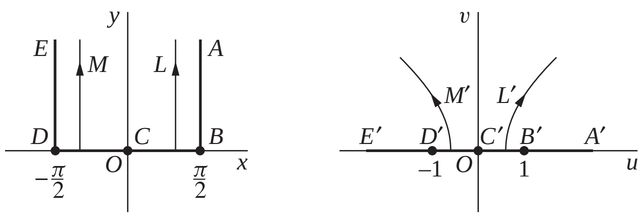

FIGURE 117
$w = \sin z$.

Now each point in the interior $-\pi/2 < x < \pi/2$, $y > 0$ of the strip lies on one of the vertical half lines $x = c_1$, $y > 0$ $(-\pi/2 < c_1 < \pi/2)$ that are shown in Fig. 117. Also, it is important to notice that the images of those half lines are distinct and constitute the entire half plane $v > 0$. More precisely, if the upper half L of a line $x = c_1$ $(0 < c_1 < \pi/2)$ is thought of as moving to the left toward the positive y axis, the right-hand branch of the hyperbola containing its image L' is opening up wider and its vertex $(\sin c_1, 0)$ is tending toward the origin $w = 0$. Hence L' tends to become the positive v axis, which we saw just prior to this example is the image of the positive y axis. On the other hand, as L approaches the segment BA of the boundary of the strip, the branch of the hyperbola closes down around the segment $B'A'$ of the u axis and its vertex $(\sin c_1, 0)$ tends toward the point $w = 1$. Similar statements can be made regarding the half line M and its image M' in Fig. 117. We may conclude that the image of each point in the interior of the strip lies in the upper half plane $v > 0$ and, furthermore, that each point in the half plane is the image of exactly one point in the interior of the strip.

This completes our demonstration that the transformation $w = \sin z$ is a one to one mapping of the strip $-\pi/2 \le x \le \pi/2$, $y \ge 0$ onto the half plane $v \ge 0$. The final result is shown in Fig. 9, Appendix 2. The right-hand half of the strip is evidently mapped onto the first quadrant of the w plane, as shown in Fig. 10, Appendix 2.

Another convenient way to find the images of certain regions when $w = \sin z$ is to consider the images of *horizontal* line segments $y = c_2$ $(-\pi \le x \le \pi)$, where $c_2 > 0$. According to equations (1), the image of such a line segment is the curve with parametric representation

$$u = \sin x \cosh c_2, \quad v = \cos x \sinh c_2 \qquad (-\pi \le x \le \pi). \tag{4}$$

That curve is readily seen to be the ellipse

$$\frac{u^2}{\cosh^2 c_2} + \frac{v^2}{\sinh^2 c_2} = 1, \tag{5}$$

whose foci lie at the points

$$w = \pm\sqrt{\cosh^2 c_2 - \sinh^2 c_2} = \pm 1.$$

The image of a point (x, c_2) moving to the right from point A to point E in Fig. 118 makes one circuit around the ellipse in the clockwise direction. Note that when smaller values of the positive number c_2 are taken, the ellipse becomes smaller but retains the same foci $(\pm 1, 0)$. In the limiting case $c_2 = 0$, equations (4) become

$$u = \sin x, \quad v = 0 \qquad (-\pi \le x \le \pi);$$

and we find that the interval $-\pi \le x \le \pi$ of the x axis is mapped onto the interval $-1 \le u \le 1$ of the u axis. The mapping is not, however, one to one, as it is when $c_2 > 0$.

The next example relies on these remarks.

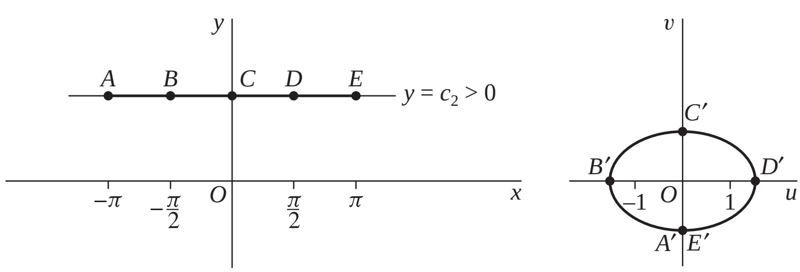

FIGURE 118
$w = \sin z$.

EXAMPLE 2. The rectangular region $-\pi/2 \le x \le \pi/2,\ 0 \le y \le b$ is mapped by $w = \sin z$ in a one to one manner onto the semi-elliptical region that is shown in Fig. 119, where corresponding boundary points are also indicated. For if L is a line segment $y = c_2$ $(-\pi/2 \le x \le \pi/2)$, where $0 < c_2 \le b$, its image L' is the top half of the ellipse (5). As c_2 decreases, L moves downward toward the x axis and the semi-ellipse L' also moves downward and tends to become the line segment $E'F'A'$ from $w = -1$ to $w = 1$. In fact, when $c_2 = 0$, equations (4) become

$$u = \sin x, \quad v = 0 \qquad \left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right);$$

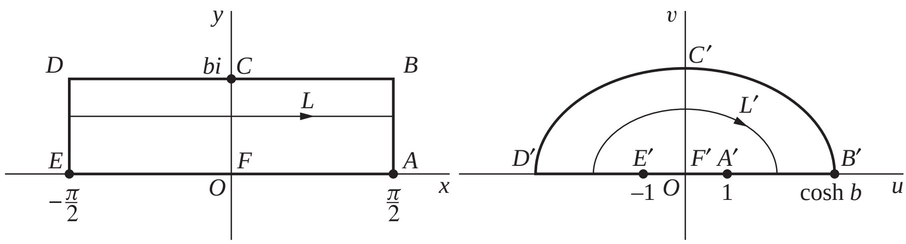

FIGURE 119
$w = \sin z$.

and this is clearly a one to one mapping of the segment EFA onto $E'F'A'$. Inasmuch as any point in the semi-elliptical region in the w plane lies on one and only one of the semi-ellipses, or on the limiting case $E'F'A'$, that point is the image of exactly one point in the rectangular region in the z plane. The desired mapping, which is also shown in Fig. 11 of Appendix 2, is now established.

Mappings by various other functions closely related to the sine function are easily obtained once mappings by the sine function are known.

EXAMPLE 3. One need only recall the identity (Sec. 34)

$$\cos z = \sin\left(z + \frac{\pi}{2}\right)$$

to see that the transformation $w = \cos z$ can be written successively as

$$Z = z + \frac{\pi}{2}, \quad w = \sin Z.$$

Hence the cosine transformation is the same as the sine transformation preceded by a translation to the right through $\pi/2$ units.

EXAMPLE 4. According to Sec. 35, the transformation $w = \sinh z$ can be written $w = -i \sin(iz)$, or

$$Z = iz, \quad W = \sin Z, \quad w = -iW.$$

It is, therefore, a combination of the sine transformation and rotations through right angles. The transformation $w = \cosh z$ is, likewise, essentially a cosine transformation since $\cosh z = \cos(iz)$.

EXERCISES

1. Show that the transformation $w = \sin z$ maps the top half $(y > 0)$ of the vertical line $x = c_1$ $(-\pi/2 < c_1 < 0)$ in a one to one manner onto the top half $(v > 0)$ of the left-hand branch of hyperbola (3), Sec. 96, as indicated in Fig. 117 of that section.

2. Show that under the transformation $w = \sin z$, a line $x = c_1$ $(\pi/2 < c_1 < \pi)$ is mapped onto the right-hand branch of hyperbola (3), Sec. 96. Note that the mapping is one to one and that the upper and lower halves of the line are mapped onto the *lower* and *upper* halves, respectively, of the branch.

3. Vertical half lines were used in Example 1, Sec. 96, to show that the transformation $w = \sin z$ is a one to one mapping of the open region $-\pi/2 < x < \pi/2$, $y > 0$ onto the half plane $v > 0$. Verify that result by using, instead, the horizontal line segments $y = c_2$ $(-\pi/2 < x < \pi/2)$, where $c_2 > 0$.

4. *(a)* Show that under the transformation $w = \sin z$, the images of the line segments forming the boundary of the rectangular region $0 \le x \le \pi/2, 0 \le y \le 1$ are the line segments and the arc $D'E'$ indicated in Fig. 120. The arc $D'E'$ is a quarter of the ellipse

$$\frac{u^2}{\cosh^2 1} + \frac{v^2}{\sinh^2 1} = 1.$$

(b) Complete the mapping indicated in Fig. 120 by using images of horizontal line segments to prove that the transformation $w = \sin z$ establishes a one to one correspondence between the interior points of the regions $ABDE$ and $A'B'D'E'$.

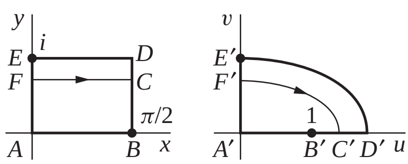

FIGURE 120
$w = \sin z$.

5. Verify that the interior of a rectangular region $-\pi \le x \le \pi, a \le y \le b$ lying above the x axis is mapped by $w = \sin z$ onto the interior of an elliptical ring which has a cut along the segment $-\sinh b \le v \le -\sinh a$ of the negative real axis, as indicated in Fig. 121. Note that while the mapping of the interior of the rectangular region is one to one, the mapping of its boundary is *not*.

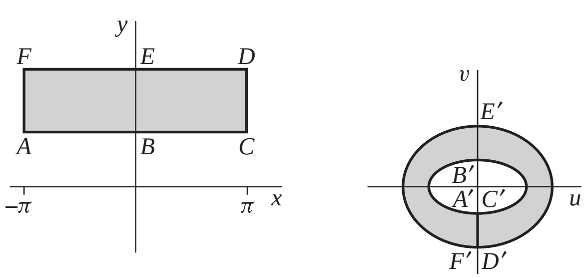

FIGURE 121
$w = \sin z$.

6. *(a)* Show that the equation $w = \cosh z$ can be written

$$Z = iz + \frac{\pi}{2}, \quad w = \sin Z.$$

(b) Use the result in part *(a)*, together with the mapping by $\sin z$ shown in Fig. 10, Appendix 2, to verify that the transformation $w = \cosh z$ maps the semi-infinite strip $x \geq 0, 0 \leq y \leq \pi/2$ in the z plane onto the first quadrant $u \geq 0, v \geq 0$ of the w plane. Indicate corresponding parts of the boundaries of the two regions.

7. Observe that the transformation $w = \cosh z$ can be expressed as a composition of the mappings

$$Z = e^z, \quad W = Z + \frac{1}{Z}, \quad w = \frac{1}{2}W.$$

Then, by referring to Figs. 7 and 16 in Appendix 2, show that when $w = \cosh z$, the semi-infinite strip $x \leq 0, 0 \leq y \leq \pi$ in the z plane is mapped onto the lower half $v \leq 0$ of the w plane. Indicate corresponding parts of the boundaries.

8. *(a)* Verify that the equation $w = \sin z$ can be written

$$Z = i\left(z + \frac{\pi}{2}\right), \quad W = \cosh Z, \quad w = -W.$$

(b) Use the result in part *(a)* here and the one in Exercise 7 to show that the transformation $w = \sin z$ maps the semi-infinite strip $-\pi/2 \leq x \leq \pi/2, y \geq 0$ onto the half plane $v \geq 0$, as shown in Fig. 9, Appendix 2. (This mapping was verified in a different way in Example 1, Sec. 96, and in Exercise 3.)

97. MAPPINGS BY z^2 AND BRANCHES OF $z^{1/2}$

In Chap 2 (Sec. 13), we considered some fairly simple mappings under the transformation $w = z^2$, written in the form

$$u = x^2 - y^2, \quad v = 2xy. \tag{1}$$

We turn now to a less elementary example and then examine related mappings $w = z^{1/2}$, where specific branches of the square root function are taken.

EXAMPLE 1. Let us use equations (1) to show that the image of the vertical strip $0 \leq x \leq 1, y \geq 0$, shown in Fig. 122, is the closed semiparabolic region indicated there.

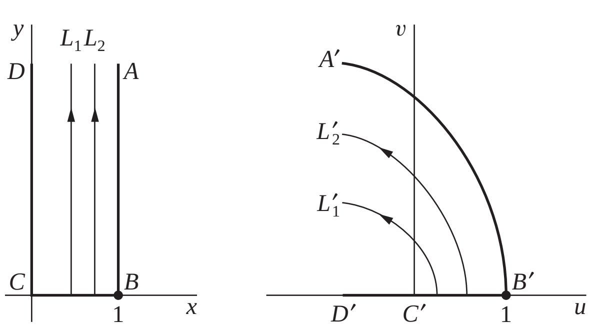

FIGURE 122
$w = z^2$.