(b) Use the result in part (a), together with the mapping by $\sin z$ shown in Fig. 10, Appendix 2, to verify that the transformation $w = \cosh z$ maps the semi-infinite strip $x \geq 0, 0 \leq y \leq \pi/2$ in the z plane onto the first quadrant $u \geq 0, v \geq 0$ of the w plane. Indicate corresponding parts of the boundaries of the two regions.

7. Observe that the transformation $w = \cosh z$ can be expressed as a composition of the mappings

$$Z = e^z, \quad W = Z + \frac{1}{Z}, \quad w = \frac{1}{2}W.$$

Then, by referring to Figs. 7 and 16 in Appendix 2, show that when $w = \cosh z$, the semi-infinite strip $x \leq 0, 0 \leq y \leq \pi$ in the z plane is mapped onto the lower half $v \leq 0$ of the w plane. Indicate corresponding parts of the boundaries.

8. (a) Verify that the equation $w = \sin z$ can be written

$$Z = i\left(z + \frac{\pi}{2}\right), \quad W = \cosh Z, \quad w = -W.$$

(b) Use the result in part (a) here and the one in Exercise 7 to show that the transformation $w = \sin z$ maps the semi-infinite strip $-\pi/2 \leq x \leq \pi/2, y \geq 0$ onto the half plane $v \geq 0$, as shown in Fig. 9, Appendix 2. (This mapping was verified in a different way in Example 1, Sec. 96, and in Exercise 3.)

97. MAPPINGS BY z^2 AND BRANCHES OF $z^{1/2}$

In Chap 2 (Sec. 13), we considered some fairly simple mappings under the transformation $w = z^2$, written in the form

$$u = x^2 - y^2, \quad v = 2xy. \tag{1}$$

We turn now to a less elementary example and then examine related mappings $w = z^{1/2}$, where specific branches of the square root function are taken.

EXAMPLE 1. Let us use equations (1) to show that the image of the vertical strip $0 \leq x \leq 1, y \geq 0$, shown in Fig. 122, is the closed semiparabolic region indicated there.

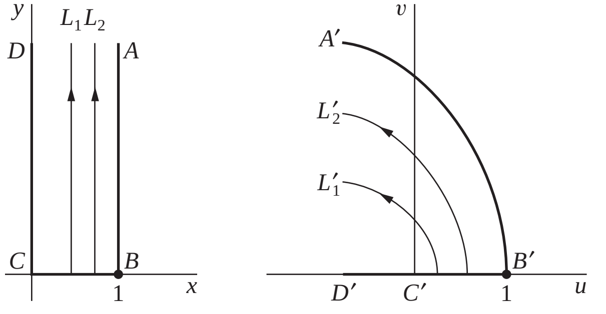

FIGURE 122
$w = z^2$.

When $0 < x_1 < 1$, the point (x_1, y) moves up a vertical half line, labeled L_1 in Fig. 122, as y increases from $y = 0$. The image traced out in the uv plane has, according to equations (1), the parametric representation

$$u = x_1^2 - y^2, \quad v = 2x_1 y \qquad (0 \le y < \infty). \tag{2}$$

Using the second of these equations to substitute for y in the first one, we see that the image points (u, v) must lie on the parabola

$$v^2 = -4x_1^2(u - x_1^2), \tag{3}$$

with vertex at $(x_1^2, 0)$ and focus at the origin. Since v increases with y from $v = 0$, according to the second of equations (2), we also see that as the point (x_1, y) moves up L_1 from the x axis, its image moves up the top half L_1' of the parabola from the u axis. Furthermore, when a number x_2 larger than x_1 but less than 1 is taken, the corresponding half line L_2 has an image L_2' that is a half parabola to the right of L_1', as indicated in Fig. 122. We note, in fact, that the image of the half line BA in that figure is the top half of the parabola $v^2 = -4(u - 1)$, labeled $B'A'$.

The image of the half line CD is found by observing from equations (1) that a typical point $(0, y)$, where $y \ge 0$, on CD is transformed into the point $(-y^2, 0)$ in the uv plane. So, as a point moves up from the origin along CD, its image moves left from the origin along the u axis. Evidently, then, as the vertical half lines in the xy plane move to the left, the half parabolas that are their images in the uv plane shrink down to become the half line $C'D'$.

It is now clear that the images of all the half lines between and including CD and BA fill up the closed semiparabolic region bounded by $A'B'C'D'$. Also, each point in that region is the image of only one point in the closed strip bounded by *ABCD*. Hence we may conclude that the semiparabolic region is the image of the strip and that there is a one to one correspondence between points in those closed regions. (Compare with Fig. 3 in Appendix 2, where the strip has arbitrary width.)

As for mappings by branches of $z^{1/2}$, we recall from Sec. 9 that the values of $z^{1/2}$ are the two square roots of z when $z \ne 0$. According to that section, if polar coordinates are used and

$$z = r\exp(i\Theta) \qquad (r > 0, -\pi < \Theta \le \pi),$$

then

$$z^{1/2} = \sqrt{r}\exp\frac{i(\Theta + 2k\pi)}{2} \qquad (k = 0, 1), \tag{4}$$

the principal root occurring when $k = 0$. In Sec. 32, we saw that $z^{1/2}$ can also be written

$$z^{1/2} = \exp\left(\frac{1}{2}\log z\right) \qquad (z \ne 0). \tag{5}$$

The *principal branch* $F_0(z)$ of the double-valued function $z^{1/2}$ is then obtained by taking the principal branch of $\log z$ and writing (see Sec. 33)

$$F_0(z) = \exp\left(\frac{1}{2}\operatorname{Log} z\right) \qquad (|z| > 0, -\pi < \operatorname{Arg} z < \pi).$$

Since

$$\frac{1}{2}\operatorname{Log} z = \frac{1}{2}(\ln r + i\Theta) = \ln\sqrt{r} + \frac{i\Theta}{2}$$

when $z = r\exp(i\Theta)$, this becomes

$$F_0(z) = \sqrt{r}\exp\frac{i\Theta}{2} \qquad (r > 0, -\pi < \Theta < \pi). \tag{6}$$

The right-hand side of this equation is, of course, the same as the right-hand side of equation (4) when $k = 0$ and $-\pi < \Theta < \pi$ there. The origin and the ray $\Theta = \pi$ form the branch cut for F_0, and the origin is the branch point.

Images of curves and regions under the transformation $w = F_0(z)$ may be obtained by writing $w = \rho\exp(i\phi)$, where $\rho = \sqrt{r}$ and $\phi = \Theta/2$. Arguments are evidently halved by this transformation, and it is understood that $w = 0$ when $z = 0$.

EXAMPLE 2. It is easy to verify that $w = F_0(z)$ is a one to one mapping of the quarter disk $0 \le r \le 2, 0 \le \theta \le \pi/2$ onto the sector $0 \le \rho \le \sqrt{2}, 0 \le \phi \le \pi/4$ in the w plane (Fig. 123). To do this, we observe that as a point $z = r\exp(i\theta_1)$ moves outward from the origin along a radius R_1 of length 2 and with angle of inclination θ_1 $(0 \le \theta_1 \le \pi/2)$, its image $w = \sqrt{r}\exp(i\theta_1/2)$ moves outward from the origin in the w plane along a radius R_1' whose length is $\sqrt{2}$ and angle of inclination is $\theta_1/2$. See Fig. 123, where another radius R_2 and its image R_2' are also shown. It is now clear from the figure that if the region in the z plane is thought of as being swept out by a radius, starting with DA and ending with DC, then the region in the w plane is swept out by the corresponding radius, starting with $D'A'$ and ending with $D'C'$. This establishes a one to one correspondence between points in the two regions.

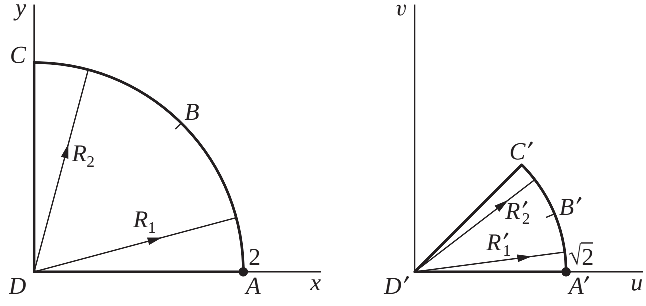

FIGURE 123
$w = F_0(z)$.

EXAMPLE 3. The transformation $w = F_0(\sin z)$ can be written

$$Z = \sin z, \qquad w = F_0(Z) \qquad (|Z| > 0, -\pi < \operatorname{Arg} Z < \pi).$$

From a remark at the end of Example 1 in Sec. 96, we know that the first transformation maps the semi-infinite strip $0 \le x \le \pi/2$, $y \ge 0$ onto the first quadrant of the Z plane. The second transformation, with the understanding that $F_0(0) = 0$, maps that quadrant onto an octant in the w plane. These successive transformations are illustrated in Fig. 124, where corresponding boundary points are shown.

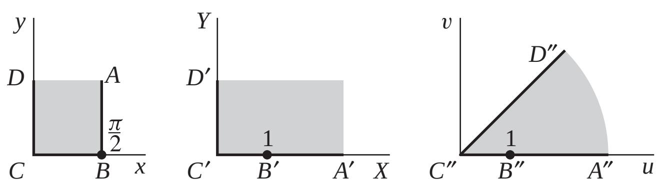

FIGURE 124
$w = F_0(\sin z)$.

When $-\pi < \Theta < \pi$ and the branch

$$\log z = \ln r + i(\Theta + 2\pi)$$

of the logarithmic function is used, equation (5) yields the branch

$$F_1(z) = \sqrt{r} \exp \frac{i(\Theta + 2\pi)}{2} \qquad (r > 0, -\pi < \Theta < \pi) \tag{7}$$

of $z^{1/2}$, which corresponds to $k = 1$ in equation (4). Since $\exp(i\pi) = -1$, it follows that $F_1(z) = -F_0(z)$. The values $\pm F_0(z)$ thus represent the totality of values of $z^{1/2}$ at all points in the domain $r > 0, -\pi < \Theta < \pi$. If, by means of expression (6), we extend the domain of definition of F_0 to include the ray $\Theta = \pi$ and if we write $F_0(0) = 0$, then the values $\pm F_0(z)$ represent the totality of values of $z^{1/2}$ in the entire z plane.

Other branches of $z^{1/2}$ are obtained by using other branches of $\log z$ in expression (5). A branch where the ray $\theta = \alpha$ is used to form the branch cut is given by the equation

$$f_\alpha(z) = \sqrt{r} \exp \frac{i\theta}{2} \qquad (r > 0, \alpha < \theta < \alpha + 2\pi). \tag{8}$$

Observe that when $\alpha = -\pi$, we have the branch $F_0(z)$ and that when $\alpha = \pi$, we have the branch $F_1(z)$. Just as in the case of F_0, the domain of definition of f_α can be extended to the entire complex plane by using expression (8) to define f_α at the nonzero points on the branch cut and by writing $f_\alpha(0) = 0$. Such extensions are, however, never continuous on the entire complex plane.

Finally, suppose that n is any positive integer, where $n \geq 2$. The values of $z^{1/n}$ are the nth roots of z when $z \neq 0$; and, according to Sec. 32, the multiple-valued function $z^{1/n}$ can be written

$$(9) \qquad z^{1/n} = \exp\left(\frac{1}{n}\log z\right) = \sqrt[n]{r}\exp\frac{i(\Theta + 2k\pi)}{n} \qquad (k = 0, 1, 2, \ldots, n-1),$$

where $r = |z|$ and $\Theta = \text{Arg } z$. The case $n = 2$ has just been considered. In the general case, each of the n functions

$$(10) \qquad F_k(z) = \sqrt[n]{r}\exp\frac{i(\Theta + 2k\pi)}{n} \qquad (k = 0, 1, 2, \ldots, n-1)$$

is a branch of $z^{1/n}$, defined on the domain $r > 0, -\pi < \Theta < \pi$. When $w = \rho e^{i\phi}$, the transformation $w = F_k(z)$ is a one to one mapping of that domain onto the domain

$$\rho > 0, \quad \frac{(2k-1)\pi}{n} < \phi < \frac{(2k+1)\pi}{n}.$$

These n branches of $z^{1/n}$ yield the n distinct nth roots of z at any point z in the domain $r > 0, -\pi < \Theta < \pi$. The principal branch occurs when $k = 0$, and further branches of the type (8) are readily constructed.

EXERCISES

1. Show, indicating corresponding orientations, that the mapping $w = z^2$ transforms horizontal lines $y = y_1$ $(y_1 > 0)$ into parabolas $v^2 = 4y_1^2(u + y_1^2)$, all with foci at the origin $w = 0$. (Compare with Example 1, Sec. 97.)

2. Use the result in Exercise 1 to show that the transformation $w = z^2$ is a one to one mapping of a horizontal strip $a \leq y \leq b$ above the x axis onto the closed region between the two parabolas

$$v^2 = 4a^2(u + a^2), \quad v^2 = 4b^2(u + b^2).$$

3. Point out how it follows from the discussion in Example 1, Sec. 97, that the transformation $w = z^2$ maps a vertical strip $0 \leq x \leq c$, $y \geq 0$ of arbitrary width onto a closed semiparabolic region, as shown in Fig. 3, Appendix 2.

4. Modify the discussion in Example 1, Sec. 97, to show that when $w = z^2$, the image of the closed triangular region formed by the lines $y = \pm x$ and $x = 1$ is the closed parabolic region bounded on the left by the segment $-2 \leq v \leq 2$ of the v axis and on the right by a portion of the parabola $v^2 = -4(u - 1)$. Verify the corresponding points on the two boundaries shown in Fig. 125.

5. By referring to Fig. 10, Appendix 2, show that the transformation $w = \sin^2 z$ maps the strip $0 \leq x \leq \pi/2$, $y \geq 0$ onto the half plane $v \geq 0$. Indicate corresponding parts of the boundaries.

Suggestion: See also the first paragraph in Example 3, Sec. 13.

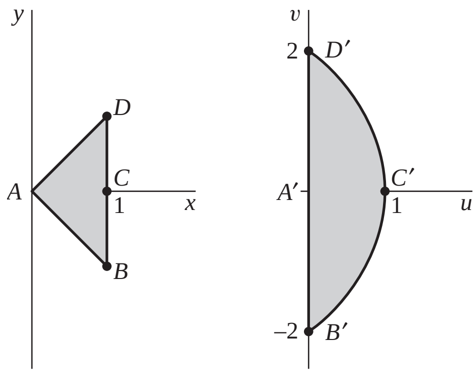

FIGURE 125
$w = z^2$.

6. Use Fig. 9, Appendix 2, to show that if $w = (\sin z)^{1/4}$ and the principal branch of the fractional power is taken, then the semi-infinite strip $-\pi/2 < x < \pi/2, y > 0$ is mapped onto the part of the first quadrant lying between the line $v = u$ and the u axis. Label corresponding parts of the boundaries.

7. According to Example 2, Sec. 95, the linear fractional transformation

$$Z = \frac{z-1}{z+1}$$

maps the x axis onto the X axis and the half planes $y > 0$ and $y < 0$ onto the half planes $Y > 0$ and $Y < 0$, respectively. Show that, in particular, it maps the segment $-1 \le x \le 1$ of the x axis onto the segment $X \le 0$ of the X axis. Then show that when the principal branch of the square root is used, the composite function

$$w = Z^{1/2} = \left(\frac{z-1}{z+1}\right)^{1/2}$$

maps the z plane, except for the segment $-1 \le x \le 1$ of the x axis, onto the right half plane $u > 0$.

8. Determine the image of the domain $r > 0, -\pi < \Theta < \pi$ in the z plane under each of the transformations $w = F_k(z)$ $(k = 0, 1, 2, 3)$, where $F_k(z)$ are the four branches of $z^{1/4}$ given by equation (10), Sec. 97, when $n = 4$. Use these branches to determine the fourth roots of i.

98. SQUARE ROOTS OF POLYNOMIALS

We now consider some mappings that are compositions of polynomials and square roots.

EXAMPLE 1. Branches of the double-valued function $(z - z_0)^{1/2}$ can be obtained by noting that it is a composition of the translation $Z = z - z_0$ with the