to define a branch of the function

(a) $(z^2-1)^{1/2}$; *(b)* $\left(\dfrac{z-1}{z+1}\right)^{1/2}$.

In each case, the branch cut should consist of the two rays $\theta_1 = 0$ and $\Theta_2 = \pi$.

6. Using the notation in Sec. 98, show that the function

$$w = \left(\frac{z-1}{z+1}\right)^{1/2} = \sqrt{\frac{r_1}{r_2}}\exp\frac{i(\theta_1-\theta_2)}{2}$$

is a branch with the same domain of definition D_z and the same branch cut as the function $w = F(z)$ in that section. Show that this transformation maps D_z onto the right half plane $\rho > 0$, $-\pi/2 < \phi < \pi/2$, where the point $w = 1$ is the image of the point $z = \infty$. Also, show that the inverse transformation is

$$z = \frac{1+w^2}{1-w^2} \qquad (\text{Re}\, w > 0).$$

(Compare with Exercise 7, Sec. 97.)

7. Show that the transformation in Exercise 6 maps the region outside the unit circle $|z| = 1$ in the upper half of the z plane onto the region in the first quadrant of the w plane between the line $v = u$ and the u axis. Sketch the two regions.

8. Write $z = r\exp(i\Theta)$, $z-1 = r_1\exp(i\Theta_1)$, and $z+1 = r_2\exp(i\Theta_2)$, where the values of all three arguments lie between $-\pi$ and π. Then define a branch of the function $[z(z^2-1)]^{1/2}$ whose branch cut consists of the two segments $x \le -1$ and $0 \le x \le 1$ of the x axis.

99. RIEMANN SURFACES

The remaining two sections of this chapter constitute a brief introduction to the concept of a mapping defined on a *Riemann surface*, which is a generalization of the complex plane consisting of more than one sheet. The theory rests on the fact that at each point on such a surface only one value of a given multiple-valued function is assigned. The material in these two sections will not be used in the chapters to follow, and the reader may skip to Chap. 9 without disruption.

Once a Riemann surface is devised for a given function, the function is single-valued on the surface and the theory of single-valued functions applies there. Complexities arising because the function is multiple-valued are thus relieved by a geometric device. However, the description of those surfaces and the arrangement of proper connections between the sheets can become quite involved. We limit our attention to fairly simple examples and begin with a surface for $\log z$.

EXAMPLE 1. Corresponding to each nonzero number z, the multiple-valued function

$$\log z = \ln r + i\theta \tag{1}$$

has infinitely many values. To describe $\log z$ as a single-valued function, we replace the z plane, with the origin deleted, by a surface on which a new point is located whenever the argument of the number z is increased or decreased by 2π, or an integral multiple of 2π.

We treat the z plane, with the origin deleted, as a thin sheet R_0 which is cut along the positive half of the real axis. On that sheet, let θ range from 0 to 2π. Let a second sheet R_1 be cut in the same way and placed in front of the sheet R_0. The lower edge of the slit in R_0 is then joined to the upper edge of the slit in R_1. On R_1, the angle θ ranges from 2π to 4π; so, when z is represented by a point on R_1, the imaginary component of $\log z$ ranges from 2π to 4π.

A sheet R_2 is then cut in the same way and placed in front of R_1. The lower edge of the slit in R_1 is joined to the upper edge of the slit in this new sheet, and similarly for sheets $R_3, R_4, \ldots$. A sheet R_{-1} on which θ varies from 0 to -2π is cut and placed behind R_0, with the lower edge of its slit connected to the upper edge of the slit in R_0; the sheets $R_{-2}, R_{-3}, \ldots$ are constructed in like manner. The coordinates r and θ of a point on any sheet can be considered as polar coordinates of the projection of the point onto the original z plane, the angular coordinate θ being restricted to a definite range of 2π radians on each sheet.

Consider any continuous curve on this connected surface of infinitely many sheets. As a point z describes that curve, the values of $\log z$ vary continuously since θ, in addition to r, varies continuously; and $\log z$ now assumes just one value corresponding to each point on the curve. For example, as the point makes a complete cycle around the origin on the sheet R_0 over the path indicated in Fig. 129, the angle changes from 0 to 2π. As it moves across the ray $\theta = 2\pi$, the point passes to the sheet R_1 of the surface. As the point completes a cycle in R_1, the angle θ varies from 2π to 4π; and as it crosses the ray $\theta = 4\pi$, the point passes to the sheet R_2.

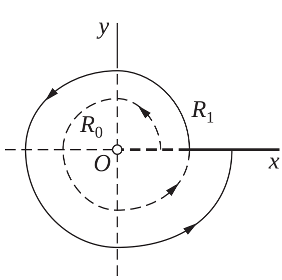

FIGURE 129

The surface described here is a Riemann surface for $\log z$. It is a connected surface of infinitely many sheets, arranged so that $\log z$ is a single-valued function of points on it.

The transformation $w = \log z$ maps the whole Riemann surface in a one to one manner onto the entire w plane. The image of the sheet R_0 is the strip $0 \leq v \leq 2\pi$ (see Example 3, Sec. 95). As a point z moves onto the sheet R_1 over the arc shown

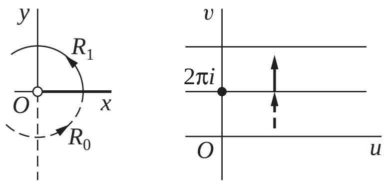

FIGURE 130

in Fig. 130, its image w moves upward across the line $v = 2\pi$, as indicated in that figure.

Note that $\log z$, defined on the sheet R_1, represents the analytic continuation (Sec. 27) of the single-valued analytic function

$$f(z) = \ln r + i\theta \qquad (0 < \theta < 2\pi)$$

upward across the positive real axis. In this sense, $\log z$ is not only a single-valued function of all points z on the Riemann surface but also an *analytic* function at all points there.

The sheets could, of course, be cut along the negative real axis or along any other ray from the origin, and properly joined along the slits, to form other Riemann surfaces for $\log z$.

EXAMPLE 2. Corresponding to each point in the z plane other than the origin, the square root function

(2) $$z^{1/2} = \sqrt{r}e^{i\theta/2}$$

has two values. A Riemann surface for $z^{1/2}$ is obtained by replacing the z plane with a surface made up of two sheets R_0 and R_1, each cut along the positive real axis and with R_1 placed in front of R_0. The lower edge of the slit in R_0 is joined to the upper edge of the slit in R_1, and the lower edge of the slit in R_1 is joined to the upper edge of the slit in R_0.

As a point z starts from the upper edge of the slit in R_0 and describes a continuous circuit around the origin in the counterclockwise direction (Fig. 131),

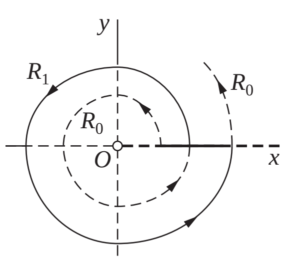

FIGURE 131

the angle θ increases from 0 to 2π. The point then passes from the sheet R_0 to the sheet R_1, where θ increases from 2π to 4π. As the point moves still further, it passes back to the sheet R_0, where the values of θ can vary from 4π to 6π or from 0 to 2π, a choice that does not affect the value of $z^{1/2}$, etc. Note that the value of $z^{1/2}$ at a point where the circuit passes from the sheet R_0 to the sheet R_1 is different from the value of $z^{1/2}$ at a point where the circuit passes from the sheet R_1 to the sheet R_0.

We have thus constructed a Riemann surface on which $z^{1/2}$ is single-valued for each nonzero z. In that construction, the edges of the sheets R_0 and R_1 are joined in pairs in such a way that the resulting surface is closed and connected. The points where two of the edges are joined are distinct from the points where the other two edges are joined. Thus it is physically impossible to build a model of that Riemann surface. In visualizing a Riemann surface, it is important to understand how we are to proceed when we arrive at an edge of a slit.

The origin is a special point on this Riemann surface. It is common to both sheets, and a curve around the origin on the surface must wind around it twice in order to be a closed curve. A point of this kind on a Riemann surface is called a *branch point.*

The image of the sheet R_0 under the transformation $w = z^{1/2}$ is the upper half of the w plane since the argument of w is $\theta/2$ on R_0, where $0 \le \theta/2 \le \pi$. Likewise, the image of the sheet R_1 is the lower half of the w plane. As defined on either sheet, the function is the analytic continuation, across the cut, of the function defined on the other sheet. In this respect, the single-valued function $z^{1/2}$ of points on the Riemann surface is analytic at all points except the origin.

EXERCISES

1. Describe the Riemann surface for $\log z$ obtained by cutting the z plane along the negative real axis. Compare this Riemann surface with the one obtained in Example 1, Sec. 99.

2. Determine the image under the transformation $w = \log z$ of the sheet R_n, where n is an arbitrary integer, of the Riemann surface for $\log z$ given in Example 1, Sec. 99.

3. Verify that under the transformation $w = z^{1/2}$, the sheet R_1 of the Riemann surface for $z^{1/2}$ given in Example 2, Sec. 99, is mapped onto the lower half of the w plane.

4. Describe the curve, on a Riemann surface for $z^{1/2}$, whose image is the entire circle $|w| = 1$ under the transformation $w = z^{1/2}$.

5. Let C denote the positively oriented circle $|z - 2| = 1$ on the Riemann surface described in Example 2, Sec. 99, for $z^{1/2}$, where the upper half of that circle lies on the sheet R_0 and the lower half on R_1. Note that for each point z on C, one can write

$$z^{1/2} = \sqrt{r}e^{i\theta/2} \quad \text{where} \quad 4\pi - \frac{\pi}{2} < \theta < 4\pi + \frac{\pi}{2}.$$

State why it follows that

$$\int_C z^{1/2}\,dz = 0.$$

Generalize this result to fit the case of the other simple closed curves that cross from one sheet to another without enclosing the branch points. Generalize to other functions, thus extending the Cauchy–Goursat theorem to integrals of multiple-valued functions.

100. SURFACES FOR RELATED FUNCTIONS

We consider here Riemann surfaces for two composite functions involving simple polynomials and the square root function.

EXAMPLE 1. Let us describe a Riemann surface for the double-valued function

$$f(z) = (z^2 - 1)^{1/2} = \sqrt{r_1 r_2}\exp\frac{i(\theta_1+\theta_2)}{2}, \tag{1}$$

where $z - 1 = r_1\exp(i\theta_1)$ and $z + 1 = r_2\exp(i\theta_2)$. A branch of this function, with the line segment P_2P_1 between the branch points $z = \pm 1$ serving as a branch cut (Fig. 132), was described in Example 2, Sec. 98. That branch is as written above, with the restrictions $r_k > 0, 0 \le \theta_k < 2\pi$ $(k = 1, 2)$ and $r_1 + r_2 > 2$. The branch is not defined on the segment P_2P_1.

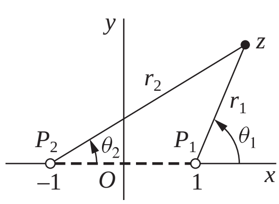

FIGURE 132

A Riemann surface for the double-valued function (1) must consist of two sheets R_0 and R_1. Let both sheets be cut along the segment P_2P_1. The lower edge of the slit in R_0 is then joined to the upper edge of the slit in R_1, and the lower edge in R_1 is joined to the upper edge in R_0.

On the sheet R_0, let the angles θ_1 and θ_2 range from 0 to 2π. If a point on the sheet R_0 describes a simple closed curve that encloses the segment P_2P_1 once in the counterclockwise direction, then both θ_1 and θ_2 change by the amount 2π upon the return of the point to its original position. The change in $(\theta_1 + \theta_2)/2$ is also 2π, and the value of f is unchanged. If a point starting on the sheet R_0 describes a path that passes twice around just the branch point $z = 1$, it crosses from the sheet