Let two sheets be cut along the line segment L_2 from $z = -1$ to $z = 0$ and along the part L_1 of the real axis to the right of the point $z = 1$. We specify that each of the three angles θ, θ_1, and θ_2 may range from 0 to 2π on the sheet R_0 and from 2π to 4π on the sheet R_1. We also specify that the angles corresponding to a point on either sheet may be changed by integral multiples of 2π in such a way that the sum of the three angles changes by an integral multiple of 4π. The value of the function f is, therefore, unaltered.

A Riemann surface for the double-valued function (2) is obtained by joining the lower edges in R_0 of the slits along L_1 and L_2 to the upper edges in R_1 of the slits along L_1 and L_2, respectively. The lower edges in R_1 of the slits along L_1 and L_2 are then joined to the upper edges in R_0 of the slits along L_1 and L_2, respectively. It is readily verified with the aid of Fig. 133 that one branch of the function is represented by its values at points on R_0 and the other branch at points on R_1.

EXERCISES

1. Describe a Riemann surface for the triple-valued function $w = (z-1)^{1/3}$, and point out which third of the w plane represents the image of each sheet of that surface.

2. Corresponding to each point on the Riemann surface described in Example 2, Sec. 100, for the function $w = f(z)$ in that example, there is just one value of w. Show that corresponding to each value of w, there are, in general, three points on the surface.

3. Describe a Riemann surface for the multiple-valued function

$$f(z) = \left(\frac{z-1}{z}\right)^{1/2}.$$

4. Note that the Riemann surface described in Example 1, Sec. 100, for $(z^2-1)^{1/2}$ is also a Riemann surface for the function

$$g(z) = z + (z^2-1)^{1/2}.$$

Let f_0 denote the branch of $(z^2-1)^{1/2}$ defined on the sheet R_0, and show that the branches g_0 and g_1 of g on the two sheets are given by the equations

$$g_0(z) = \frac{1}{g_1(z)} = z + f_0(z).$$

5. In Exercise 4, the branch f_0 of $(z^2-1)^{1/2}$ can be described by means of the equation

$$f_0(z) = \sqrt{r_1 r_2}\left(\exp\frac{i\theta_1}{2}\right)\left(\exp\frac{i\theta_2}{2}\right),$$

where θ_1 and θ_2 range from 0 to 2π and

$$z - 1 = r_1 \exp(i\theta_1), \quad z + 1 = r_2 \exp(i\theta_2).$$

Note that

$$2z = r_1 \exp(i\theta_1) + r_2 \exp(i\theta_2),$$

and show that the branch g_0 of the function $g(z) = z + (z^2 - 1)^{1/2}$ can be written in the form

$$g_0(z) = \frac{1}{2}\left(\sqrt{r_1}\exp\frac{i\theta_1}{2} + \sqrt{r_2}\exp\frac{i\theta_2}{2}\right)^2.$$

Find $g_0(z)\overline{g_0(z)}$ and note that $r_1 + r_2 \geq 2$ and $\cos[(\theta_1 - \theta_2)/2] \geq 0$ for all z, to prove that $|g_0(z)| \geq 1$. Then show that the transformation $w = z + (z^2 - 1)^{1/2}$ maps the sheet R_0 of the Riemann surface onto the region $|w| \geq 1$, the sheet R_1 onto the region $|w| \leq 1$, and the branch cut between the points $z = \pm 1$ onto the circle $|w| = 1$. Note that the transformation used here is an inverse of the transformation

$$z = \frac{1}{2}\left(w + \frac{1}{w}\right).$$