and that when w is in the neighborhood,

$$f[g(w)] = \exp(\log w) = w.$$

Also

$$g'(w) = \frac{d}{dw} \log w = \frac{1}{w} = \frac{1}{\exp z},$$

in accordance with equation (1).

Note that if the point $z_0 = 0$ is chosen, one can use the principal branch

$$\text{Log } w = \ln \rho + i\phi \qquad (\rho > 0, -\pi < \phi < \pi)$$

of the logarithmic function to define g. In this case, $g(1) = 0$.

EXERCISES

1. Determine the angle of rotation at the point $z_0 = 2 + i$ when $w = z^2$, and illustrate it for some particular curve. Show that the scale factor at that point is $2\sqrt{5}$.

2. What angle of rotation is produced by the transformation $w = 1/z$ at the point

(a) $z_0 = 1$; *(b)* $z_0 = i$?

Ans. (a) π; *(b)* 0.

3. Show that under the transformation $w = 1/z$, the images of the lines $y = x - 1$ and $y = 0$ are the circle $u^2 + v^2 - u - v = 0$ and the line $v = 0$, respectively. Sketch all four curves, determine corresponding directions along them, and verify the conformality of the mapping at the point $z_0 = 1$.

4. Show that the angle of rotation at a nonzero point $z_0 = r_0 \exp(i\theta_0)$ under the transformation $w = z^n$ $(n = 1, 2, \ldots)$ is $(n - 1)\theta_0$. Determine the scale factor of the transformation at that point.

Ans. nr_0^{n-1}.

5. Show that the transformation $w = \sin z$ is conformal at all points except

$$z = \frac{\pi}{2} + n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Note that this is in agreement with the mapping of directed line segments shown in Figs. 9, 10, and 11 of Appendix 2.

6. Find the local inverse of the transformation $w = z^2$ at the point

(a) $z_0 = 2$; *(b)* $z_0 = -2$; *(c)* $z_0 = -i$.

Ans. (a) $w^{1/2} = \sqrt{\rho}\, e^{i\phi/2} \quad (\rho > 0, -\pi < \phi < \pi)$;
(c) $w^{1/2} = \sqrt{\rho}\, e^{i\phi/2} \quad (\rho > 0, 2\pi < \phi < 4\pi)$.

7. In Sec. 103, it was pointed out that the components $x(u, v)$ and $y(u, v)$ of the inverse function $g(w)$ defined by equation (6) there are continuous and have continuous first-order partial derivatives in a neighborhood N. Use equations (5), Sec. 103, to show that the Cauchy–Riemann equations $x_u = y_v, x_v = -y_u$ hold in N. Then conclude that $g(w)$ is analytic in that neighborhood.

8. Show that if $z = g(w)$ is the local inverse of a conformal transformation $w = f(z)$ at a point z_0, then

$$g'(w) = \frac{1}{f'(z)}$$

at points w in a neighborhood N where g is analytic (Exercise 7).

Suggestion: Start with the fact that $f[g(w)] = w$, and apply the chain rule for differentiating composite functions.

9. Let C be a smooth arc lying in a domain D throughout which a transformation $w = f(z)$ is conformal, and let Γ denote the image of C under that transformation. Show that Γ is also a smooth arc.

10. Suppose that a function f is analytic at z_0 and that

$$f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0, \quad f^{(m)}(z_0) \neq 0$$

for some positive integer m ($m \geq 1$). Also, write $w_0 = f(z_0)$.

(a) Use the Taylor series for f about the point z_0 to show that there is a neighborhood of z_0 in which the difference $f(z) - w_0$ can be written

$$f(z) - w_0 = (z - z_0)^m \frac{f^{(m)}(z_0)}{m!} [1 + g(z)],$$

where $g(z)$ is continuous at z_0 and $g(z_0) = 0$.

(b) Let Γ be the image of a smooth arc C under the transformation $w = f(z)$, as shown in Fig. 134 (Sec. 101), and note that the angles of inclination θ_0 and ϕ_0 in that figure are limits of $\arg(z - z_0)$ and $\arg[f(z) - w_0]$, respectively, as z approaches z_0 along the arc C. Then use the result in part *(a)* to show that θ_0 and ϕ_0 are related by the equation

$$\phi_0 = m\theta_0 + \arg f^{(m)}(z_0).$$

(c) Let α denote the angle between two smooth arcs C_1 and C_2 passing through z_0, as shown on the left in Fig. 135 (Sec. 101). Show how it follows from the relation obtained in part *(b)* that the corresponding angle between the image curves Γ_1 and Γ_2 at the point $w_0 = f(z_0)$ is $m\alpha$. (Note that the transformation is conformal at z_0 when $m = 1$ and that z_0 is a critical point when $m \geq 2$.)

104. HARMONIC CONJUGATES

We saw in Sec. 26 that if a function

$$f(z) = u(x, y) + iv(x, y)$$

is analytic in a domain D, then the real-valued functions u and v are harmonic in that domain. That is, they have continuous partial derivatives of the first and second order in D and satisfy Laplace's equation there:

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0. \tag{1}$$