polynomials are called *rational functions* and are defined at each point z where $Q(z) \neq 0$. Polynomials and rational functions constitute elementary, but important, classes of functions of a complex variable.

A generalization of the concept of function is a rule that assigns more than one value to a point z in the domain of definition. These *multiple-valued functions* occur in the theory of functions of a complex variable, just as they do in the case of a real variable. When multiple-valued functions are studied, usually just one of the possible values assigned to each point is taken, in a systematic manner, and a (single-valued) function is constructed from the multiple-valued function.

EXAMPLE 4. Let z denote any nonzero complex number. We know from Sec. 9 that $z^{1/2}$ has the two values

$$z^{1/2} = \pm\sqrt{r}\exp\left(i\frac{\Theta}{2}\right),$$

where $r = |z|$ and Θ $(-\pi < \Theta \leq \pi)$ is the *principal value* of $\arg z$. But, if we choose only the positive value of $\pm\sqrt{r}$ and write

$$f(z) = \sqrt{r}\exp\left(i\frac{\Theta}{2}\right) \qquad (r > 0, -\pi < \Theta \leq \pi), \tag{3}$$

the (single-valued) function (3) is well defined on the set of nonzero numbers in the z plane. Since zero is the only square root of zero, we also write $f(0) = 0$. The function f is then well defined on the entire plane.

EXERCISES

1. For each of the functions below, describe the domain of definition that is understood:

(a) $f(z) = \dfrac{1}{z^2+1}$; *(b)* $f(z) = \operatorname{Arg}\left(\dfrac{1}{z}\right)$;

(c) $f(z) = \dfrac{z}{z+\overline{z}}$; *(d)* $f(z) = \dfrac{1}{1-|z|^2}$.

Ans. (a) $z \neq \pm i$; *(c)* $\operatorname{Re} z \neq 0$.

2. Write the function $f(z) = z^3 + z + 1$ in the form $f(z) = u(x, y) + iv(x, y)$.

Ans. $f(z) = (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)$.

3. Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions (see Sec. 5)

$$x = \frac{z+\overline{z}}{2} \quad \text{and} \quad y = \frac{z-\overline{z}}{2i}$$

to write $f(z)$ in terms of z, and simplify the result.

Ans. $f(z) = \overline{z}^2 + 2iz$.

4. Write the function

$$f(z) = z + \frac{1}{z} \qquad (z \neq 0)$$

in the form $f(z) = u(r, \theta) + iv(r, \theta)$.

$$\textit{Ans.}\ f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

13. MAPPINGS

Properties of a real-valued function of a real variable are often exhibited by the graph of the function. But when $w = f(z)$, where z and w are complex, no such convenient graphical representation of the function f is available because each of the numbers z and w is located in a plane rather than on a line. One can, however, display some information about the function by indicating pairs of corresponding points $z = (x, y)$ and $w = (u, v)$. To do this, it is generally simpler to draw the z and w planes separately.

When a function f is thought of in this way, it is often referred to as a *mapping*, or transformation. The *image* of a point z in the domain of definition S is the point $w = f(z)$, and the set of images of all points in a set T that is contained in S is called the image of T. The image of the entire domain of definition S is called the *range* of f. The *inverse image* of a point w is the set of all points z in the domain of definition of f that have w as their image. The inverse image of a point may contain just one point, many points, or none at all. The last case occurs, of course, when w is not in the range of f.

Terms such as *translation, rotation,* and *reflection* are used to convey dominant geometric characteristics of certain mappings. In such cases, it is sometimes convenient to consider the z and w planes to be the same. For example, the mapping

$$w = z + 1 = (x + 1) + iy,$$

where $z = x + iy$, can be thought of as a translation of each point z one unit to the right. Since $i = e^{i\pi/2}$, the mapping

$$w = iz = r\exp\left[i\left(\theta + \frac{\pi}{2}\right)\right],$$

where $z = re^{i\theta}$, rotates the radius vector for each nonzero point z through a right angle about the origin in the counterclockwise direction; and the mapping

$$w = \overline{z} = x - iy$$

transforms each point $z = x + iy$ into its reflection in the real axis.

More information is usually exhibited by sketching images of curves and regions than by simply indicating images of individual points. In the following three examples, we illustrate this with the transformation $w = z^2$. We begin by finding the images of some *curves* in the z plane.