Our final example here uses the images of *horizontal* lines to find the image of a horizontal strip.

EXAMPLE 3. When $w = e^z$, the image of the infinite strip $0 \le y \le \pi$ is the upper half $v \ge 0$ of the w plane (Fig. 22). This is seen by recalling from Example 1 how a horizontal line $y = c$ is transformed into a ray $\phi = c$ from the origin. As the real number c increases from $c = 0$ to $c = \pi$, the y intercepts of the lines increase from 0 to π and the angles of inclination of the rays increase from $\phi = 0$ to $\phi = \pi$. This mapping is also shown in Fig. 6 of Appendix 2, where corresponding points on the boundaries of the two regions are indicated.

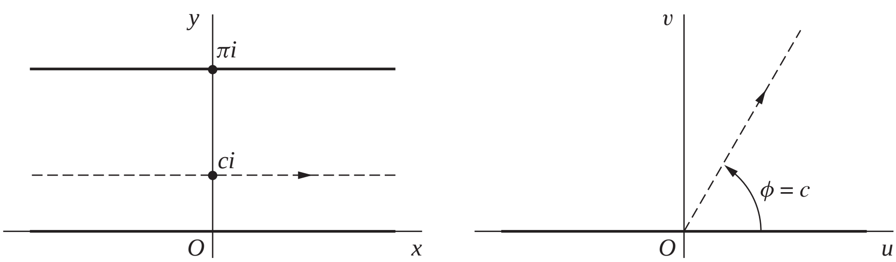

FIGURE 22
$w = \exp z$.

EXERCISES

1. By referring to Example 1 in Sec. 13, find a domain in the z plane whose image under the transformation $w = z^2$ is the square domain in the w plane bounded by the lines $u = 1, u = 2, v = 1$, and $v = 2$. (See Fig. 2, Appendix 2.)

2. Find and sketch, showing corresponding orientations, the images of the hyperbolas
$$x^2 - y^2 = c_1 \ (c_1 < 0) \quad \text{and} \quad 2xy = c_2 \ (c_2 < 0)$$
under the transformation $w = z^2$.

3. Sketch the region onto which the sector $r \le 1, 0 \le \theta \le \pi/4$ is mapped by the transformation *(a)* $w = z^2$; *(b)* $w = z^3$; *(c)* $w = z^4$.

4. Show that the lines $ay = x$ $(a \ne 0)$ are mapped onto the spirals $\rho = \exp(a\phi)$ under the transformation $w = \exp z$, where $w = \rho \exp(i\phi)$.

5. By considering the images of *horizontal* line segments, verify that the image of the rectangular region $a \le x \le b, c \le y \le d$ under the transformation $w = \exp z$ is the region $e^a \le \rho \le e^b, c \le \phi \le d$, as shown in Fig. 21 (Sec. 14).

6. Verify the mapping of the region and boundary shown in Fig. 7 of Appendix 2, where the transformation is $w = \exp z$.

7. Find the image of the semi-infinite strip $x \ge 0, 0 \le y \le \pi$ under the transformation $w = \exp z$, and label corresponding portions of the boundaries.

8. One interpretation of a function $w = f(z) = u(x, y) + iv(x, y)$ is that of a *vector field* in the domain of definition of f. The function assigns a vector w, with components $u(x, y)$ and $v(x, y)$, to each point z at which it is defined. Indicate graphically the vector fields represented by *(a)* $w = iz$; *(b)* $w = z/|z|$.

15. LIMITS

Let a function f be defined at all points z in some deleted neighborhood (Sec. 11) of z_0. The statement that the *limit* of $f(z)$ as z approaches z_0 is a number w_0, or that

$$\lim_{z \to z_0} f(z) = w_0, \tag{1}$$

means that the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it. We now express the definition of limit in a precise and usable form.

Statement (1) means that for each positive number ε, there is a positive number δ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta. \tag{2}$$

Geometrically, this definition says that for each ε neighborhood $|w - w_0| < \varepsilon$ of w_0, there is a deleted δ neighborhood $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w lying in the ε neighborhood (Fig. 23). Note that even though all points in the deleted neighborhood $0 < |z - z_0| < \delta$ are to be considered, their images need not fill up the entire neighborhood $|w - w_0| < \varepsilon$. If f has the constant value w_0, for instance, the image of z is always the center of that neighborhood. Note, too, that once a δ has been found, it can be replaced by any smaller positive number, such as $\delta/2$.

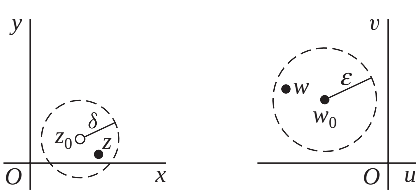

FIGURE 23

It is easy to show that *when a limit of a function $f(z)$ exists at a point z_0, it is unique.* To do this, we suppose that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} f(z) = w_1.$$

Then, for each positive number ε, there are positive numbers δ_0 and δ_1 such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_0$$