To prove this, we assume that the function f in equation (5) is continuous and note how it follows that the function

$$\sqrt{[u(x, y)]^2 + [v(x, y)]^2}$$

is continuous throughout R and thus reaches a maximum value M somewhere in R.* Inequality (6) thus holds, and we say that *f is bounded on R*.

EXERCISES

1. Use definition (2), Sec. 15, of limit to prove that

(a) $\lim_{z \to z_0} \operatorname{Re} z = \operatorname{Re} z_0$; *(b)* $\lim_{z \to z_0} \overline{z} = \overline{z_0}$; *(c)* $\lim_{z \to 0} \dfrac{\overline{z}^2}{z} = 0$.

2. Let a, b, and c denote complex constants. Then use definition (2), Sec. 15, of limit to show that

(a) $\lim_{z \to z_0} (az + b) = az_0 + b$; *(b)* $\lim_{z \to z_0} (z^2 + c) = z_0^2 + c$;

(c) $\lim_{z \to 1-i} [x + i(2x + y)] = 1 + i \quad (z = x + iy)$.

3. Let n be a positive integer and let $P(z)$ and $Q(z)$ be polynomials, where $Q(z_0) \neq 0$. Use Theorem 2 in Sec. 16, as well as limits appearing in that section, to find

(a) $\lim_{z \to z_0} \dfrac{1}{z^n} \quad (z_0 \neq 0)$; *(b)* $\lim_{z \to i} \dfrac{iz^3 - 1}{z + i}$; *(c)* $\lim_{z \to z_0} \dfrac{P(z)}{Q(z)}$.

Ans. (a) $1/z_0^n$; *(b)* 0; *(c)* $P(z_0)/Q(z_0)$.

4. Use mathematical induction and property (9), Sec. 16, of limits to show that

$$\lim_{z \to z_0} z^n = z_0^n$$

when n is a positive integer $(n = 1, 2, \ldots)$.

5. Show that the limit of the function

$$f(z) = \left(\frac{z}{\overline{z}}\right)^2$$

as z tends to 0 does not exist. Do this by letting nonzero points $z = (x, 0)$ and $z = (x, x)$ approach the origin. [Note that it is not sufficient to simply consider points $z = (x, 0)$ and $z = (0, y)$, as it was in Example 2, Sec. 15.]

6. Prove statement (8) in Theorem 2 of Sec. 16 using

(a) Theorem 1 in Sec. 16 and properties of limits of real-valued functions of two real variables;

(b) definition (2), Sec. 15, of limit.

*See, for instance, A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 125–126 and p. 529, 1983.

7. Use definition (2), Sec. 15, of limit to prove that

$$\text{if} \quad \lim_{z \to z_0} f(z) = w_0, \quad \text{then} \quad \lim_{z \to z_0} |f(z)| = |w_0|.$$

Suggestion: Observe how the first of inequalities (9), Sec. 4, enables one to write

$$||f(z)| - |w_0|| \leq |f(z) - w_0|.$$

8. Write $\Delta z = z - z_0$ and show that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{if and only if} \quad \lim_{\Delta z \to 0} f(z_0 + \Delta z) = w_0.$$

9. Show that

$$\lim_{z \to z_0} f(z)g(z) = 0 \quad \text{if} \quad \lim_{z \to z_0} f(z) = 0$$

and if there exists a positive number M such that $|g(z)| \leq M$ for all z in some neighborhood of z_0.

10. Use the theorem in Sec. 17 to show that

(a) $\displaystyle\lim_{z \to \infty} \frac{4z^2}{(z-1)^2} = 4$; *(b)* $\displaystyle\lim_{z \to 1} \frac{1}{(z-1)^3} = \infty$; *(c)* $\displaystyle\lim_{z \to \infty} \frac{z^2+1}{z-1} = \infty$.

11. With the aid of the theorem in Sec. 17, show that when

$$T(z) = \frac{az+b}{cz+d} \qquad (ad - bc \neq 0),$$

(a) $\displaystyle\lim_{z \to \infty} T(z) = \infty$ if $c = 0$;

(b) $\displaystyle\lim_{z \to \infty} T(z) = \frac{a}{c}$ and $\displaystyle\lim_{z \to -d/c} T(z) = \infty$ if $c \neq 0$.

12. State why limits involving the point at infinity are unique.

13. Show that a set S is unbounded (Sec. 11) if and only if every neighborhood of the point at infinity contains at least one point in S.

19. DERIVATIVES

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \varepsilon$ of a point z_0. The *derivative* of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{1}$$

and the function f is said to be *differentiable* at z_0 when $f'(z_0)$ exists.

By expressing the variable z in definition (1) in terms of the new complex variable

$$\Delta z = z - z_0 \quad (z \neq z_0),$$