where we must stipulate that $z \neq z_0$ so that we are not dividing by zero. As already noted, f is continuous at z_0 and Φ is continuous at the point $w_0 = f(z_0)$. Hence the composition $\Phi[f(z)]$ is continuous at z_0; and since $\Phi(w_0) = 0$,

$$\lim_{z \to z_0} \Phi[f(z)] = 0.$$

So equation (10) becomes equation (6) in the limit as z approaches z_0.

EXERCISES

1. Use results in Sec. 20 to find $f'(z)$ when

(a) $f(z) = 3z^2 - 2z + 4$; *(b)* $f(z) = (1 - 4z^2)^3$;

(c) $f(z) = \dfrac{z-1}{2z+1}$ $(z \neq -1/2)$; *(d)* $f(z) = \dfrac{(1+z^2)^4}{z^2}$ $(z \neq 0)$.

2. Using results in Sec. 20, show that

(a) a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

of degree n $(n \geq 1)$ is differentiable everywhere, with derivative

$$P'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1};$$

(b) the coefficients in the polynomial $P(z)$ in part *(a)* can be written

$$a_0 = P(0), \quad a_1 = \frac{P'(0)}{1!}, \quad a_2 = \frac{P''(0)}{2!}, \quad \ldots, \quad a_n = \frac{P^{(n)}(0)}{n!}.$$

3. Apply definition (3), Sec. 19, of derivative to give a direct proof that

$$\frac{dw}{dz} = -\frac{1}{z^2} \quad \text{when} \quad w = \frac{1}{z} \qquad (z \neq 0).$$

4. Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Use definition (1), Sec. 19, of derivative to show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

5. Derive formula (3), Sec. 20, for the derivative of the sum of two functions.

6. Derive expression (2), Sec. 20, for the derivative of z^n when n is a positive integer by using

(a) mathematical induction and formula (4), Sec. 20, for the derivative of the product of two functions;

(b) definition (3), Sec. 19, of derivative and the binomial formula (Sec. 3).

7. Prove that expression (2), Sec. 20, for the derivative of z^n remains valid when n is a negative integer $(n = -1, -2, \ldots)$, provided that $z \neq 0$.

Suggestion: Write $m = -n$ and use the formula for the derivative of a quotient of two functions.

8. Use the method in Example 2, Sec. 19, to show that $f'(z)$ does not exist at any point z when

(a) $f(z) = \operatorname{Re} z$; *(b)* $f(z) = \operatorname{Im} z$.

9. Let f denote the function whose values are

$$f(z) = \begin{cases} \bar{z}^2/z & \text{when} \quad z \neq 0, \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Show that if $z = 0$, then $\Delta w/\Delta z = 1$ at each nonzero point on the real and imaginary axes in the Δz, or $\Delta x\ \Delta y$, plane. Then show that $\Delta w/\Delta z = -1$ at each nonzero point $(\Delta x, \Delta x)$ on the line $\Delta y = \Delta x$ in that plane. Conclude from these observations that $f'(0)$ does not exist. Note that to obtain this result, it is not sufficient to consider only horizontal and vertical approaches to the origin in the Δz plane. (Compare with Example 2, Sec. 19.)

21. CAUCHY–RIEMANN EQUATIONS

In this section, we obtain a pair of equations that the first-order partial derivatives of the component functions u and v of a function

$$f(z) = u(x, y) + iv(x, y) \tag{1}$$

must satisfy at a point $z_0 = (x_0, y_0)$ when the derivative of f exists there. We also show how to express $f'(z_0)$ in terms of those partial derivatives.

We start by writing

$$z_0 = x_0 + iy_0, \quad \Delta z = \Delta x + i\Delta y,$$

and

$$\begin{aligned} \Delta w &= f(z_0 + \Delta z) - f(z_0) \\ &= [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]. \end{aligned}$$

Assuming that the derivative

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \tag{2}$$

exists, we know from Theorem 1 in Sec. 16 that

$$f'(z_0) = \lim_{(\Delta x, \Delta y) \to (0,0)} \left(\operatorname{Re} \frac{\Delta w}{\Delta z} \right) + i \lim_{(\Delta x, \Delta y) \to (0,0)} \left(\operatorname{Im} \frac{\Delta w}{\Delta z} \right). \tag{3}$$