and since the other conditions in the theorem are satisfied, the derivative $f'(z)$ exists at each point where $f(z)$ is defined. The theorem tells us, moreover, that

$$f'(z) = e^{-i\theta}\left[\frac{1}{3(\sqrt[3]{r})^2}\cos\frac{\theta}{3} + i\frac{1}{3(\sqrt[3]{r})^2}\sin\frac{\theta}{3}\right],$$

or

$$f'(z) = \frac{e^{-i\theta}}{3(\sqrt[3]{r})^2}e^{i\theta/3} = \frac{1}{3(\sqrt[3]{r}e^{i\theta/3})^2} = \frac{1}{3[f(z)]^2}.$$

Note that when a specific point z is taken in the domain of definition of f, the value $f(z)$ is one value of $z^{1/3}$ (see Sec. 9). Hence this last expression for $f'(z)$ can be put in the form

$$\frac{d}{dz}z^{1/3} = \frac{1}{3(z^{1/3})^2}$$

when that value is taken. Derivatives of such power functions will be elaborated on in Chap. 3 (Sec. 33).

EXERCISES

1. Use the theorem in Sec. 21 to show that $f'(z)$ does not exist at any point if

(a) $f(z) = \overline{z}$; *(b)* $f(z) = z - \overline{z}$;
(c) $f(z) = 2x + ixy^2$; *(d)* $f(z) = e^x e^{-iy}$.

2. Use the theorem in Sec. 22 to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and find $f''(z)$ when

(a) $f(z) = iz + 2$; *(b)* $f(z) = e^{-x}e^{-iy}$;
(c) $f(z) = z^3$; *(d)* $f(z) = \cos x \cosh y - i \sin x \sinh y$.

Ans. *(b)* $f''(z) = f(z)$; *(d)* $f''(z) = -f(z)$.

3. From results obtained in Secs. 21 and 22, determine where $f'(z)$ exists and find its value when

(a) $f(z) = 1/z$; *(b)* $f(z) = x^2 + iy^2$; *(c)* $f(z) = z \operatorname{Im} z$.

Ans. *(a)* $f'(z) = -1/z^2$ $(z \neq 0)$; *(b)* $f'(x + ix) = 2x$; *(c)* $f'(0) = 0$.

4. Use the theorem in Sec. 23 to show that each of these functions is differentiable in the indicated domain of definition, and also to find $f'(z)$:

(a) $f(z) = 1/z^4$ $(z \neq 0)$;

(b) $f(z) = \sqrt{r}e^{i\theta/2}$ $(r > 0, \alpha < \theta < \alpha + 2\pi)$;

(c) $f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$ $(r > 0, 0 < \theta < 2\pi)$.

Ans. *(b)* $f'(z) = \dfrac{1}{2f(z)}$; *(c)* $f'(z) = i\dfrac{f(z)}{z}$.

5. Show that when $f(z) = x^3 + i(1-y)^3$, it is legitimate to write

$$f'(z) = u_x + iv_x = 3x^2$$

only when $z = i$.

6. Let u and v denote the real and imaginary components of the function f defined by means of the equations

$$f(z) = \begin{cases} \overline{z}^2/z & \text{when} \quad z \neq 0, \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Verify that the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at the origin $z = (0, 0)$. [Compare with Exercise 9, Sec. 20, where it is shown that $f'(0)$ nevertheless fails to exist.]

7. Solve equations (2), Sec. 23 for u_x and u_y to show that

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}, \quad u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r}.$$

Then use these equations and similar ones for v_x and v_y to show that in Sec. 23 equations (4) are satisfied at a point z_0 if equations (6) are satisfied there. Thus complete the verification that equations (6), Sec. 23, are the Cauchy–Riemann equations in polar form.

8. Let a function $f(z) = u + iv$ be differentiable at a nonzero point $z_0 = r_0 \exp(i\theta_0)$. Use the expressions for u_x and v_x found in Exercise 7, together with the polar form (6), Sec. 23, of the Cauchy–Riemann equations, to rewrite the expression

$$f'(z_0) = u_x + iv_x$$

in Sec. 22 as

$$f'(z_0) = e^{-i\theta}(u_r + iv_r),$$

where u_r and v_r are to be evaluated at (r_0, θ_0).

9. *(a)* With the aid of the polar form (6), Sec. 23, of the Cauchy–Riemann equations, derive the alternative form

$$f'(z_0) = \frac{-i}{z_0}(u_\theta + iv_\theta)$$

of the expression for $f'(z_0)$ found in Exercise 8.

(b) Use the expression for $f'(z_0)$ in part *(a)* to show that the derivative of the function $f(z) = 1/z \ (z \neq 0)$ in Example 1, Sec. 23, is $f'(z) = -1/z^2$.

10. *(a)* Recall (Sec. 5) that if $z = x + iy$, then

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i}.$$

By *formally* applying the chain rule in calculus to a function $F(x, y)$ of two real variables, derive the expression

$$\frac{\partial F}{\partial \overline{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

(b) Define the operator

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

suggested by part *(a)*, to show that if the first-order partial derivatives of the real and imaginary components of a function $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy–Riemann equations, then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}[(u_x - v_y) + i(v_x + u_y)] = 0.$$

Thus derive the *complex form* $\partial f/\partial \overline{z} = 0$ *of the Cauchy–Riemann equations.*

24. ANALYTIC FUNCTIONS

We are now ready to introduce the concept of an analytic function. A function f of the complex variable z is *analytic at a point* z_0 if it has a derivative at each point in some neighborhood of z_0.* It follows that if f is analytic at a point z_0, it must be analytic at each point in some neighborhood of z_0. A function f is *analytic in an open set* if it has a derivative everywhere in that set. If we should speak of a function f that is analytic in a set S which is not open, it is to be understood that f is analytic in an open set containing S.

Note that the function $f(z) = 1/z$ is analytic at each nonzero point in the finite plane. But the function $f(z) = |z|^2$ is not analytic at any point since its derivative exists only at $z = 0$ and not throughout any neighborhood. (See Example 3, Sec. 19.)

An *entire* function is a function that is analytic at each point in the entire finite plane. Since the derivative of a polynomial exists everywhere, it follows that *every polynomial is an entire function.*

If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0, then z_0 is called a *singular point*, or singularity, of f. The point $z = 0$ is evidently a singular point of the function $f(z) = 1/z$. The function $f(z) = |z|^2$, on the other hand, has no singular points since it is nowhere analytic.

A necessary, but by no means sufficient, condition for a function f to be analytic in a domain D is clearly the continuity of f throughout D. Satisfaction of the Cauchy–Riemann equations is also necessary, but not sufficient. Sufficient conditions for analyticity in D are provided by the theorems in Secs. 22 and 23.

Other useful sufficient conditions are obtained from the differentiation formulas in Sec. 20. The derivatives of the sum and product of two functions exist wherever

*The terms *regular* and *holomorphic* are also used in the literature to denote analyticity.