where c is a real constant. If $c = 0$, it follows that $f(z) = 0$ everywhere in D. If $c \neq 0$, the fact that (see Sec. 5)

$$f(z)\overline{f(z)} = c^2$$

tells us that $f(z)$ is never zero in D. Hence

$$\overline{f(z)} = \frac{c^2}{f(z)} \quad \text{for all } z \text{ in } D,$$

and it follows from this that $\overline{f(z)}$ is analytic everywhere in D. The main result in Example 3 just above thus ensures that $f(z)$ is constant throughout D.

EXERCISES

1. Apply the theorem in Sec. 22 to verify that each of these functions is entire:

(a) $f(z) = 3x + y + i(3y - x)$; *(b)* $f(z) = \sin x \cosh y + i \cos x \sinh y$;

(c) $f(z) = e^{-y} \sin x - ie^{-y} \cos x$; *(d)* $f(z) = (z^2 - 2)e^{-x}e^{-iy}$.

2. With the aid of the theorem in Sec. 21, show that each of these functions is nowhere analytic:

(a) $f(z) = xy + iy$; *(b)* $f(z) = 2xy + i(x^2 - y^2)$; *(c)* $f(z) = e^y e^{ix}$.

3. State why a composition of two entire functions is entire. Also, state why any *linear combination* $c_1 f_1(z) + c_2 f_2(z)$ of two entire functions, where c_1 and c_2 are complex constants, is entire.

4. In each case, determine the singular points of the function and state why the function is analytic everywhere except at those points:

(a) $f(z) = \dfrac{2z + 1}{z(z^2 + 1)}$; *(b)* $f(z) = \dfrac{z^3 + i}{z^2 - 3z + 2}$; *(c)* $f(z) = \dfrac{z^2 + 1}{(z + 2)(z^2 + 2z + 2)}$.

Ans. *(a)* $z = 0, \pm i$; *(b)* $z = 1, 2$; *(c)* $z = -2, -1 \pm i$.

5. According to Exercise 4*(b)*, Sec. 23, the function

$$g(z) = \sqrt{r}e^{i\theta/2} \qquad (r > 0, -\pi < \theta < \pi)$$

is analytic in its domain of definition, with derivative

$$g'(z) = \frac{1}{2\,g(z)}.$$

Show that the composite function $G(z) = g(2z - 2 + i)$ is analytic in the half plane $x > 1$, with derivative

$$G'(z) = \frac{1}{g(2z - 2 + i)}.$$

Suggestion: Observe that $\operatorname{Re}(2z - 2 + i) > 0$ when $x > 1$.

6. Use results in Sec. 23 to verify that the function

$$g(z) = \ln r + i\theta \qquad (r > 0, 0 < \theta < 2\pi)$$

is analytic in the indicated domain of definition, with derivative $g'(z) = 1/z$. Then show that the composite function $G(z) = g(z^2 + 1)$ is analytic in the quadrant $x > 0, y > 0$, with derivative

$$G'(z) = \frac{2z}{z^2 + 1}.$$

Suggestion: Observe that $\text{Im}(z^2 + 1) > 0$ when $x > 0, y > 0$.

7. Let a function f be analytic everywhere in a domain D. Prove that if $f(z)$ is real-valued for all z in D, then $f(z)$ must be constant throughtout D.

26. HARMONIC FUNCTIONS

A real-valued function H of two real variables x and y is said to be *harmonic* in a given domain of the xy plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$H_{xx}(x, y) + H_{yy}(x, y) = 0, \tag{1}$$

known as *Laplace's equation.*

Harmonic functions play an important role in applied mathematics. For example, the temperatures $T(x, y)$ in thin plates lying in the xy plane are often harmonic. A function $V(x, y)$ is harmonic when it denotes an electrostatic potential that varies only with x and y in the interior of a region of three-dimensional space that is free of charges.

EXAMPLE 1. It is easy to verify that the function $T(x, y) = e^{-y} \sin x$ is harmonic in any domain of the xy plane and, in particular, in the semi-infinite vertical strip $0 < x < \pi, y > 0$. It also assumes the values on the edges of the strip that are indicated in Fig. 31. More precisely, it satisfies all of the conditions

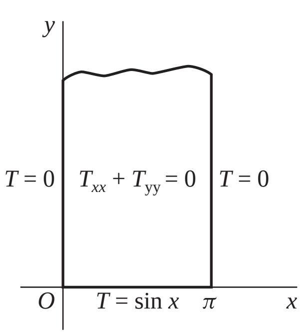

FIGURE 31