EXAMPLE 5. We now illustrate one method of obtaining a harmonic conjugate of a given harmonic function. The function

$$u(x, y) = y^3 - 3x^2 y \tag{5}$$

is readily seen to be harmonic throughout the entire xy plane. Since a harmonic conjugate $v(x, y)$ is related to $u(x, y)$ by means of the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x, \tag{6}$$

the first of these equations tells us that

$$v_y(x, y) = -6xy.$$

Holding x fixed and integrating each side here with respect to y, we find that

$$v(x, y) = -3xy^2 + \phi(x) \tag{7}$$

where ϕ is, at present, an arbitrary function of x. Using the second of equations (6), we have

$$3y^2 - 3x^2 = 3y^2 - \phi'(x),$$

or $\phi'(x) = 3x^2$. Thus $\phi(x) = x^3 + C$, where C is an arbitrary real number. According to equation (7), then, the function

$$v(x, y) = -3xy^2 + x^3 + C \tag{8}$$

is a harmonic conjugate of $u(x, y)$.

The corresponding analytic function is

$$f(z) = (y^3 - 3x^2 y) + i(-3xy^2 + x^3 + C). \tag{9}$$

The form $f(z) = i(z^3 + C)$ of this function is easily verified and is suggested by noting that when $y = 0$, expression (9) becomes $f(x) = i(x^3 + C)$.

EXERCISES

1. Show that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$ when

(a) $u(x, y) = 2x(1 - y)$; *(b)* $u(x, y) = 2x - x^3 + 3xy^2$;
(c) $u(x, y) = \sinh x \sin y$; *(d)* $u(x, y) = y/(x^2 + y^2)$.

Ans. (a) $v(x, y) = x^2 - y^2 + 2y$; *(b)* $v(x, y) = 2y - 3x^2 y + y^3$;
(c) $v(x, y) = -\cosh x \cos y$; *(d)* $v(x, y) = x/(x^2 + y^2)$.

2. Show that if v and V are harmonic conjugates of $u(x, y)$ in a domain D, then $v(x, y)$ and $V(x, y)$ can differ at most by an additive constant.

3. Suppose that v is a harmonic conjugate of u in a domain D and also that u is a harmonic conjugate of v in D. Show how it follows that both $u(x, y)$ and $v(x, y)$ must be constant throughout D.

4. Use Theorem 2 in Sec. 26 to show that v is a harmonic conjugate of u in a domain D if and only if $-u$ is a harmonic conjugate of v in D. (Compare with the result obtained in Exercise 3.)

Suggestion: Observe that the function $f(z) = u(x, y) + iv(x, y)$ is analytic in D if and only if $-if(z)$ is analytic there.

5. Let the function $f(z) = u(r, \theta) + iv(r, \theta)$ be analytic in a domain D that does not include the origin. Using the Cauchy–Riemann equations in polar coordinates (Sec. 23) and assuming continuity of partial derivatives, show that throughout D the function $u(r, \theta)$ satisfies the partial differential equation

$$r^2 u_{rr}(r, \theta) + r u_r(r, \theta) + u_{\theta\theta}(r, \theta) = 0,$$

which is the *polar form of Laplace's equation*. Show that the same is true of the function $v(r, \theta)$.

6. Verify that the function $u(r, \theta) = \ln r$ is harmonic in the domain $r > 0, 0 < \theta < 2\pi$ by showing that it satisfies the polar form of Laplace's equation, obtained in Exercise 5. Then use the technique in Example 5, Sec. 26, but involving the Cauchy–Riemann equations in polar form (Sec. 23), to derive the harmonic conjugate $v(r, \theta) = \theta$. (Compare with Exercise 6, Sec. 25.)

7. Let the function $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D, and consider the families of *level curves* $u(x, y) = c_1$ and $v(x, y) = c_2$, where c_1 and c_2 are arbitrary real constants. Prove that these families are orthogonal. More precisely, show that if $z_0 = (x_0, y_0)$ is a point in D which is common to two particular curves $u(x, y) = c_1$ and $v(x, y) = c_2$ and if $f'(z_0) \neq 0$, then the lines tangent to those curves at (x_0, y_0) are perpendicular.

Suggestion: Note how it follows from the pair of equations $u(x, y) = c_1$ and $v(x, y) = c_2$ that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0.$$

8. Show that when $f(z) = z^2$, the level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ of the component functions are the hyperbolas indicated in Fig. 32. Note the orthogonality of the two families, described in Exercise 7. Observe that the curves $u(x, y) = 0$ and $v(x, y) = 0$ intersect at the origin but are not, however, orthogonal to each other. Why is this fact in agreement with the result in Exercise 7?

9. Sketch the families of level curves of the component functions u and v when $f(z) = 1/z$, and note the orthogonality described in Exercise 7.

10. Do Exercise 9 using polar coordinates.

11. Sketch the families of level curves of the component functions u and v when

$$f(z) = \frac{z-1}{z+1},$$

and note how the result in Exercise 7 is illustrated here.

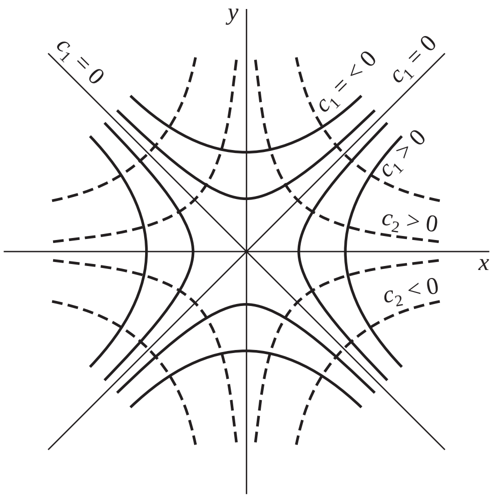

FIGURE 32

27. UNIQUELY DETERMINED ANALYTIC FUNCTIONS

We conclude this chapter with two sections dealing with how the values of an analytic function in a domain D are affected by its values in a subdomain of D or on a line segment lying in D. While these sections are of considerable theoretical interest, they are not central to our development of analytic functions in later chapters. The reader may pass directly to Chap. 3 at this time and refer back when necessary.

Lemma. *Suppose that*

(a) a function f is analytic throughout a domain D;

(b) $f(z) = 0$ at each point z of a domain or line segment contained in D.

Then $f(z) \equiv 0$ in D; that is, $f(z)$ is identically equal to zero throughout D.

To prove this lemma, we let f be as stated in its hypothesis and let z_0 be any point of the subdomain or line segment where $f(z) = 0$. Since D is a *connected* open set (Sec. 11), there is a polygonal line L, consisting of a finite number of line segments joined end to end and lying entirely in D, that extends from z_0 to any other point P in D. We let d be the shortest distance from points on L to the boundary of D, unless D is the entire plane; in that case, d may be any positive number. We then form a finite sequence of points

$$z_0, z_1, z_2, \ldots, z_{n-1}, z_n$$

along L, where the point z_n coincides with P (Fig. 33) and where each point is sufficiently close to adjacent ones that

$$|z_k - z_{k-1}| < d \qquad (k = 1, 2, \ldots, n).$$