EXERCISES

1. Show that

(a) $\exp(2 \pm 3\pi i) = -e^2$; *(b)* $\exp\left(\frac{2+\pi i}{4}\right) = \sqrt{\frac{e}{2}}\,(1+i)$;

(c) $\exp(z + \pi i) = -\exp z$.

2. State why the function $f(z) = 2z^2 - 3 - ze^z + e^{-z}$ is entire.

3. Use the Cauchy–Riemann equations and the theorem in Sec. 21 to show that the function $f(z) = \exp \bar{z}$ is not analytic anywhere.

4. Show in two ways that the function $f(z) = \exp(z^2)$ is entire. What is its derivative?

Ans. $f'(z) = 2z \exp(z^2)$.

5. Write $|\exp(2z + i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$$|\exp(2z + i) + \exp(iz^2)| \le e^{2x} + e^{-2xy}.$$

6. Show that $|\exp(z^2)| \le \exp(|z|^2)$.

7. Prove that $|\exp(-2z)| < 1$ if and only if $\operatorname{Re} z > 0$.

8. Find all values of z such that

(a) $e^z = -2$; *(b)* $e^z = 1 + \sqrt{3}i$; *(c)* $\exp(2z - 1) = 1$.

Ans. *(a)* $z = \ln 2 + (2n+1)\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $z = \ln 2 + \left(2n + \frac{1}{3}\right)\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$;

(c) $z = \frac{1}{2} + n\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$.

9. Show that $\overline{\exp(iz)} = \exp(i\bar{z})$ if and only if $z = n\pi \ (n = 0, \pm 1, \pm 2, \ldots)$. (Compare with Exercise 4, Sec. 28.)

10. *(a)* Show that if e^z is real, then $\operatorname{Im} z = n\pi \ (n = 0, \pm 1, \pm 2, \ldots)$.

(b) If e^z is pure imaginary, what restriction is placed on z?

11. Describe the behavior of $e^z = e^x e^{iy}$ as *(a)* x tends to $-\infty$; *(b)* y tends to ∞.

12. Write $\operatorname{Re}(e^{1/z})$ in terms of x and y. Why is this function harmonic in every domain that does not contain the origin?

13. Let the function $f(z) = u(x, y) + iv(x, y)$ be analytic in some domain D. State why the functions

$$U(x, y) = e^{u(x,y)} \cos v(x, y), \quad V(x, y) = e^{u(x,y)} \sin v(x, y)$$

are harmonic in D and why $V(x, y)$ is, in fact, a harmonic conjugate of $U(x, y)$.

14. Establish the identity

$$(e^z)^n = e^{nz} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

in the following way.

(a) Use mathematical induction to show that it is valid when $n = 0, 1, 2, \ldots$.

(b) Verify it for negative integers n by first recalling from Sec. 7 that

$$z^n = (z^{-1})^m \qquad (m = -n = 1, 2, \ldots)$$

when $z \neq 0$ and writing $(e^z)^n = (1/e^z)^m$. Then use the result in part *(a)*, together with the property $1/e^z = e^{-z}$ (Sec. 29) of the exponential function.

30. THE LOGARITHMIC FUNCTION

Our motivation for the definition of the logarithmic function is based on solving the equation

$$e^w = z \tag{1}$$

for w, where z is any *nonzero* complex number. To do this, we note that when z and w are written $z = re^{i\Theta}$ $(-\pi < \Theta \leq \pi)$ and $w = u + iv$, equation (1) becomes

$$e^u e^{iv} = re^{i\Theta}.$$

According to the statement in italics at the beginning of Sec. 9 about the equality of two complex numbers expressed in exponential form, this tells us that

$$e^u = r \quad \text{and} \quad v = \Theta + 2n\pi$$

where n is any integer. Since the equation $e^u = r$ is the same as $u = \ln r$, it follows that equation (1) is satisfied if and only if w has one of the values

$$w = \ln r + i(\Theta + 2n\pi) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Thus, if we write

$$\log z = \ln r + i(\Theta + 2n\pi) \qquad (n = 0, \pm 1, \pm 2, \ldots), \tag{2}$$

equation (1) tells us that

$$e^{\log z} = z \qquad (z \neq 0), \tag{3}$$

which serves to motivate expression (2) as the *definition* of the (multiple-valued) logarithmic function of a nonzero complex variable $z = re^{i\Theta}$.

EXAMPLE 1. If $z = -1 - \sqrt{3}i$, then $r = 2$ and $\Theta = -2\pi/3$. Hence

$$\log(-1 - \sqrt{3}i) = \ln 2 + i\left(-\frac{2\pi}{3} + 2n\pi\right) = \ln 2 + 2\left(n - \frac{1}{3}\right)\pi i$$
$$(n = 0, \pm 1, \pm 2, \ldots).$$