EXERCISES

1. Show that

(a) $\mathrm{Log}(-ei) = 1 - \dfrac{\pi}{2}i$; *(b)* $\mathrm{Log}(1-i) = \dfrac{1}{2}\ln 2 - \dfrac{\pi}{4}i$.

2. Show that

(a) $\log e = 1 + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $\log i = \left(2n + \dfrac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(c) $\log(-1 + \sqrt{3}i) = \ln 2 + 2\left(n + \dfrac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

3. Show that

(a) $\mathrm{Log}(1+i)^2 = 2\,\mathrm{Log}(1+i)$; *(b)* $\mathrm{Log}(-1+i)^2 \neq 2\,\mathrm{Log}(-1+i)$.

4. Show that

(a) $\log(i^2) = 2\log i$ when $\log z = \ln r + i\theta \left(r > 0, \dfrac{\pi}{4} < \theta < \dfrac{9\pi}{4}\right)$;

(b) $\log(i^2) \neq 2\log i$ when $\log z = \ln r + i\theta \left(r > 0, \dfrac{3\pi}{4} < \theta < \dfrac{11\pi}{4}\right)$.

5. Show that

(a) the set of values of $\log(i^{1/2})$ is

$$\left(n + \frac{1}{4}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and that the same is true of $(1/2)\log i$;

(b) the set of values of $\log(i^2)$ is *not* the same as the set of values of $2\log i$.

6. Given that the branch $\log z = \ln r + i\theta\ (r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is analytic at each point z in the stated domain, obtain its derivative by differentiating each side of the identity (Sec. 30)

$$e^{\log z} = z \qquad (z \neq 0)$$

and using the chain rule.

7. Find all roots of the equation $\log z = i\pi/2$.

Ans. $z = i$.

8. Suppose that the point $z = x + iy$ lies in the horizontal strip $\alpha < y < \alpha + 2\pi$. Show that when the branch $\log z = \ln r + i\theta\ (r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is used, $\log(e^z) = z$. [Compare with equation (4), Sec. 30.]

9. Show that

(a) the function $f(z) = \mathrm{Log}(z - i)$ is analytic everywhere except on the portion $x \leq 0$ of the line $y = 1$;

(b) the function

$$f(z) = \frac{\mathrm{Log}(z+4)}{z^2 + i}$$

is analytic everywhere except at the points $\pm(1-i)/\sqrt{2}$ and on the portion $x \le -4$ of the real axis.

10. Show in two ways that the function $\ln(x^2+y^2)$ is harmonic in every domain that does not contain the origin.

11. Show that

$$\operatorname{Re}[\log(z-1)] = \frac{1}{2}\ln[(x-1)^2+y^2] \qquad (z \neq 1).$$

Why must this function satisfy Laplace's equation when $z \neq 1$?

32. SOME IDENTITIES INVOLVING LOGARITHMS

If z_1 and z_2 denote any two nonzero complex numbers, it is straightforward to show that

$$\log(z_1 z_2) = \log z_1 + \log z_2. \tag{1}$$

This statement, involving a multiple-valued function, is to be interpreted in the same way that the statement

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \tag{2}$$

was in Sec. 8. That is, if values of two of the three logarithms are specified, then there is a value of the third such that equation (1) holds.

The verification of statement (1) can be based on statement (2) in the following way. Since $|z_1 z_2| = |z_1||z_2|$ and since these moduli are all positive real numbers, we know from experience with logarithms of such numbers in calculus that

$$\ln|z_1 z_2| = \ln|z_1| + \ln|z_2|.$$

So it follows from this and equation (2) that

$$\ln|z_1 z_2| + i\arg(z_1 z_2) = (\ln|z_1| + i\arg z_1) + (\ln|z_2| + i\arg z_2). \tag{3}$$

Finally, because of the way in which equations (1) and (2) are to be interpreted, equation (3) is the same as equation (1).

EXAMPLE. To illustrate statement (1), write $z_1 = z_2 = -1$ and recall from Examples 2 and 3 in Sec. 30 that

$$\log 1 = 2n\pi i \quad \text{and} \quad \log(-1) = (2n+1)\pi i,$$

where $n = 0, \pm 1, \pm 2, \ldots$. Noting that $z_1 z_2 = 1$ and using the values

$$\log(z_1 z_2) = 0 \quad \text{and} \quad \log z_1 = \pi i,$$