#3e) \[ \int_\gamma \frac{e^{-z}}{(z+1)^2} \, dz \]

- \( e^{-z} \) is entire, singularities occur @ \( z = -1 \) (twice)

- \( f(z) = e^{-z} \quad f'(z) = -e^{-z} \quad z_0 = -1 \)

\[ \int_\gamma \frac{e^{-z}}{(z+1)^2} \, dz = 2\pi i f'(z_0) = 2\pi i \left(-e^{-1}\right) = -2\pi i e \]

#7 \[ \int_\gamma \frac{\cos z}{z^2(z-3)} \, dz \]

- \( \cos z \) is entire, singularities occur at \( z = 0, 3 \) @ zero is twice
- \( z = 3 \) is once

Notice that \( z = 0 \) is in the contour
- \( z = 3 \) is not in the contour

\[ f(z) = \frac{\cos z}{z-3} \quad f'(z) = -\frac{\sin z}{z-3} - \frac{\cos z}{(z-3)^2} \]

\[ \int \frac{f(z)}{z^2} \, dz = 2\pi i f'(z_0) = 2\pi i f'(0) = -\frac{2\pi i}{9} \]
\[ f(z) = \frac{1}{(1-z)^2} \]  

Want to verify that \( \max_{|z| = R} |f(z)| = \frac{1}{(1-R)^2} \)

\[ z = R e^{i\theta} \]

\[ |f(z)| = \left| \frac{1}{(1-z)^2} \right| \]

\[ = \left| \frac{1}{(1-R e^{i\theta})^2} \right| \]

\[ \max_{|z| = R} |f(z)| = \frac{1}{|1-R|^2} = \frac{1}{(1-R)^2} \]

Now verify \( f^{(n)}(0) = (n+1)! \)

If we use \( f(z) = \frac{1}{(1-z)^2} \)

\[ f'(z) = \frac{2}{(1-z)^3} \]

\[ f''(z) = \frac{6}{(1-z)^4} = \frac{3 \cdot 2}{(1-z)^4} \]

\[ f^{(n)}(z) = \frac{(n+1)!}{(1-z)^{n+2}} \]

\[ f^{(n+1)}(z) = \frac{(n+1)! (n+2)}{(1-z)^{n+3}} \]

\[ \therefore \text{By induction } f^{(n)}(z) = \frac{(n+1)!}{(1-z)^{n+2}} \text{ is what we want.} \]

\[ \Rightarrow f^{(n)}(0) = \frac{(n+1)!}{1} = (n+1)! \]
#1. Using Thm 20. on pg 215

\[ f^{(n)}(z_0) \leq \frac{\frac{n! \cdot M}{R^n}}{R^n} \]

\[ f^{(n)}(0) = (n+1)! \leq \frac{n! \cdot M}{R^n} \]

where we found M to be \( |\frac{1}{(1-R)^2}| \)

\[ = (n+1)! \leq \frac{n!}{R^n (1-R)^2} \]
Let $f$ be entire & suppose
\[ \text{Re } f(z) \leq M \text{ for all } z. \]

Prove that $f$ must be a const. funct.

Since $f$ is entire & using Liouville's Thm which states that

i) $f(z)$ must be analytic

ii) $f(z)$ is bounded $f(z) \leq M$ for some const.

then $f(z)$ must be a constant.

Pf. if we use Cauchy's inequality

\[ |f^{(n)}(z)| \leq \frac{n!M}{R^n} \quad \text{say } n=1 \]

\[ |f'(z)| \leq \frac{M}{R^n} \]

let $r \to \infty$ $|f'(z)| = 0$ so $f'(z) = 0$

\[ \therefore f(z) \text{ must be a constant} \]

\[ \Rightarrow \therefore f(z) = \text{const.} \]
7. If \( f(z) \) is entire \( \Rightarrow \) \(|f(z)| \leq |z|^2 \)

if we bound \(|f(z)| \leq M|z|^n\) for any \(|z| > r\)

where \( n \) is a nonnegative integer.

b. Using the maximum modulus principle on pg 217

\[ \Rightarrow \text{Suppose that } f(z) \text{ is analytic in a disk centered at } z_0 \text{ and its maximum value of } |f(z)| \text{ over this disk is } |f(z_0)|. \text{ Then } |f(z)| \text{ is a constant in the disk.} \]

\[ \Rightarrow \text{If } f \text{ is analytic in a domain } D \text{ and } |f(z)| \text{ achieves its maximum value at a point } z_0 \text{ in } D, \text{ then } f \text{ is constant in } D. \]

\[ |z_1| < R \]

\[ f(z_0) = i \]

\[ |f(z)| \leq 1 = M. \]

17. Find \( \max_{|z| \leq 1} \left| (z-1)(z + \frac{1}{2}) \right| \)

\[ = |z^2 - \frac{1}{2}z - \frac{1}{2}| \]

\[ \leq |z^2| + |\frac{1}{2}z| + |\frac{1}{2}| \]

\[ \leq 2 \]
#18.

Let $z = a$ be a pole of order (multiplicity) $m$ of $p$

zero $z = p$ of order ( ) of $n$

Given $\Rightarrow \frac{1}{2\pi i} \int_{C} \frac{p'(z)}{p(z)} \, dz$

$\Rightarrow p = a (z-z_1)^{n_1} \cdots (z-z_k)^{n_k}$

$n_1 + \cdots + n_k = n$

$\frac{p'}{p} = \frac{n_1}{z-z_1} + \cdots + \frac{n_k}{z-z_k}$

$\Rightarrow \frac{1}{2\pi i} \int_{c} \frac{p'(z)}{p(z)} \, dz = \sum_{r=1}^{k} \frac{1}{2\pi i} \int_{C_r} \frac{p'(z)}{p(z)} \, dz + \sum_{r=1}^{k} \frac{1}{2\pi i} \int_{C_r} \frac{p'(z)}{p(z)} \, dz$

$= \sum_{r=1}^{k} n_r - \sum_{r=1}^{k} p_r$

$= n - p.$