Solutions, 316-VII

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1 Problem 4.11.5

The motion of a mass-spring system with damping is governed by

\[ y'' + 10y' + ky = 0 \quad ; \quad y(0) = 1, \quad y'(0) = 0. \]

Find the equation of motion and sketch its graph for \( k = 20, 25, 30 \). Solution:

\[
\begin{align*}
ky &= ke^{rt} \\
+ &+ \quad + \\
10y' &= 10re^{rt} \\
+ &+ \quad + \\
y'' &= r^2e^{rt} \\
&= \quad = \\
0 &= (r^2 + 10r + k)e^{rt}
\]

and factoring:

\[
\begin{align*}
 r_{\pm} &= \frac{-10 \pm \sqrt{100 - 4k}}{2} \\
&= -5 \pm \sqrt{25 - k}
\end{align*}
\]

so that the equation has two real, distinct roots for \( k < 25 \), two real equal roots (double root) for \( k = 25 \) and two complex conjugate roots for \( k > 25 \). Therefore the solutions for the 3 cases given \( (k = 20, 25, 30) \) are:

\[ k = 20 \Rightarrow r_+ = -5 + \sqrt{5}, \quad r_- = -5 - \sqrt{5} \]
\[ y(t) = Ae^{-(5-\sqrt{5})t} + Be^{-(5+\sqrt{5})t} \quad ; \quad y(0) = 1 \]
\[ y'(t) = -A(5 - \sqrt{5})e^{-(5-\sqrt{5})t} - B(5 + \sqrt{5})e^{-(5+\sqrt{5})t} \quad ; \quad y'(0) = 0 \]
\[ A + B = 1, \quad -A(5 - \sqrt{5}) - B(5 + \sqrt{5}) = -5(A + B) + \sqrt{5}(A - B) = 0 \]
\[ \Rightarrow A = \frac{1 + \sqrt{5}}{2}, \quad B = \frac{1 - \sqrt{5}}{2} \]
\[ y(t) = \frac{1 + \sqrt{5}}{2} e^{-(5-\sqrt{5})t} + \frac{1 - \sqrt{5}}{2} e^{-(5+\sqrt{5})t} \].

\[ k = 25 \Rightarrow r_{\pm} = -5 \text{(double root)} \]
\[ y(t) = Ae^{-5t} + Bte^{-5t} \quad ; \quad y(0) = 1 \]
\[ y'(t) = -5Ae^{-5t} + Be^{-5t}(1 - 5t) \quad ; \quad y'(0) = 0 \]
\[ A = 1, \quad -5A + B = 0 \quad \Rightarrow A = 1, \quad B = 5 \]
\[ y(t) = e^{-5t} + 5te^{-5t} \].

\[ k = 30 \Rightarrow r_{\pm} = -1 \pm i\sqrt{5} \]
\[ y(t) = Ae^{-t} \cos(\sqrt{5}t) + Be^{-t} \sin(\sqrt{5}t) \quad ; \quad y(0) = 1, \quad y'(0) = 0 \]
\[ y'(t) = A e^{-t} \left( - \cos(\sqrt{5}t) - \sqrt{5} \sin(\sqrt{5}t) \right) + B e^{-t} \left( - \sin(\sqrt{5}t) + \sqrt{5} \cos(\sqrt{5}t) \right) \]
\[ A = 1, \quad -A + \sqrt{5}B = 0 \quad \Rightarrow A = 1, \quad B = \frac{\sqrt{5}}{5} \]
\[ y(t) = e^{-t} \cos\left(\sqrt{5}t\right) + \frac{\sqrt{5}}{5} e^{-t} \sin\left(\sqrt{5}t\right) \]

We now use Maple to plot all three functions for \(0 \leq t \leq 2\pi\).

> restart:

\[ m := 1 \]
\[ Om\theta := 5. \]
> Y1 := t -> (1+sqrt(5))*exp(-(5-sqrt(5))*t/2)/2 + (1-sqrt(5))*exp(-(5+sqrt(5))*t/2)/2;
> Y1 := t -> \frac{1}{2} (1 + \sqrt{5}) e^{(-1/2(5-\sqrt{5})t)} + \frac{1}{2} (1 - \sqrt{5}) e^{(-1/2(5+\sqrt{5})t)}
> Y2 := t -> exp(-5*t)*(1 + 5*t); Y3 := t -> exp(-t)*(cos(srgb(5)*t) + (1/sqrt(5))*sin(srgb(5)*t));
\[ Y_2 := t \to e^{-5t} (1 + 5t) \]
\[ Y_3 := t \to e^{-t} (\cos(\sqrt{5} t) + \frac{\sin(\sqrt{5} t)}{\sqrt{5}}) \]

```plaintext
g> plot([Y1(t), Y2(t), Y3(t)], t = 0 .. 2*Pi, color=[blue, black, red], labels=[t, y], legend=[overdamped, critical, underdamped], linestyle=[DASHDOT, SOLID, DASH]);
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Legend:
- Dashed Line: Overdamped
- Solid Line: Critical
- Dashed-Dotted Line: Underdamped
2 Problem 4.11.7

A 1/8-kg mass is attached to a spring with stiffness 16 N/m. The damping constant for the system is 2 N-sec/m. If the mass is pulled 75 cm to the right of equilibrium and given an initial leftward velocity of 2 m/sec, determine the equation of motion of the mass and give its damping factor, quasiperiod and quasifrequency.

Solution: We must solve:

\[
\frac{1}{8}y'' + 2y' + 16y = 0 ,
\]

The roots of the characteristic equation \((r^2 + 16r + 128 = 0)\) are

\[
r_{\pm} = u \pm iv = -8 \pm i8
\]

so that the general solution is

\[
y(t) = e^{-8t} (C_1 \cos (8t) + C_2 \sin (8t)) .
\]

Solving the IVP we find \(C_1 = -\frac{3}{4} \), \(C_2 = -1\), and the solution

\[
y(t) = -\frac{1}{4} e^{-8t} (3 \cos (8t) + 4 \sin (8t)) .
\]

Since \(3^2 + 4^2 = 5^2\) we have

\[
y(t) = -\frac{5}{4} e^{-8t} \left( \frac{3}{5} \cos (8t) + \frac{4}{5} \sin (8t) \right) = -\frac{5}{4} e^{-8t} \cos (8t - \phi) ,
\]

where

\[
\phi = \tan^{-1} \left( \frac{4}{3} \right) .
\]

The damping factor is \(e^{-8t}\). The quasiperiod is

\[
T = \frac{\pi}{4},
\]

so that the quasifrequency is

\[
\nu = \frac{4}{\pi} .
\]
3 Problem 4.11.11

A 1-kg mass is attached to a spring with stiffness 100 N/m. The damping constant for the system is 0.2 N-sec/m. If the mass is pushed rightward from the equilibrium position with a velocity of 1 m/sec, when will it attain its maximum displacement to the right?

**Solution:** We must solve:

\[ y'' + 0.2y' + 100y = 0, \]

The roots of the characteristic equation \((r^2 + 0.2r + 100 = 0)\) are

\[ r_{\pm} = u \pm iv = -0.1 \pm \frac{\sqrt{9999}}{10} \]

so that the general solution is

\[ y(t) = e^{ut} (C_1 \cos (vt) + C_2 \sin (vt)) . \]

Solving the IVP we find \(C_1 = 0, C_2 = .1\), and the solution

\[ y(t) = \frac{1}{10} e^{ut} \sin (vt) . \]

For the maximum, solve

\[ y'(t) = \frac{1}{10} e^{ut} (u \sin (vt) + v \cos (vt)) = 0 \]

to find

\[ \tan (vt) = -\frac{v}{u} = 10 \frac{\sqrt{9999}}{10} = \sqrt{9999} \]

i.e.

\[ t = \frac{10}{\sqrt{9999}} \tan^{-1} \sqrt{9999} \approx 0.15608742057822 \]
4 Problem 4.2.13b

Given that $y_1(x) = e^{2x} \cos x$ and $y_2(x) = e^{2x} \sin x$ are solutions to the homogeneous equation

$$y'' - 4y' + 5y = 0,$$

find solutions to this equation that satisfy the following initial conditions:
(b) $y(\pi) = 4e^{2\pi}$, $y'(\pi) = 5e^{2\pi}$.

**Solution:**

We have the general solution:

$$y(x) = C_1 e^{2x} \cos x + C_2 e^{2x} \sin x$$

so that

$$y'(x) = C_1 \left(2e^{2x} \cos x - e^{2x} \sin x\right) + C_2 \left(2e^{2x} \sin x + e^{2x} \cos x\right)$$

and setting $x = \pi$:

$$y(\pi) = C_1 e^{2\pi} \cos \pi + C_2 e^{2\pi} \sin \pi = -C_1 e^{2\pi} = 4e^{2\pi}$$

$$y'(\pi) = C_1 e^{2\pi} (2 \cos \pi - \sin \pi) + C_2 e^{2\pi} (2 \sin \pi + \cos \pi) = -2C_1 e^{2\pi} - C_2 e^{2\pi} = 5e^{2\pi}$$

so that solving:

$$C_1 = -4, \quad C_2 = 3.$$

Then, the solution is:

$$y(x) = e^{2x} (-4 \cos x + 3 \sin x).$$

This can be written in compact form if we let

$$\cos \phi = \frac{-4}{5}, \quad \sin \phi = \frac{3}{5}$$

or

$$\phi = \tan^{-1} \left(\frac{3}{-4}\right)$$

then

$$y(x) = 5e^{2x} \cos (x - \phi).$$
5 Problem 4.2.14a

Given that $y_1(x) = e^{2x}$ and $y_2(x) = e^{-x}$ are solutions to the homogeneous equation

$$y'' - y' - 2y = 0,$$

find solutions to this equation that satisfy the following initial conditions:
(a) $y(0) = -1$, $y'(0) = 4$.

**Solution:**

We have the general solution:

$$y(x) = C_1 e^{2x} + C_2 e^{-x}$$

so that

$$y'(x) = 2C_1 e^{2x} - C_2 e^{-x}$$

and setting $x = 0$:

$$y(0) = C_1 + C_2 = -1$$
$$y'(0) = 2C_1 - C_2 = 4$$

so that solving:

$$C_1 = 1 \ , \ C_2 = -2 \ .$$

Then, the solution is:

$$y(x) = e^{2x} - 2e^{-x} \ .$$