# Memoir on the Theory of the Articulated Octahedron

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## 1 (Introduction)

Mr. C. Stephanos posed the following question in the "*Intermédiaire des Mathématiciens* " [3]:
"Do there exist polyhedra with invariant facets that are susceptible to an infinite family of transformations that only alter solid angles and dihedrals?"
I announced, in the same Journal [4], a special concave octahedron possessing the required property. Cauchy, on the other hand, has proved [5] that there do not exist convex polyhedra that are deformable under the prescribed conditions.
In this Memoir I propose to extend the above mentioned result, by resolving the problem of Mr. Stephanos in general for octahedra of triangular facets. Following Cauchy's theorem, all the octahedra which I shall establish as deformable will be of necessity concave by virtue of the fact that they possess reentrant dihedrals or, in fact, facets that intercross, in the manner of facets of polyhedra in higher dimensional spaces.

*Translation from the French original of Raoul Bricard's 1897 memoir on articulated octahedra [2]. (E. A. Coutsias, coutsias@unm.edu, Mathematics Dept., University of New Mexico ©). Footnotes in the original are listed in the bibliography. Notes by the translator are shown in parentheses. Section titles (in parentheses) were provided by the translator. Translations of the replies to the problem of C. Stephanos by R. Bricard and C. Juel in the Intermédiaire des Mathématiciens are also provided in the bibliography. Only equations numbered are those numbered in the original. Changes in equation numbering are as follows: $(4, 4') \to (4a, 4b); (5, 5') \to (5a, 5b); (7', 8', 9') \to (12, 13, 14); (7'', 8'', 9'') \to (15, 16, 17); (12, 13) \to (18, 19)$.

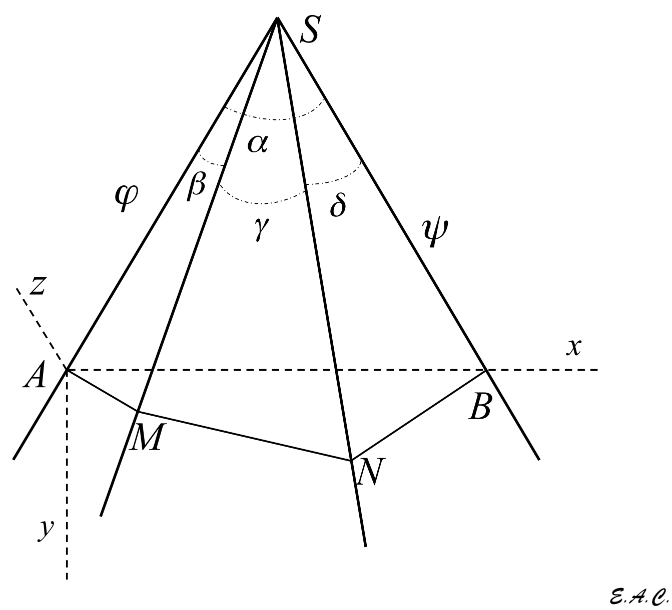


Figure 1:

## 2 (The tetrahedral equation and its decomposition)

I shall begin by establishing certain properties pertaining to the deformation of a tetrahedral angle, whose four faces remain invariant. This deformation is analogous to that of a planar articulated quadrilateral. Given (fig. 1) the tetrahedral $SABNM$, articulated along its four edges and having fixed-size faces

$$\angle ASB = \alpha \ , \ \angle ASM = \beta \ , \ \angle MSN = \gamma \ , \ \angle NSB = \delta \ (0 < \alpha \ , \ \beta \ , \ \gamma \ , \ \delta < \pi) \ ,$$

let us find the relation between the dihedrals $SA = \phi$ and $SB = \psi$. One may suppose that the face $ASB$ maintains a fixed position. We associate with it an orthogonal coordinate system of 3 axes defined as follows: the origin is placed at a point $A$ of $SA$, so that $SA = 1$. The axis $Ax$ ($Ay$) is parallel to the exterior (interior) bisectrix of the angle $\angle ASB$ and oriented so that the point $S$ has a negative ordinate and the point $B$ a positive abscissa. The direction of positive $z$ may be taken arbitrarily.

With their axes thus chosen, the points $M$ and $N$, lying on the edges $SM$, $SN$ and such that the angles $\angle SAM$, $\angle SBN$ are right angles, have as coordinates, respectively:

$$\mathbf{M}\left\{\begin{array}{lcl} x_1 &=& \tan\beta\cos\frac{\alpha}{2}\cos\phi\ , \\ y_1 &=& \tan\beta\sin\frac{\alpha}{2}\cos\phi\ , \\ z_1 &=& \tan\beta\sin\phi\ , \end{array}\right. \quad \mathbf{N}\left\{\begin{array}{lcl} x_2 &=& 2\sin\frac{\alpha}{2}-\tan\delta\cos\frac{\alpha}{2}\cos\psi\ , \\ y_2 &=& \tan\delta\sin\frac{\alpha}{2}\cos\psi\ , \\ z_2 &=& \tan\delta\sin\psi\ . \end{array}\right.$$

Equating the value of $\bar{MN}^2$ that results from these expressions to that found by considering the triangle $\triangle SMN$, we get

$$\begin{aligned} &\frac{1}{\cos^2\beta}+\frac{1}{\cos^2\delta}-\frac{2\cos\gamma}{\cos\beta\cos\delta} \\ &= \left(\tan\beta\cos\frac{\alpha}{2}\cos\phi+\tan\delta\cos\frac{\alpha}{2}\cos\psi-2\sin\frac{\alpha}{2}\right)^2 \\ &\quad +\left(\tan\beta\sin\frac{\alpha}{2}\cos\phi-\tan\delta\sin\frac{\alpha}{2}\cos\psi\right)^2 \\ &\quad +(\tan\beta\sin\phi-\tan\delta\sin\psi)^2 \qquad , \end{aligned}$$

and, after reductions,

$$\begin{aligned} &\sin\beta\sin\delta\cos\alpha\cos\phi\cos\psi-\sin\beta\sin\delta\sin\phi\sin\psi \\ &-\sin\alpha\sin\beta\cos\delta\cos\phi-\sin\alpha\sin\delta\cos\beta\cos\psi \\ &+\cos\gamma-\cos\alpha\cos\beta\cos\delta \quad = \quad 0\ . \end{aligned}$$

Let us now introduce the variables

$$t=\tan\frac{\phi}{2}\ \text{ and }\ u=\tan\frac{\psi}{2}\ .$$

We have

$$\begin{aligned} \cos\phi &= \frac{1-t^2}{1+t^2}\ , \qquad \sin\phi = \frac{2t}{1+t^2}\ , \\ \cos\psi &= \frac{1-u^2}{1+u^2}\ , \qquad \sin\psi = \frac{2u}{1+u^2}\ . \end{aligned}$$

We arrive thus at the following relationship, which I shall term "*equation of the tetrahedral angle*":

$$At^2u^2+Bt^2+2Ctu+Du^2+E=0\ , \tag{1}$$

where

$$\left.\begin{array}{rcl}
A & = & \sin\beta\sin\delta\cos\alpha+\sin\beta\cos\delta\sin\alpha \\
 & & +\sin\delta\cos\beta\sin\alpha-\cos\alpha\cos\beta\cos\delta+\cos\gamma \\
 & = & \cos\gamma-\cos\left(\alpha+\beta+\delta\right)\ , \\
B & = & -\sin\beta\sin\delta\cos\alpha+\sin\beta\cos\delta\sin\alpha \\
 & & -\sin\delta\cos\beta\sin\alpha-\cos\alpha\cos\beta\cos\delta+\cos\gamma \\
 & = & \cos\gamma-\cos\left(\alpha+\beta-\delta\right)\ , \\
C & = & -2\sin\beta\sin\delta\ , \\
D & = & -\sin\beta\sin\delta\cos\alpha-\sin\beta\cos\delta\sin\alpha \\
 & & +\sin\delta\cos\beta\sin\alpha-\cos\alpha\cos\beta\cos\delta+\cos\gamma \\
 & = & \cos\gamma-\cos\left(\alpha-\beta+\delta\right)\ , \\
E & = & \sin\beta\sin\delta\cos\alpha-\sin\beta\cos\delta\sin\alpha \\
 & & -\sin\delta\cos\beta\sin\alpha-\cos\alpha\cos\beta\cos\delta+\cos\gamma \\
 & = & \cos\gamma-\cos\left(\alpha-\beta-\delta\right)\ .
\end{array}\right\} \qquad (2)$$

It should be easy to anticipate the form of relation (1).

In fact, to a fixed value of $t=\tan\left(\phi/2\right)$ there corresponds a unique position of the face $ASM$, the angle $\phi$ being determined in this way up to an integral multiple of $2\pi$. This face being thus fixed, the construction of the tetrahedral angle can be achieved in two ways: there exist, in fact, two positions of the line $SN$, symmetric with respect to the plane $BSM$ and such that we have

$$\angle MSN=\gamma\ ,\ \angle BSN=\delta\ .$$

To every position of the face $BSN$ there corresponds a single value of the variable $u$. Therefore the relationship that couples $t$ and $u$ must be of the second degree with respect to $u$.

For the same reason, that relationship should be of the second degree with respect to $t$. At last it can be seen that if this relation is satisfied by the values $t$, $u$, it is equally true for $-t$ and $-u$. It is therefore necessarily of the form (1).

*Cases of decomposition of equation (1).*– We just saw that to a value of $t$ there correspond two values of $u$. It is useful to investigate under what conditions relation (1) decomposes, in a way that these values of $u$ are rational functions of $t$.

For that to be so, it is necessary and sufficient that the polynomial

$$C^2t^2-\left(At^2+D\right)\left(Bt^2+E\right)=-ABt^4+\left(C^2-AE-BD\right)t^2-DE\ ,$$

that appears under the radical entering in the expression for $u$ as a function of $t$, is a perfect square. This can be achieved in two ways:

1. There holds
$$\Big(C^2 - AE - BD\Big)^2 - 4ABDE = 0 \ .$$
It is found, by an easy calculation, that the left hand side of this equality reduces to the expression
$$16 \sin^2 \alpha \sin^2 \beta \sin^2 \gamma \sin^2 \delta \ .$$
This condition is not satisfied unless one of the angles $\alpha$, $\beta$, $\gamma$ or $\delta$ is reduced to 0 or $\pi$, which is impossible. It would seem that this case presents itself when the vertex $S$ is moved to infinity and the tetrahedral degenerates to a prismatic solid. But then one can see that the values of the coefficients $A$, $B$, $C$, $D$, $E$, are all reduced to 0, and the preceding reasoning no longer applies. Any right section of the prismatic solid is an articulated quadrilateral, whose deformation is governed by an equation of the same form as equation (1):
$$A't^2u^2 + B't^2 + 2C'tu + D'u^2 + E' = 0 \ .$$
The coefficients $A'$, $B'$, $C'$, $D'$, $E'$ have, in terms of the edges $a$, $b$, $c$, $d$ of the quadrilateral, expressions which can be obtained from the forms of $A$, $B$, $C$, $D$, $E$ by allowing in the latter the angles $\alpha$, $\beta$, $\gamma$, $\delta$ to approach zero in such a way that
$$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\gamma}{c} = \frac{\delta}{d} \ .$$
One can then see that the condition
$$\Big(C'^2 - A'E' - B'D'\Big)^2 - 4A'B'D'E' = 0 \ ,$$
implies the impossible relationship
$$abcd = 0 \ .$$

2. There holds
$$AB = 0 \quad \text{with} \quad DE = 0 \ .$$
These relationships imply one of the following cases:
$$\left.\begin{array}{ll} B = 0 \ , & E = 0 \ , \\ A = 0 \ , & D = 0 \ , \\ B = 0 \ , & D = 0 \ , \\ A = 0 \ , & E = 0 \ . \end{array}\right\} \qquad (3)$$

Let us consider for example the equalities

$$A = 0 \ , \qquad D = 0 \ .$$

There results:

$$\begin{aligned} \text{let} \ \ \gamma &= \alpha + \beta + \delta + 2k\pi \ \ \text{with} \ \ \gamma = \alpha - \beta + \delta + 2k'\pi \\ \text{let} \ \ \gamma &= \alpha + \beta + \delta + 2k\pi \ \ \text{with} \ \ \gamma = -\alpha + \beta - \delta + 2k'\pi \\ \text{let} \ \ \gamma &= -\alpha - \beta - \delta + 2k\pi \ \ \text{with} \ \ \gamma = \alpha - \beta + \delta + 2k'\pi \\ \text{let} \ \ \gamma &= -\alpha - \beta - \delta + 2k\pi \ \ \text{with} \ \ \gamma = -\alpha + \beta - \delta + 2k'\pi \ . \end{aligned}$$

The first pair of relationships is incompatible with the hypotheses on the magnitudes of the angles $\alpha$, $\beta$, $\gamma$, $\delta$. It can be seen, in fact, that

$$\beta + (k - k')\,\pi = 0 \ ,$$

which is impossible.

In examining the other pairs, it is seen that only the third is admissible and that it has as necessary consequences

$$\alpha + \delta = \pi \ , \quad \beta + \gamma = \pi \ .$$

It will be argued similarly on the remaining equalities (3). I will rewrite them anew, listing next to each the relationship implied among the faces of the tetrahedral:

$$\begin{array}{llll} B = 0 \ , & E = 0 \ , & \delta = \alpha \ , & \gamma = \beta \\ A = 0 \ , & D = 0 \ , & \delta = \pi - \alpha \ , & \gamma = \pi - \beta \\ B = 0 \ , & D = 0 \ , & \delta = \beta \ , & \gamma = \alpha \\ A = 0 \ , & E = 0 \ , & \delta = \pi - \beta \ , & \gamma = \pi - \alpha \ . \end{array}$$

We are now able to recognize two cases of decomposing equation (1):

1. The tetrahedral has its adjacent faces equal or supplementary in pairs. Its equation reduces to

$$At^2u + 2Ct + Du = 0 \ ,$$

or

$$Bt^2 + 2Ctu + E = 0 \ .$$

(by omitting in the first the factor $u = 0$ which corresponds to the uninteresting case where the adjacent faces of the tetrahedral remain coincident during the deformation). I will say that a tetrahedral of this nature is *rhomboidal*.

2. The tetrahedral has opposing angles equal or supplementary in pairs. Its equation is then

$$At^2u^2 + 2Ctu + E = 0 \ ,$$

or

$$Bt^2 + 2Ctu + Du^2 = 0 \ .$$

These equations are written, respectively, by introducing the explicit forms of their coefficients,

$$[\cos\alpha - \cos(\alpha + 2\beta)]\, t^2u^2 - 4\sin^2\beta tu + \cos\alpha - \cos(\alpha - 2\beta) = 0 \ ,$$

or

$$\sin(\alpha+\beta)t^2u^2 - 2\sin\beta tu - \sin(\alpha-\beta) = 0$$

and

$$[\cos(\alpha + 2\beta) - \cos\alpha]\, t^2 - 2\sin^2\beta tu + [\cos(\alpha - 2\beta) - \cos\alpha]\, u^2 = 0 \ ,$$

or

$$\sin(\alpha+\beta)t^2 + 2\sin\beta tu - \sin(\alpha-\beta)u^2 = 0 \ .$$

They decompose, the first into

$$\left.\begin{aligned} tu &= \frac{\sin\beta + \sin\alpha}{\sin(\alpha+\beta)} = \frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}} \qquad (a) \\ &\text{and} \\ tu &= \frac{\sin\beta - \sin\alpha}{\sin(\alpha+\beta)} = \frac{\sin\frac{\beta-\alpha}{2}}{\sin\frac{\alpha+\beta}{2}} \qquad (b) \end{aligned}\right\} \qquad (4)$$

the second into

$$\left.\begin{aligned} \frac{t}{u} &= \frac{-\sin\beta + \sin\alpha}{\sin(\alpha+\beta)} = \frac{\sin\frac{\alpha-\beta}{2}}{\sin\frac{\alpha+\beta}{2}} \qquad (a) \\ &\text{and} \\ \frac{t}{u} &= \frac{-\sin\beta - \sin\alpha}{\sin(\alpha+\beta)} = -\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}} \qquad (b) \end{aligned}\right\} \qquad (5)$$

It is not unhelpful to summarize the previous discussion: we can distinguish three types of articulated tetrahedral angles:

1. The *general* tetrahedral angles whose faces have no special relation. Its equation is irreducible, of the form that to each value of one of the variables $t$, $u$ there correspond two values of the other variable, that are not rational functions of the first;

2. The *rhomboidal* tetrahedral angles. To one value of $t$ there corresponds a single value of $u$, which is a rational function of $t$, but the converse is not true;

3. The tetrahedral angles whose opposite faces are *equal or supplementary in pairs*. To one value of $t$ there corresponds a unique value of $u$ and conversely ([6]).

# 3 (Reconstruction and equivalence)

Since equation (1) involves four arbitrary parameters, any equation of similar form

$$At^2u^2 + Bt^2 + 2Ctu + Du^2 + E = 0 \ ,$$

may be considered to define the deformation of an articulated tetrahedral angle.

The elements of this tetrahedral angle are given by the relationships

$$\begin{aligned} \frac{\cos\gamma - \cos(\alpha+\beta+\delta)}{A} &= \frac{\cos\gamma - \cos(\alpha+\beta-\delta)}{B} = \frac{-2\sin\beta\sin\delta}{C} \\ &= \frac{\cos\gamma - \cos(\alpha-\beta+\delta)}{D} = \frac{\cos\gamma - \cos(\alpha-\beta-\delta)}{E} \end{aligned}$$

from which we can show

$$\begin{aligned} \frac{-2\sin\beta\sin\delta}{C} &= \frac{4\sin\beta\sin\delta\cos\alpha}{A-B-D+E} = \frac{4\sin\delta\cos\beta\sin\alpha}{A-B+D-E} \\ &= \frac{4\sin\beta\cos\delta\sin\alpha}{A+B-D-E} = \frac{4\,(\cos\gamma - \cos\alpha\cos\beta\cos\delta)}{A+B+D+E} \ . \end{aligned}$$

We have, as a result:

$$\left.\begin{aligned} \cos\alpha = -\frac{A-B-D+E}{2C} \ , \quad \tan\beta = \frac{-2C\sin\alpha}{A-B+D-E} \ , \\ \tan\delta = \frac{-2C\sin\alpha}{A+B-D-E} \ , \\ \cos\gamma = \cos\alpha\cos\beta\cos\delta - \frac{A+B+D+E}{2C}\sin\beta\sin\delta \ , \end{aligned}\right\} \qquad (6)$$

formulas which permit the successive calculation of the angles $\alpha$, $\beta$, $\delta$, $\gamma$. Clearly, certain reality conditions need to be satisfied; however they are quite complicated and it will not be of interest to state them here.

Since it can be assumed that

$$\alpha\ ,\ \beta\ ,\ \gamma\ ,\ \delta\ <\ \pi\ ,$$

the previous formulas define only *two* systems of values for the angles (neglecting the second case of decomposition to which I shall return shortly). It is seen immediately that if one of the systems is formed from the values

$$\alpha\ ,\ \beta\ ,\ \gamma\ ,\ \delta\ \ ,$$

those of the other system are

$$\alpha\ ,\ \pi-\beta\ ,\ \gamma\ ,\ \pi-\delta\ \ .$$

There results thus a theorem which we will find extremely useful in the sequel: *If two articulated tetrahedral angles* $\mathbf{T}$ *and* $\mathbf{T}_1$ *can be subjected to a continuous deformation in such a manner that two adjacent dihedrals of* $\mathbf{T}$ *remain equal or supplementary to two adjacent dihedrals of* $\mathbf{T}_1$*, these two tetrahedral angles have all their faces pairwise equal or supplementary.*

Let us reserve the previous symbols for the elements of the tetrahedral angle $\mathbf{T}$ and let us designate the corresponding elements of $\mathbf{T}_1$ by the same subscripted letters. The theorem posits three cases, all of which are established in similar fashion.

Let us suppose, for example, that we have invariably

$$\phi=\phi_1\ ,\ \psi=\pi-\psi_1\ ,$$

from which

$$t_1=t\ ,\ u_1=\frac{1}{u}\ .$$

We have, during the deformation of $\mathbf{T}_1$, the relation

$$A_1t_1^2u_1^2+B_1t_1^2+2C_1t_1u_1+D_1u_1^2+E_1=0\ .$$

Upon substituting $t$ and $\frac{1}{u}$ for $t_1$ and $u_1$, respectively, this becomes

$$A_1\frac{t^2}{u^2}+B_1t^2+2C_1\frac{t}{u}+D_1\frac{1}{u^2}+E_1=0\ ,$$

or

$$B_1 t^2 u^2 + A_1 t^2 + 2C_1 tu + E_1 u^2 + D_1 = 0 \ .$$

This relationship must be identical to (1). We have therefore

$$\frac{B_1}{A} = \frac{A_1}{B} = \frac{C_1}{C} = \frac{E_1}{D} = \frac{D_1}{E} \ .$$

Let us now apply the formulas in (6) to the tetrahedral angle $\mathbf{T}_1$. We find

$$\alpha_1 = \pi - \alpha \ , \ \beta_1 = \beta \text{ or } \pi - \beta \ , \ \delta_1 = \pi - \delta \text{ or } \delta \ , \ \gamma_1 = \gamma \ ,$$

in agreement with the theorem stated above.

This proposition is still true when the tetrahedral angles $\mathbf{T}$ and $\mathbf{T}_1$ are rhomboidal, but it ceases to be if they have each of their opposite faces equal or supplementary in pairs. There exists in fact an infinity of such tetrahedral angles whose deformation is governed by the same equation

$$tu = k \text{ or } \frac{t}{u} = k' \ .$$

If $\alpha$ and $\beta$ denote two adjacent faces of a tetrahedral angle which satisfy the first relationship, e.g., we must have

$$\frac{\cos\frac{\beta-\alpha}{2}}{\cos\frac{\beta+\alpha}{2}} = k \text{ or, alternatively } \frac{\sin\frac{\beta-\alpha}{2}}{\sin\frac{\beta+\alpha}{2}} = k$$

from which

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{k-1}{k+1} \text{ or, alternatively } \frac{\tan\frac{\alpha}{2}}{\tan\frac{\beta}{2}} = \frac{1-k}{k+1} \ ,$$

equalities which are satisfied for an infinity of values of $\alpha$ and $\beta$.

# 4 (Covariance of opposite dihedrals in a tetrahedral)

In order to complete these generalities, I shall establish now the following property of the articulated tetrahedral angle, analogous to a well known property of the articulated quadrilateral.

*When an articulated tetrahedral angle is deformed, there exists a linear relationship between the cosines of two opposite dihedrals.*

In effect, maintaining the previous notation and designating additionally by $\theta$ the dihedral $ON$, we have, by the fundamental formulas of spherical Trigonometry,

$$\begin{aligned} \cos BOM &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\phi \\ &= \cos\gamma\cos\delta + \sin\gamma\sin\delta\cos\theta\ , \end{aligned}$$

that is, a relation of the form

$$A\cos\phi + B\cos\theta + C = 0\ .$$

When the tetrahedral angle fits any of the cases of decomposition listed above, this relationship reduces to

$$\cos\phi = \cos\theta$$

from which

$$\phi = \theta\ ,$$

if we do not allow for the dihedrals of the tetrahedral angle other than values between 0 and $\pi$. Conversely, if an articulated tetrahedral angle is deformed in such a fashion that two opposite dihedrals remain always equal, this tetrahedral angle is rhomboidal or it has its opposite faces equal or supplementary pairwise. In fact, we have now the relations

$$\cos\alpha\cos\beta = \cos\gamma\cos\delta\ ,$$

$$\sin\alpha\sin\beta = \sin\gamma\sin\delta\ ,$$

from which

$$\cos(\alpha\pm\beta) = \cos(\gamma\pm\delta)\ ,$$

which admit the four sets of solutions

$$\begin{aligned} \gamma = \alpha\ ,\quad \delta = \beta\ ;\qquad \gamma = \beta\ ,\quad \delta = \alpha\ ; \\ \gamma = \pi-\alpha\ ,\quad \delta = \pi-\beta\ ;\qquad \gamma = \pi-\beta\ ,\quad \delta = \pi-\alpha\ . \end{aligned}$$

## 5 (Deformability in general)

We shall now undertake the study of the deformability conditions for an octahedron with triangular facets, with edges of fixed lengths.

It must first be noted that such an octahedron, although concave, is generally rigid. This follows from Legendre's theorem by which the number of conditions necessary for the determination of a polyhedron are exactly equal to the number of its edges. In effect, the proof of that theorem relies entirely on the fact that the polyhedron is determined by the relation of Euler (or of Descartes) and does not depend on its convexity or concavity. That is, an octahedron with triangular facets is well defined by that relation, whatever the disposition of its facets may be.

An octahedron whose edges are given is therefore in general completely determined, and therefore undeformable. Our approach is to examine if in certain cases, by means of particular relations existing between its edges, that determinacy ceases to hold. Then the octahedron will be deformable.

Let us suppose that this happens for the octahedron $ABCDEF$ (fig. 2). We can see then that the twelve dihedrals of this octahedron are of necessity all variable, when it is subjected to the deformation to which it is amenable.

Let us assume in fact that, during the deformation of the octahedron one of its dihedrals, for example $AB$, remains constant in magnitude. The tetrahedral angle formed by the four faces having the point $A$ as common vertex will be completely rigid, since one of its dihedrals is invariant. Consideration of the tetrahedral angles having their vertices at $F$, $E$, $D$ shows then that all the dihedrals of the octahedron have constant magnitude, which is contrary to the hypothesis.

The octahedron therefore is comprised of six tetrahedral angles which all deform while keeping their faces constant. Three cases must be distinguished, according to whether these tetrahedral angles are *general* (in the sense given in §II of this paper), *rhomboidal* or *having opposite faces pairwise equal or supplementary.*

## 6 (The case of a general tetrahedral)

We examine now the first case. Among the six tetrahedral angles, let us consider those whose vertices are at $A$, $B$, $C$. Their deformations are governed by three equations that are similar to (1) and *indecomposable*:

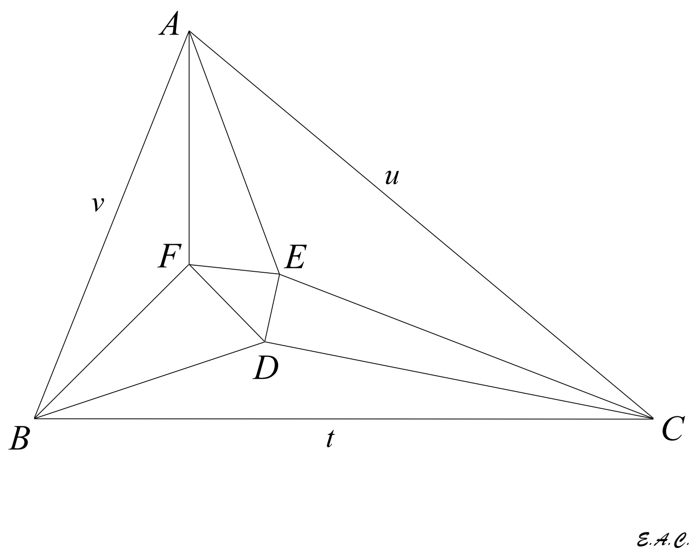


Figure 2:

$$At^2u^2 + Bt^2 + 2Ctu + Du^2 + E = 0 \ , \tag{7}$$

$$A't^2v^2 + B't^2 + 2C'tv + D'v^2 + E' = 0 \ , \tag{8}$$

$$A''u^2v^2 + B''u^2 + 2C''uv + D''v^2 + E'' = 0 \ , \tag{9}$$

by designating as $t$, $u$, $v$ the tangents of the semi-dihedrals $BC$, $CA$, $AB$ and as $A$, $B$, …, $E''$ constants that depend on the faces of the three tetrahedral angles and, consequently, on the edges of the octahedron.

The preceding equations ought to be satisfied by an infinity of sets of values of $t$, $u$ and $v$. Therefore, equations (8) and (9), in terms of $v$, ought to have *one* or *two* common roots for an infinity of values of $t$ and of $u$ that satisfy equation (7).

However it is impossible that equations (8) and (9) have their two roots always equal: indeed if it was so, we would have

$$\frac{A't^2 + D'}{A''u^2 + D''} = \frac{C't}{C''u} = \frac{B't^2 + E'}{B''u^2 + E''} \ ,$$

from which one obtains two equations of third degree in $t$ and $u$, which ought to have, with equation (7), an infinity of common solutions; however this is impossible, since the latter was supposed indecomposable.

Thus equations (8) and (9) have, in general, a unique common root in $v$. This root is therefore expressible as a rational function of the coefficients of these equations and, consequently, of $t$ and $u$. Now, one can derive from equations (7) and (8)

$$\begin{aligned} u &= \frac{-Ct \pm \sqrt{F(t)}}{At^2 + D} \, , \\ v &= \frac{-C't \pm \sqrt{F_1(t)}}{A't^2 + D'} \, , \end{aligned}$$

defining

$$\begin{aligned} F(t) &= C^2t^2 - \left(At^2 + D\right)\left(Bt^2 + E\right) \, , \\ F_1(t) &= C'^2t^2 - \left(A't^2 + D'\right)\left(B't^2 + E'\right) \, , \end{aligned}$$

and conveniently choosing the signs placed in front of the radicals. One therefore has

$$\frac{-C't \pm \sqrt{F_1(t)}}{A't^2 + D'} = \phi\left[t, \frac{-Ct \pm \sqrt{F(t)}}{At^2 + D}\right] \, ,$$

where $\phi(x, y)$ denotes a rational function ([8]).

The second member of this relationship may be put in the form

$$\frac{L + M\sqrt{F(t)}}{N} \, ,$$

$L$, $M$, $N$ being polynomials in $t$. One arrives finally at the identity

$$P\sqrt{F(t)} + Q\sqrt{F_1(t)} + R = 0 \, ,$$

where $P$, $Q$, $R$ are also polynomials in $t$. One extracts from that

$$F(t)F_1(t) = \left[\frac{R^2 - P^2F(t) - Q^2F_1(t)}{2PQ}\right]^2 \, .$$

The product of the polynomials $F(t)$ and $F_1(t)$ must therefore be the square of a rational function and, consequently, of a polynomial in $t$. It there follows that $F(t)$ and $F_1(t)$ are identical up to a constant factor.

Indeed $F(t)$ and $F_1(t)$ are two biquadratic polynomials that are not perfect squares, else equations (7) and (8) would be reducible, contrary to the stated hypothesis. One may therefore set

$$\begin{aligned} F(t) &= -A\,B\,(t-\lambda)(t+\lambda)(t-\mu)(t+\mu)\,, \qquad \lambda \neq \mu\,, \\ F_1(t) &= -A'B'(t-\lambda')(t+\lambda')(t-\mu')(t+\mu')\,, \qquad \lambda' \neq \mu', \end{aligned}$$

and their product cannot be a perfect square unless one has

$$\lambda = \pm\lambda', \qquad \mu = \pm\mu',$$

or

$$\lambda = \pm\mu', \qquad \mu = \pm\lambda',$$

which establish the above assertion. We deduce from this the following important consequence:

The equations (7) and (8), respectively in terms of $u$ and $v$, have their roots equal for the same values of $t$.

I could pursue the algebraic study of the system (7), (8), (9), whose consideration ought to be by itself sufficient, as one can see easily, to give the conditions of deformability of the octahedron. But this would lead to rather involved calculations, by reason of the complicated dependence of the coefficients $A$, $B$, $C, \ldots$ on the elements of the octahedron. Therefore I shall take a different route. I shall state however the following theorem, since it may find application in other investigations:

*For equations (7), (8), (9), to have an infinity of common solutions, it is necessary and sufficient that they result from the successive elimination of $t$, $u$, $v$ between the 2 equations*

$$\begin{aligned} luv + mvt + ntu + p &= 0, \\ l't + m'u + n'v + p'tuv &= 0, \end{aligned}$$

*where $l, m, n, p, l', m', n', p'$ are arbitrary coefficients.*

Let us return then to the consideration of the octahedron $ABCDEF$ and let us interpret geometrically the last result that was obtained.

When $t$ assumes such a value that equation (7) in $u$ has its roots equal, the dihedral $CE$ will become evidently equal to 0 or $\pi$. Similarly, when equation (8) in $v$ has its roots equal, the dihedral $BF$ will become equal to 0 or $\pi$. Therefore the dihedrals $CE$ and $BF$ are such that if one of them becomes 0 or $\pi$, the other assumes one of these values simultaneously.

Now, during the deformation of the octahedron, there exists a linear relationship between the cosines of these two dihedrals. On has in effect (IV) a linear relationship between the cosine of each of these dihedrals and that of the dihedral $BC$, which opposes them in the articulated tetrahedral angles with vertices respectively at $C$ and $B$

$$\begin{aligned} l\,\cos CE + m\,\cos BC + n &= 0\,, \\ l'\cos BF + m'\cos BC + n' &= 0 \end{aligned}$$

from which there arises one more relation

$$l''\cos CE + m''\cos BF + n'' = 0 \tag{10}$$

with constant coefficients $l''\,, m''\,, n''$.
Let us successively set in this relation the dihedral $CE$ equal to 0 or $\pi$. The dihedral $BF$, as we have stated, will assume each time one of these values; it cannot assume the value 0 or $\pi$ twice, because one must also have

$$\begin{aligned} l'' \pm m'' + n'' &= 0\,, \\ -l'' \pm m'' + n'' &= 0\,, \end{aligned}$$

$m''$ having the same sign in each right side and, as a result,

$$l'' = 0\,, \qquad m'' = \pm n''\,.$$

Relationship (10) will reduce then to

$$\cos BF = \pm 1\,,$$

which is impossible, since all the dihedrals of the octahedron are variable. The dihedral $BF$ must therefore assume once the value 0 and once the value $\pi$, in the same order as the dihedral $CE$ or in the reverse order. One has then

$$\begin{aligned} l'' \pm m'' + n'' &= 0\,, \\ -l'' \mp m'' + n'' &= 0\,, \end{aligned}$$

with correspondence of the signs in the two right hand sides. One obtains from this

$$n'' = 0\,, \qquad l'' = \pm m''\,,$$

and relation (10) becomes

$$\cos CE = \pm \cos BF\,.$$

Thus, during the deformation of the octahedron, *the dihedrals $CE$ and $BF$ are invariably equal or supplementary.*

We may employ the same reasoning in considering three tetrahedral angles having as vertices those of a facet other than $ABC$. Since all these tetrahedrals were assumed general, the conclusion will remain the same, and we may state:

*During the deformation of the octahedron, its opposing dihedrals will remain equal or supplementary in pairs.*

Let us then envision two opposing tetrahedral angles, for example those having their vertices respectively at $A$ and $D$. These can be deformed in such a manner that their dihedrals remain invariably equal or supplementary in pairs. Following the theorem of Sec. III, this implies that their faces must be pairwise equal or supplementary. It is likewise for the faces of the other three couples of opposed tetrahedral angles that constitute the octahedron.

It is easy to see that, solely, the equality of corresponding faces is admissible. Let us consider, in effect, two opposing facets of the octahedron, $\triangle ABC$ and $\triangle DEF$ for example. One has, as a result of the preceding discussion,

$$\begin{aligned}
\angle A = \angle D \quad &\text{or} \quad \angle A + \angle D = \pi\,,\\
\angle B = \angle E \quad &\text{or} \quad \angle B + \angle E = \pi\,,\\
\angle C = \angle F \quad &\text{or} \quad \angle C + \angle F = \pi\,,
\end{aligned}$$

and, as it is shown in elementary geometry in order to establish the similarity of two triangles having their sides pairwise parallel or perpendicular, the equalities written at the begining of each line are the only ones that may hold true.

The octahedron is thus such that its opposing facets are similar triangles, with homologous sides having always opposing edges. One has, as a result, the sequence of equalities

$$\begin{aligned}
\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}\,, \qquad\qquad & \frac{CA}{FD} = \frac{AE}{DB} = \frac{EC}{BF}\,,\\
\frac{EC}{BF} = \frac{CD}{FA} = \frac{DE}{AB}\,, \qquad\qquad & \frac{CD}{FA} = \frac{DB}{AE} = \frac{BC}{EF}\,,
\end{aligned}$$

from which there follows

$$\begin{cases} AB = DE\,, \quad BC = EF\,, \quad CA = FD\,, \\ AE = BD\,, \quad BF = CE\,, \quad CD = AF\,. \end{cases} \tag{11}$$

*Therefore the octahedron has its opposing edges pairwise equal.*

I shall now show that these relationships suffice to ensure the deformability of the octahedron (which does not follow from the preceding analysis), *if one assumes additionally certain conditions related to the relative placement of the facets*. There exist in fact, convex octahedra for which the opposing edges are pairwise equal, and which, by Cauchy's theorem, cannot be deformable.

# 7 (Flexibility of Type I: axis of rotational symmetry)

Let us consider to this effect a system of four invariant triangles $\triangle AFB$, $\triangle DFB$, $\triangle ACE$, $\triangle DCE$ (*fig.* 3), joined at the points $A$ and $D$ and along the lines $BF$ and $CE$. One may suppose

$$AF = DC\,, AB = DE\,, DB = AE\,, DF = AC\,, BF = CE\ .$$

The figure formed by the two last triangles is evidently superimposable either onto the figure formed by the first two, or to the one symmetric to this with respect to an arbitrary plane. This leads to two cases that must be examined.

Let us suppose then that the two systems of triangles form superimposable figures. Now the four triangles $\triangle ADF$, $\triangle ADB$, $\triangle ADE$, $\triangle ADC$ are projected, on a plane perpendicular to $AD$, respectively along the lines $OF'$, $OB'$, $OE'$, $OC'$; such that the angles $\angle F'OB'$, $\angle F'OC'$ will be equal and that the same sense of rotation brings $OF'$ inot coincidence with $OB'$, $OE'$ into coincidence with $OC'$. One then has

$$\angle F'OE' = \angle B'OC'\ .$$

From this there results that the two trihedrals $A(FDE)$ and $D(CAB)$, that have the same orientation, are equal by virtue of having an equal dihedral comprised of two faces equal each to each. In effect, the preceding equality

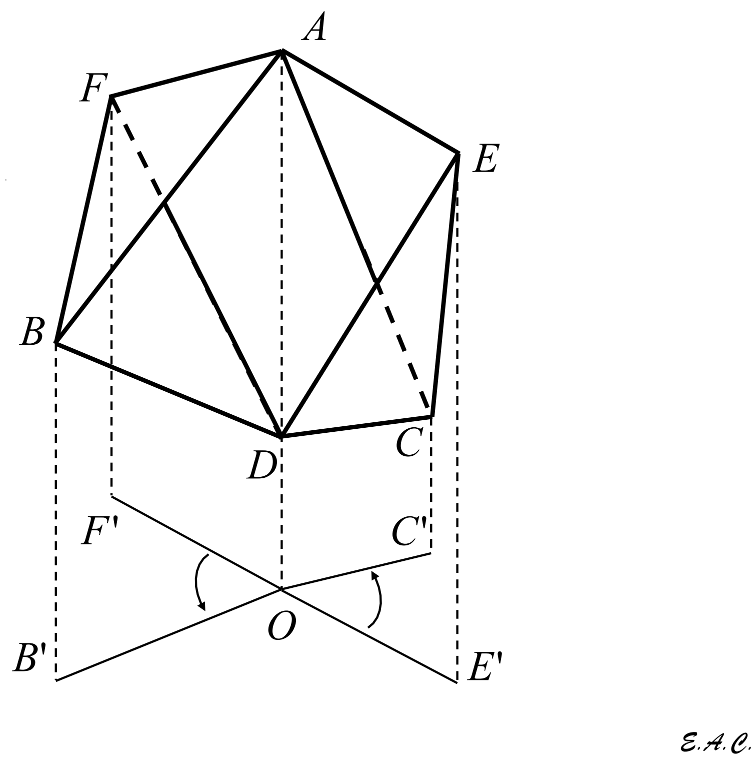


Figure 3:

expresses that property of those dihedrals having $AD$ as a common edge. One has, on the other hand,

$$\angle FAD = \angle CDA \ ,$$

$$\angle DAE = \angle ADB \ .$$

One is led from this to the equality of the angles $\angle FAE$, $\angle CDB$. The two triangles $\triangle FAE$, $\triangle BDC$ are therefore equal, and one has

$$FE = BC \ .$$

This equality is true during all the deformations of which our system of triangles is susceptible, under the condition, I repeat, that the set of the latter two is at all times superimposable to that of the former two.

Now, this deformation is such that the complete determination of the system depends on two parameters (for which one may take for example the distance $AD$ and the angle $\angle F'OE'$). This deformation is thus still possible if the system is subjected to the supplementary condition that the distance $EF$ remains constant: it will then be the same as the distance $BC$.

The set of the figure will exhibit eight invariant triangles, constituting a defomable octahedron whose opposing edges are pairwise equal.

This octahedron admits one axis of symmetry: let us draw, in effect, in the bisecting plane of the dihedral projected on $B'OE'$, a straight line $L$, perpendicular to $AD$ and passing through the midpoint of that segment. It is clear that the points $A$, $B$, $F$ are respectively symmetric to the points $D$, $E$, $C$, with respect to $L$. On may then bring the octahedron into coincidence with itself by making it turn by two right angles about $L$. On may see also that the three diagonals of the octahedron are perpendicular to the same line $L$ which divides each of them in two equal parts [7].

One may construct such an octahedron by means of cardboard triangles appropriately cut and assembled with sticky tape. It is necessary to leave empty the facets $ABC$, $DEF$, which are only realised by their contour.

The model thus obtained is represented in Fig. 4.

If, now, returning to Fig. 3, one assumes that the figure formed by the two triangles $\triangle ACE$, $\triangle DCE$, is superimposable to the figure symmetric to that formed by the two triangles $\triangle AFB$, $\triangle DFB$, one will conclude, by an argument entirely similar to the preceding one that, if the distance $EF$ remains constant, the distance $BC$ is necessarily variable: the octahedron

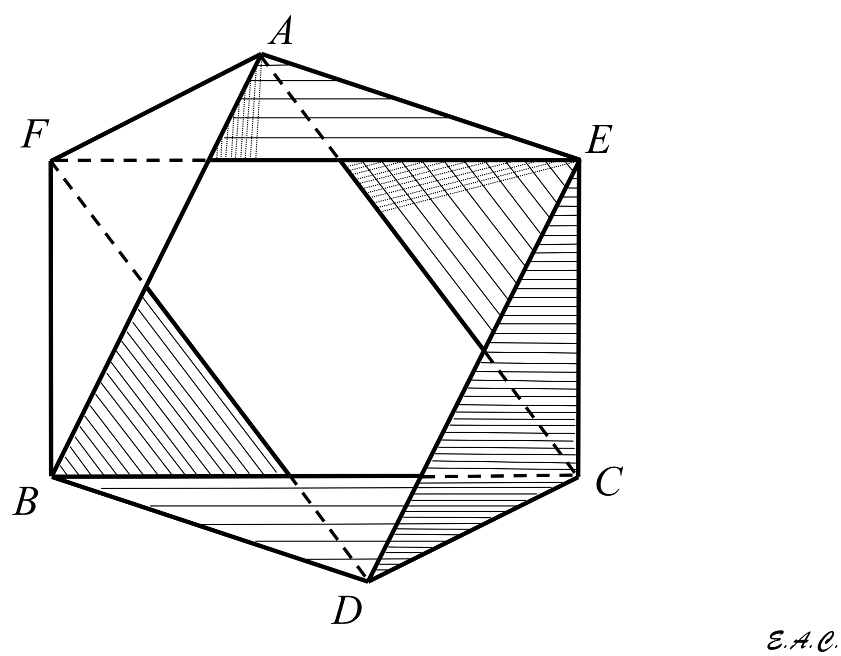


Figure 4: Octahedron whose opposite edges are equal in pairs. The facets are $ABC$, $DEF$, $BCD$, $CAE$, $ABF$, $AEF$, $BFD$, $CDE$

$ABCDEF$ is not deformable. This establishes the assertion made at the end of section VI. One sees that the relations (11) are not independent, but that one of them implies the other five.

# 8 (The case of a rhomboidal tetrahedral)

I pass to the case where one among the tetrahedral angles is rhomboidal, so that only one of them has its adjacent faces equal or supplementary pairwise. Let us suppose that the tetrahedral angle $C$ has this special property, that is it has, for example:

$$\angle BCD = \angle DCE\,, \qquad \angle BCA = \angle ACE\,.$$

The system (7), (8), (9), becomes then

$$At^2u + 2Ct + Du = 0\ , \tag{12}$$

$$A't^2v^2 + B't^2 + 2C'tv + D'v^2 + E' = 0\ , \tag{13}$$

$$A''u^2v^2 + B''u^2 + 2C''uv + D''v^2 + E'' = 0\ . \tag{14}$$

We may reason as before: the equations (13) and (14) can have at all times the two roots in $v$ common, or instead they may only have one of these roots in common.

I will demonstrate shortly that the first assumption is inadmissible. The equations (13) and (14) have thus a single common root in $v$; this root is a rational function of $t$ and $u$, and in terms of $t$, equation (13) reduces necessarily to one of the forms

$$A't^2v + 2C't + D'v = 0\ ,\ \ B't^2 + 2C'tv + E' = 0\ .$$

In other words, the tetrahedral angle $B$ is rhomboidal, and it has

$$\text{dihedral } BF = \text{dihedral } BC = \text{dihedral } CE\ .$$

It also follows that the equations (12) and (14), respectively in $t$ and in $v$, have double roots for the same values of $u$. One then concludes that the dihedrals $CD$ and $AF$ remain equal or supplementary at all times. The same is true for the dihedrals $BD$, $AE$. Continuing in this way, it will be true that the opposing pairs of the dihedrals of the octahedron all possess the same property.

The conclusions of Section VI are not therefore changed.

It remains to show that the equations (13) and (14) cannot have their two roots in $v$ identically equal at all times.

In effect, if it is so, one must have

$$\frac{A't^2+D'}{A''u^2+D''}=\frac{C't}{C''u}=\frac{B't^2+E'}{B''u^2+E''}$$

or

$$\begin{aligned} A'C''t^2u - C'A''tu^2 - C'D''t + D'C''u &= 0 \ , \\ B'C''t^2u - C'B''tu^2 - C'E''t + E'C''u &= 0 \ . \end{aligned}$$

These last equations must be identical with equation (12). One then has

$$C'A''=0 \ , \ C'B''=0 \ ;$$

from which

$$C'=0 \ \text{ or instead } \ A''=0 \ , \quad B''=0 \ .$$

Now, if it is true that

$$C'=0 \ ,$$

there follows from the discussion of Sec. II that equation (13) has of necessity one of the forms

$$A't^2v^2+E'=0 \ , \ B't^2+D'v^2=0 \ .$$

The tetrahedral angle $B$ will then have its opposing faces equal or supplementary pairwise, and we have excluded such tetrahedral angles in examining the present case.

One then has

$$A''=0 \ , \quad B''=0 \ ,$$

and equation (14) reduces to

$$2C''uv+D''v^2+E''=0 \ .$$

The tetrahedral angle $A$ is rhomboidal (Sec.II), and one has

$$\angle EAC=\pi-\angle BAC \ , \quad \angle FAE=\pi-\angle BAF \ ,$$

$$\text{dihedral}AE = \text{dihedral}AB \ .$$

Since one also has

$$\text{dihedral}CE = \text{dihedral}CB \ ,$$

the two tetrahedral angles having their vertices at $B$ and $E$ must deform in such a way that two dihedrals adjacent to the first are at all times equal, respectively, to two dihedrals adjacent to the second tetrahedral angle. It must also be true (Sec. III) that their faces are pairwise equal or supplementary. One has also

$$\text{dihedral}FE = \text{dihedral}FB \ ,$$

$$\text{dihedral}DE = \text{dihedral}DB \ .$$

The tetrahedral angles $F$ and $D$ are also rhomboidal. Combining these results, one can see that the triangles

$$\triangle AFE \quad \text{and} \quad \triangle AFB \ ,$$

$$\triangle ACB \quad \text{and} \quad \triangle ACE \ ,$$

$$\triangle BFD \quad \text{and} \quad \triangle EFD \ ,$$

$$\triangle BCD \quad \text{and} \quad \triangle ECD$$

have, pairwise, their angles equal or supplementary. The equality alone is possible. One has in particular

$$\angle BAF = \angle EAF \ ,$$

$$\angle BAC = \angle EAC \ .$$

We have concluded, on the other hand, that these angles, faces of the tetrahedral angle $A$, are pairwise supplementary. The may not all be also equal unless they are all $\pi/2$, which is visibly impossible.

# 9 (The case of a unicursal tetrahedral)

There remains to examine the case where at least one of the tetrahedral angles has its opposing faces equal or supplementary in pairs.

Let us suppose this is so for the tetrahedral angle $C$. The variables $t$ and $u$ satisfy one of the relations

$$tu = k \ , \ \frac{t}{u} = k \ .$$

I shall admit the existence of the first, the reasoning being the same for the second case. Assuming this, two distinct hypotheses need to be examined:

1. None of the tetrahedral angles whose vertices are located at the ends of the edges issuing from $C$ has its opposing faces equal or supplementary.
2. One of these tetrahedral angles prossesses that property.

In the first case, the tetrahedral angles $A$ and $B$ are general or rhomboidal. Let us suppose them general, for example. The system of relations among $t, u, v$ is then

$$tu = k \ , \tag{15}$$

$$A't^2v^2 + B't^2 + 2C'tv + D'v^2 + E' = 0 \ , \tag{16}$$

$$A''u^2v^2 + B''u^2 + 2C''uv + D''v^2 + E'' = 0 \ . \tag{17}$$

Equation (16) in $t$ and equation (17) in $u$ have their roots equal for the same values of $v$: it then follows that the dihedrals $AE$ and $BD$ are at all times equal or supplementary.

Concurrent consideration of the tetrahedral angles $A, C, E$ shows similarly that the dihedrals $AB$, $DE$ remain at all times equal or supplementary. One then may construct fig. 5, where the dihedrals marked by the same number exhibit the same relationship (the dihedrals 1 and the dihedrals 2 due to the nature of the tetrahedral angle $C$).

If one applies to the tetrahedral angles $A$ and $D$ on the one hand, $B$ and $E$ on the other, the theorem which has already been utilised several times, one sees:

1. That the dihedrals $FA$ and $FD$, on the one hand, $FB$ and $FE$ on the other, are at all times equal, and that the tetrahedral angle $F$ has, consequently, its opposing faces equal or supplementary in pairs;
2. That the facets of the octahedron are subdivided into four pairs of triangles having their angles equal in pairs:

$$\triangle AFE \text{ and } \triangle DFB,$$

$$\triangle AFB \text{ and } \triangle DFE,$$

$$\triangle AEC \text{ and } \triangle DBC,$$

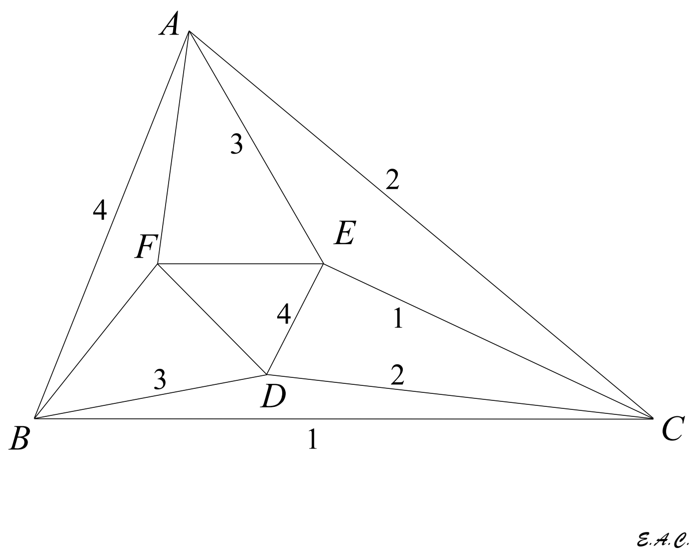


Figure 5:

$$\triangle ABC \quad \text{and} \quad \triangle DEC.$$

For each pair the homologous vertices are written in the same order. One has therefore the sequence of equalities

$$\frac{AF}{DF} = \frac{FE}{FD} = \frac{EA}{BD} \ , \quad \frac{AF}{DF} = \frac{FB}{FE} = \frac{BA}{ED} \ ,$$

$$\frac{AE}{DB} = \frac{EC}{BC} = \frac{BA}{ED} \ , \quad \frac{AB}{DE} = \frac{BC}{EC} = \frac{CA}{CD} \ ,$$

from which it is seen that

$$\begin{cases} FA = FD \ , \quad FE = FB \ , \quad CA = CD \ , \\ CB = CE \ , \quad AE = DB \ , \quad AB = DE \ . \end{cases} \tag{18}$$

*Therefore the edges of the octahedron must again be pairwise equal, but the equal edges are not all pairwise opposite, a fact that distinguishes conditions (18) from conditions (11).*

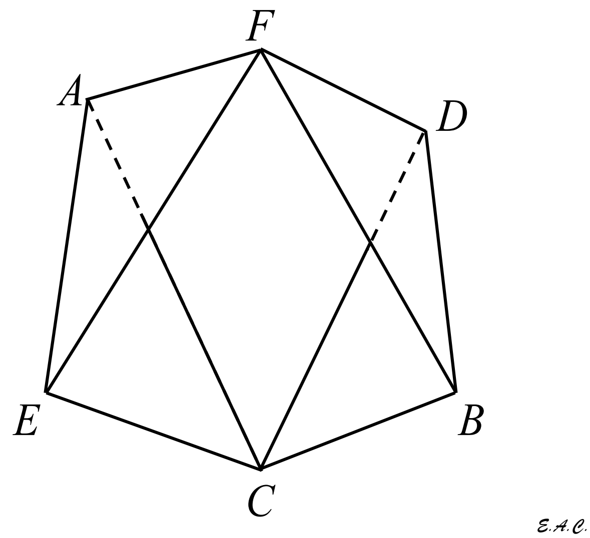


Figure 6:

## 10 (Flexibility of Type II: Opposing unicursal tetrahedrals and a plane of symmetry)

It must now be shown that all octahedra whose edges satisfy relations (18) are deformable, provided certain conditions, related to the placement of the facets, are satisfied.

For this, let us consider (fig. 6) the system of four rigid triangles $\triangle FAE$, $\triangle CAE$, $\triangle FDB$, $\triangle CDB$, joined at the points $F$, $C$, and along the lines $AE$, $BD$. Assume the equalities

$$FA = FD, FE = FB, CA = CD, CE = CB, AE = DB.$$

The system of the last two triangles is superimposample on that of the first two, or instead it is symmetric to that with respect to a plane passing through $FC$.

In the first case it will be true, by arguments analogous to those in Sec. VII, that one cannot have at all times that $AB = DE$. The octahedron formed by connecting the points $A$ and $B$ on the one hand, $D$ and $E$ on the other, not satisfying all of the relationships (18) and moreover not satisfying

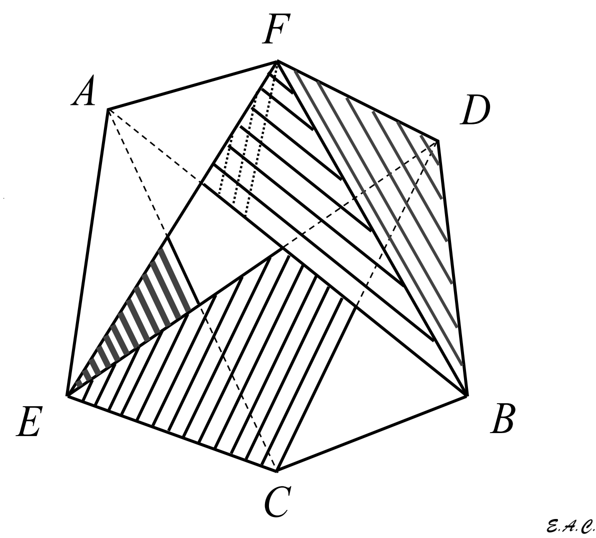


Figure 7: Octahedron possessing a plane of symmetry passing through two opposite vertices. The facets are $ABC$, $DEF$, $BCD$, $CAE$, $ABF$, $AEF$, $BFD$, $CDE$.

relations (11), is not deformable.

On the contrary, if the systems of triangles are symmetric, it is evident that, whatever their relative position, one has always

$$AB = DE \ .$$

The reasoning will proceed like that in Sec. VII, and it will establish that relations (18), just like relations (11), suffice, under the indicated restrictions, to ensure the deformability of an octahedron.

This second octahedron may be realised like the first, by leaving empty the facets $ABC$ and $DEF$. The model thus obtained is represented in fig. 7.

# 11 Flexibility of Type III: adjacent unicursal tetrahedrals in proportion)

I arrive finally at the case where two tetrahedral angles having their vertices adjacent, the tetrahedral angle $C$ and the tetrahedral angle $B$, for example, have each of their opposite faces equal or supplementary in pairs.

One can see immediately that this must be similarly true for all the tetrahedral angles of the octahedron.

In effect, from the relations

$$tu \;\text{ or }\; \frac{t}{u} = k,$$

$$tv \;\text{ or }\; \frac{t}{v} = k',$$

one shows

$$\frac{u}{v} \;\text{ or }\; uv = k'',$$

which establishes the proposition for the tetrahedral angle $A$; this will be established similarly for the other tetrahedral angles $D$, $E$, $F$.

If then the dihedral $BD$ becomes equal to 0 or $\pi$, $t$ becoming zero or infinity, the variables $u$, $v$ are also zero or infinite, and the dihedrals $AC$, $AB$, equal to 0 or $\pi$. It is similarly so for all the other dihedrals. In other words, the octahedron may be completely flatened onto the facet $ABC$.

Let us represent it in that position. There may be several cases of this figure, for which the reasoning is identical.

I will suppose, for example, that the arrangement is that of fig. 8. One has

$$\begin{aligned} \angle FAE &= \angle BAC \;, \\ \angle DCE &= \angle ACB \;, \\ \angle DBF &= \pi - \angle ABC \;. \end{aligned}$$

During the deformation of the octahedron, one may have various systems of relationships among $t, u, v$. For example (sec. II), let

$$\begin{aligned} uv &= \frac{\cos\dfrac{\angle BAF - \angle BAC}{2}}{\cos\dfrac{\angle BAF + \angle BAC}{2}} \;, \\ \frac{t}{v} &= \frac{\sin\dfrac{\angle ABC - \angle DBC}{2}}{\sin\dfrac{\angle ABC + \angle DBC}{2}} \;, \end{aligned}$$

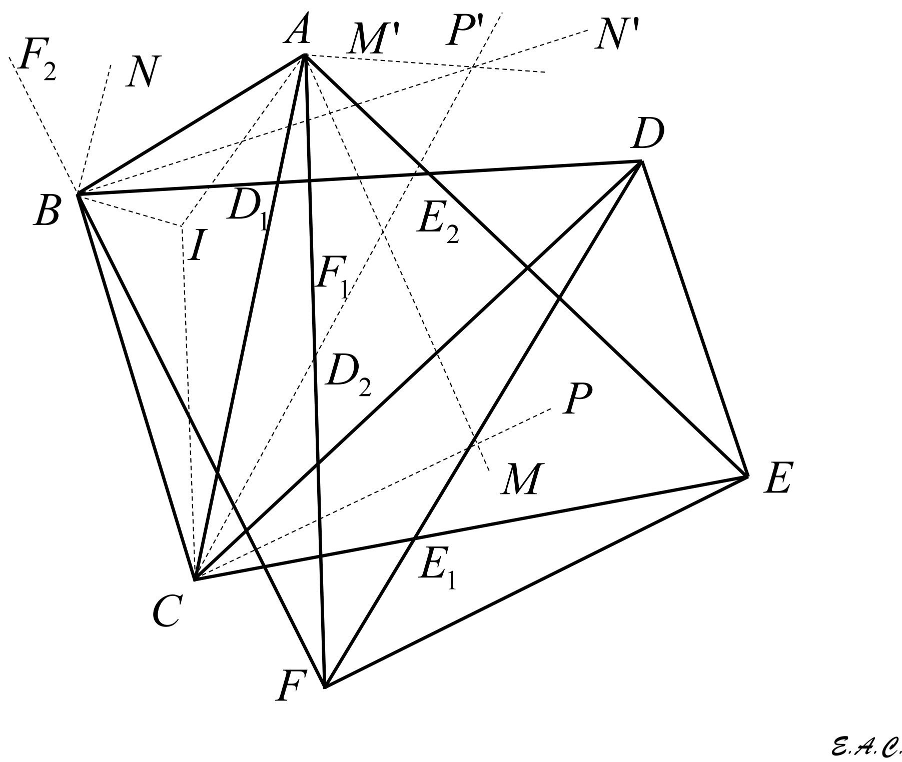


Figure 8:

$$tu \;=\; \frac{\sin\dfrac{\angle DCB-\angle ACB}{2}}{\sin\dfrac{\angle DCB+\angle ACB}{2}}\;.$$

These equations must be satisfied by an infinity of sets of values of $t, u, v$. One has then

$$\frac{\cos\dfrac{\angle BAF-\angle BAC}{2}}{\cos\dfrac{\angle BAF+\angle BAC}{2}}\,\frac{\sin\dfrac{\angle ABC-\angle DBC}{2}}{\sin\dfrac{\angle ABC+\angle DBC}{2}}\,\frac{\sin\dfrac{\angle DCB+\angle ACB}{2}}{\sin\dfrac{\angle DCB-\angle ACB}{2}}=1\;. \qquad (19)$$

This is the necessary and sufficient condition for the octahedron $ABCDEF$ to be deformable.

In order to give to this condition a geometric form, let us draw the *interior* bisectrices of the triangle $\triangle ABC$: $AI$, $BI$, $CI$. Let us draw also the lines $AM$, $CP$, $BN$, the first two, *interior* bisectrices of the angles $\angle EAF$, $\angle DCE$, the third, *exterior* bisectrice of the angle $\angle FBD$. Let finally $FM'$ be the *exterior* bisectrice of the angle $\angle IAM$, $BN'$ and $CP'$ the *exterior* bisectrices to the angles $\angle IBN$, $\angle ICP$. One has:

$$\begin{aligned}\frac{\cos\dfrac{\angle BAF-\angle BAC}{2}}{\cos\dfrac{\angle BAF+\angle BAC}{2}} &= \frac{\cos\left(\dfrac{\angle IAM}{2}-\angle IAC\right)}{\cos\left(\dfrac{\angle IAM}{2}+\angle BAI\right)} = \frac{\cos\left(\angle IAM'-\dfrac{\pi}{2}-\angle IAC\right)}{\cos\left(\angle IAM'-\dfrac{\pi}{2}+\angle BAI\right)}\\ &= \frac{\sin\left(\angle IAM'-\angle IAC\right)}{\sin\left(\angle IAM'+\angle BAI\right)} = \frac{\sin\angle CAM'}{\sin\angle BAM'}\;,\end{aligned}$$

$$\frac{\sin\dfrac{\angle ABC-\angle DBC}{2}}{\sin\dfrac{\angle ABC+\angle DBC}{2}} = \frac{\sin\left(\angle IBA-\dfrac{\angle IBN}{2}\right)}{\sin\left(\angle CBI+\dfrac{\angle IBN}{2}\right)} = \frac{\sin\left(\angle IBA-\angle IBN'\right)}{\sin\left(\angle CBI-\angle IBN'\right)} = \frac{\sin\angle N'BA}{\sin\angle CBN'}$$

$$\frac{\sin\dfrac{\angle DCB+\angle ACB}{2}}{\sin\dfrac{\angle DCB-\angle ACB}{2}} = \frac{\sin\left(\dfrac{\angle PCI}{2}+\angle ICB\right)}{\sin\left(\dfrac{\angle PCI}{2}-\angle ACI\right)} = \frac{\sin\left(\angle P'CI+\angle ICB\right)}{\sin\left(\angle P'CI-\angle ACI\right)} = \frac{\sin\angle P'CB}{\sin\angle P'CA}\;.$$

Relation (19) becomes then

$$\frac{\sin\angle CAM'}{\sin\angle BAM'}\,\frac{\sin\angle N'BA}{\sin\angle CBN'}\,\frac{\sin\angle P'CN}{\sin\angle P'CA}=1\;;$$

from which it follows, by virtue of a well known theorem, that the lines $AM'$,$BN'$,$CP'$, meet at a point (are concurrent).

I have made a particular hypothesis on the form of the relationships that exist among $t, u, v$. It is clear that in all of the other cases, one will arrive at a similar result, and it may be pronounced as the general rule for constructing an articulated octahedron all of whose tetrahedral angles have their opposite faces pairwise equal or supplementary.

*Construct an arbitrary triangle $\triangle ABC$, whose interior bisectrices are the lines $AI$, $BI$, $CI$, and from the vertices of that triangle draw three concurrent lines $AM'$, $BN'$, $CP'$. Trace the lines $AM$, $BN$, $CP$, respectively symmetric to the lines $AI$, $BI$, $CI$, with respect to the lines $AM'$, $BN'$, $CP'$.*

*Construct then the angles $\angle F_1AE_2$, $\angle D_1BF_2$, $\angle E_1CD_2$, obtained by rotating in their own planes the angles $\angle BAC$, $\angle CBA$, $\angle ACB$, about their vertices by angles equal in magnitude and sign respectively to $\angle IAM$, $\angle IBN$, $\angle ICP$. Let $D$, $E$, $F$ be the points of intersection, respectively, of the lines (suitably elongated through the vertices of the triangle) $BD_1$ and $CD_2$, $CE_1$ and $AE_2$, $AF_1$ and $BF_2$.*

*Assuming the triangles $\triangle ABC$, $\triangle BCD$, $\triangle CAE$, $\triangle ABF$, $\triangle AEF$, $\triangle BFD$, $\triangle CDE$, $\triangle DEF$, are constructed and articulated in pairs along their common edges, these triangles are the facets of a deformable octahedron.*

Certain difficulties may be encountered during the construction of a model of this latter octahedron: which facets should be left open? from which side should the concavity of each dihedral be oriented? These are problems that may not be resolved in each particular case without a careful examination of the relations among $t, u, v$ and the signs imposed on each variable. It would be too lengthy to discuss here the detailed reasoning. I shall content myself by indicating how to carry out the construction of the octahedron of fig. 8.

I have represented it (fig. 9, right) in the same position as in the preceding figure; the facets $AEC$, $DBF$ are empty. The dihedral $AF$ has its concavity facing forward from the plane of the figure; that of the dihedral $DC$ is facing backward.

The broken lines indicate unambiguously in which order they are superimposed on the facets: in particular, the facet $DEF$ is behind the facet $AEF$.

When deforming this octahedron, by setting the facet $ABF$ on the plane of the figure, the vertices $D$, $E$, $C$ are displaced in front of this plane, and pursuing the deformation one arrives at a new flattened position (fig. 10). Fig. 11 represents an intermediate position.

If the reader takes the pains of constructing this model, by following the

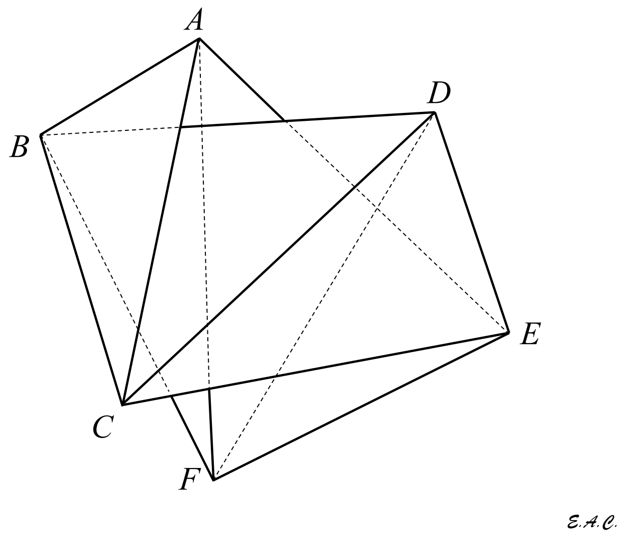


Figure 9:

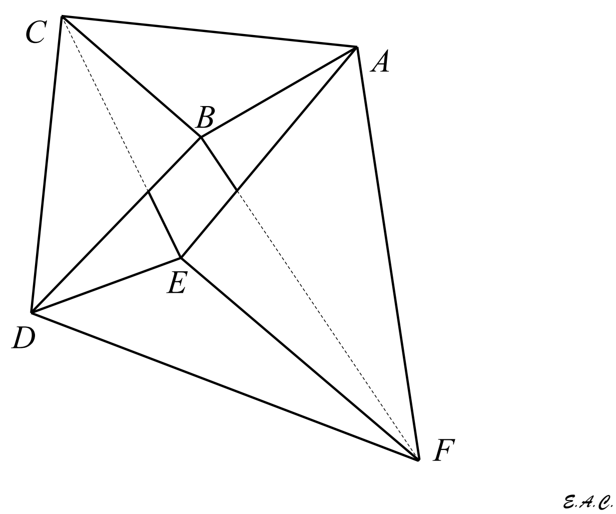


Figure 10:

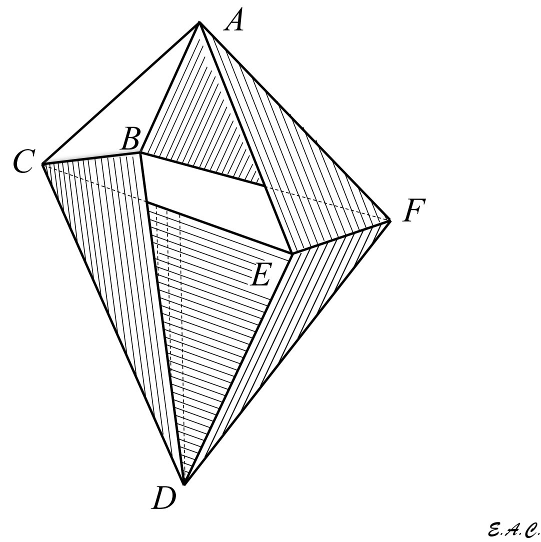


Figure 11: Octahedron all of whose tetrahedral angles have their opposing faces pairwise equal or supplementary. The facets are $ABC$, $DEF$, $BCD$, $CAE$, $ABF$, $AEF$, $BFD$, $CDE$.

preceding instructions, I shall reiterate that the proportions of the figure must be preserved in the most precise fashion.

## 12 (Discussion)

In summary, the preceding study has shown that there exist *three* types of articulated octahedra with invariant facets. All these polyhedra are concave or, to be more precise, possess certain facets that intercross.

Octahedra of type I (11) and II (18) have simple definitions: the first possess an axis of symmetry and, as a result, are such that the figure formed by four of their facets with a common vertex is superimposable on the figure formed by the other four; those of type II have a plane of symmetry, passing through two opposite vertices. (These definitions are not *absolutely* sufficient, but one can only complete them by lengthy discussion, and I feel that an examination of figures 4 and 7 makes that unnecessary).

About octahedra of type III (19), we have seen that their definition is more complicated; their deformability is far from being as intuitive as that of the former types, and in this sense they ought to be considered as the most interesting.

I shall also remark that the problem which I have discussed is identical to the following two problems:

1$^o$ *What are the deformable oblique hexagons with constant sides and angles?*

If, in fact, an oblique hexagon is deformable under these conditions, the segments that join its next to nearest vertices have constant lengths and the eight triangles formed by these segments and by the sides of the hexagon are the facets of a deformable octahedron with constant edges.

A deformable octahedron exhibits, on the other hand, four such hexagons. These are (fig. 4, 7 or 10) the hexagons

$$ABCDEF\ ,$$

$$ABFDEC\ ,$$

$$AECDBF\ ,$$

$$AEFDBC\ .$$

2$^o$ *Under what conditions is a system of six planes 1, 2, 3, 4, 5, 6, where each is articulated with the previous along a line serving as a hinge, with plane 6 being articulated with planes 1 and 5, susceptible to deformation?*

In fact, the lines along which these planes are articulated form a deformable hexagon with fixed sides and angles.

The planar facets of one of the octahedra of figs. 4, 7, 11 constitute such a system.

# 13 (The planar articulated quadrilateral)

It is not without interest to note the analogy between the theory of the articulated octahedron with that of systems of articulated quadrilaterals studied by Mesrs. Hart ([9]) and Kempe ([10]) for particular cases and Mr. Darboux ([11]) for the general case. One of the problems studied by these geometers is, in effect, the following: given three planar quadrilaterals $ABCD$, $AEFG$, $BEHI$ arranged as shown in fig. 12, under what conditions does their system, generally rigid, become susceptible to deformation? If the tangents of

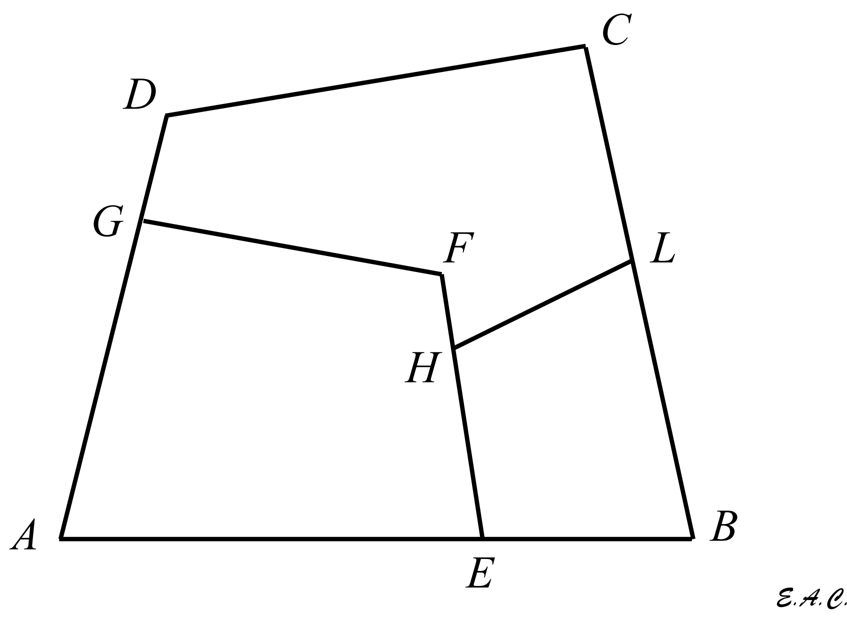


Figure 12:

the halves of the angles $\angle BAD$, $\angle ABC$, $\angle BEH$ are denoted by $t, u, v$, there exist among these quantities three relations that have the same form as relations (7), (8), (9) and they ought to have an infinity of solutions. It is the same problem that presents itself in the theory of the articulated octahedron [12].

# References


[1] (Raoul Bricard was born March 23, 1870. In 1908 he was appointed professor of Applied Geometry at the Paris Observatoire des Arts et Metiers; Science, 17 July 1908, vol. 28, No. 707, p.85-86. He was awarded the 1932 Poncelet prize for his work in geometry by the Paris Academy of Sciences; Nature, 4 February 1933, vol. 131, No. 3301, p.174-175.)

[2] (Bricard, Raoul, Mémoire sur la théorie de l'octaèdre articulé, J.Math.Pures Appl. 1897, 3, 113-150.)

[3] Stephanos, Cyparissos (Athens); *L' Intermédiaire des Mathématiciens* vol. 1, p. 228, problem 376:
*Do there exist polyhedra with invariant facets susceptible to continuous deformations that solely alter solid angles and dihedral angles?"*.

[4] Bricard, Raoul; *L' Intermédiaire des Mathématiciens* vol. 2 (1895), p. 243-4.:
**376***(C. Stephanos).*– Here is how an octahedron may be constructed that is susceptible to deformations while its facets remain unchanged.

Let $\triangle ABC$, $\triangle DBC$ be two equal triangles ($AB = CD$, $AC = BD$) joined along the edge $BC$. Imagine that the system of these triangles is rotated about the line $AD$, so that it assumes the position $AB'C'D$. The new system is deformable if we impose on the angle $\angle BAB'$ the condition that it assumes a fixed value.

Indeed, the two isosceles trihedrals $ABB'D$, $DBB'A$ are equal, having corresponding facets equal, as can be easily seen. We have then

$$\angle CDC' = \angle BAB' = \text{constant} .$$

Figure (13) exhibits the 8 invariant triangles, arranged in groups of 4 about the 6 vertices $A$, $B$, $C$, $D$, $B'$, $C'$ whose collection may be

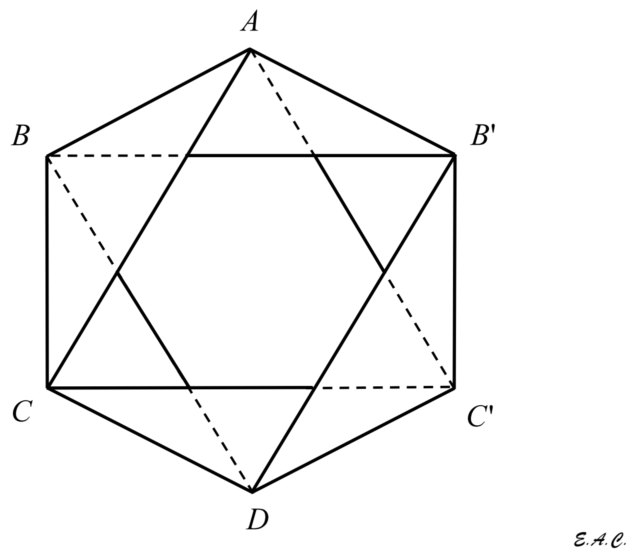


Figure 13:

deformed. These 8 triangles constitute the facets of a concave octahedron answering the question.

In the construction of such an octahedron (by means, for example, of business cards suitably cut and joined by tape) one must leave open the facets $ACC'$ and $DBB'$, which are only realized by their perimeter.
RAOUL BRICARD

It seems to me that Cauchy has treated this problem completely in the *Journal de l'École Polytechnique, (XVI Cahier, 1813)* [4], and that he answered the question posed, in the negative.
C. JUEL (Copenhagen).

[5] Cauchy, Augustine Louis; Deuxième Mémoire sur les polygones et les polyèdres, *Journal de l'École Impériale Polytechnique, XVI Cahier*, p.87-99, 1813.

[6] (Such angles are called *unicursal* as the tetrahedral equation decomposes to two rational functions, each tracing a complete circle in each variable).

[7] These interesting comments were communicated to me by Mr. Mannheim.

[8] (In the original, the first argument of $\phi$ in the equation above was omitted; it is clear that it expresses the relation $v = \phi(t, u)$.)

[9] (see, e.g. Hart, Harry; On some cases of Parallel Motion (read April 12, 1877), Proceedings of the London Mathematical Society, 286-289, vol. s1-8, Issue 1, November 1876; for earlier references to Hart's construction, see Kempe[10]: *”(6) This paper is printed in extenso in the Cambridge Messenger of Mathematics, 1875, vol. iv, p.82-116, and contains much valuable matter about the mathematical part of the subject.”*).

[10] (Kempe, Alfred Bray; *How to draw a straight line*, lecture delivered in the summer of 1876 at London's South Kensington Museum. Available in electronic form at

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/t/text/text-idx?c=math;cc= math;rgn=main;view=text;idno=kemp009

A discussion of Hart and Kempe's work on linkages can be found in: Sylvester, James J.; History of the Plagiograph, Nature, vol.12, No.298, July 15 1875, p.214-216)

[11] (Darboux, Gaston; De l'emploi des fonctions elliptiques dans la théorie du quadrilatère plan, Bulletin des sciences mathématiques et astronomiques, 2$^{e}$ série, tome 3, no.1 (1879), p.109-128).

[12] The models of the three octahedra, constructed from cardboard leaves, have been offered to the Collections of the École Polytechnique.

# A Comments on the preceding Mémoire

**by Mr. Amédée Mannheim.**

The Mémoire of Mr. Bricard, very interesting in itself, is of particular value from the point of view of *Kinematic Geometry*, because it constitutes a chapter in the study of the displacement of a triangle in space.

Mr. Bricard, by discovering the deformable octahedra, proved thus that under certain conditions a triangle of fixed size may be displaced in such a manner that its vertices describe circular arcs.

In effect, if we assume one of the facets $ABC$ of one of the deformable octahedra to remain fixed, during the deformation the vertices $D$, $E$, $F$ of the opposite facet revolve as a result around the edges of the facet $ABC$ and the triangle $\triangle DEF$ is displaced. In the case of an arbitrary octahedron, with $ABC$ fixed, the displacement of the triangle $\triangle DEF$ is not possible. This is understood easily by noticing that the vertices of that triangle must each describe circles, each of which is constrained by two conditions. The triangle itself must then be constrained by six conditions: it is therefore immobile.

Let us consider (2) a deformable octahedron whose facet $ABC$ is fixed. The triangle $\triangle DEF$ may therefore be displaced. The well known properties, related to its displacement, lead to the properties of the octahedron. I will give a single example.

Let us apply this theorem:

*The normal planes to the trajectories of points of a plane pass through a point of that plane.*

The plane perpendicular to the trajectory of the point $D$ is the plane of the facet $BDC$. Similarly, the planes of the facets that pass through $AB$, $AC$ are the normal planes to the trajectories of the vertices $F$, $E$. These three planes meet therefore at a point of the plane $DEF$.

According to this, the following theorem can be stated:

*The planes of the facets of a deformable octahedron, which pass through the sides of one of the facets of that polyhedron, meet at a point of the plane of the opposite facet.*

This way, the properties of deformable octahedra may be found, but independently of these properties, there appear additional questions related to a mobile triangle.

For an infinitesimally small displacement of the triangle, there exists an axis of displacement: what is the locus of these axes if the displacement of the triangle is continued? What is the locus of the envelopes of the plane of the mobile triangle? etc., etc.

It is seen that following the discovery of deformable octahedra, the study of displacement of a triangle is presented with special conditions and it requires new investigations by geometers.