

Effects of thermal spread on the space charge limit of an electron beam

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An asymptotic analysis is carried out to calculate the effects of a small thermal spread in the injection energy of an electron beam on its space charge limit. It is found that the space charge limit is lowered proportionally to the beam temperature T near $T = 0$.

1. Introduction

Recent applications of intense charged particle beams in such areas as inertial confinement fusion and microwave generation (Coutsias & Sullivan 1983, and references therein) has necessitated a deepening of our understanding of the basic physics of space charge limited flows. An excellent review of our present state of knowledge can be found in Miller (1982).

In particular, many authors have carried out calculations of the space charge limit (SCL) of electron and ion beams (Voronin, Zozulya & Lebedev 1972; Read & Nation 1975; Genoni & Proctor 1980). These are concerned mostly with monoenergetic beams in various geometries. Although the importance of thermal effects is recognized, no analytical estimates of the effect of thermal spread on the beam kinetic energy at injection have appeared.

Here we present an asymptotic method to estimate the modification of the SCL due to a small thermal spread. For simplicity we limit our discussion to one-dimensional, classical motion. However, the method can be applied to any of the other situations for which SCL estimates exist and produce appropriate corrections.

2. Effect of temperature on the space charge limit

An adequate discussion of the properties of a non-relativistic electron beam in one dimension can be found in Coutsiias & Sullivan (1983). Here we shall treat the beam as a one-dimensional electron gas flowing between two conducting grid planes, at fixed potential. For the density range we consider ($\simeq 10^{13}$ cm $^{-3}$) the usual collisionless approximation is valid, and thus the electron distribution function satisfies the Vlasov equation

$$\partial_t f + u \partial_x f + (e/m) E \partial_u f = 0 \quad (1)$$

where t , x , u are the time, position and velocity variables, respectively, and E is the self-consistent electric field.

The electron charge density is given by

$$n = \int_u f(x, u, t) du, \quad (2)$$

and the electric field is found from

$$\partial_x E = (1/\epsilon_0) n, \quad (3)$$

while the potential $\phi(x, t)$ is given by

$$\partial_x \phi = -E. \quad (4)$$

The boundary conditions are given for the potential

$$\phi(0, t) = \phi_0 > 0, \quad \phi(l, t) = 0, \quad (5)$$

and the distribution function

$$\left. \begin{array}{l} f(0, u, t) \text{ specified for } u > 0 \text{ (incoming flow at } x = 0) \\ f(l, u, t) = 0 \text{ for } u < 0 \text{ (no incoming flow at } x = l) \end{array} \right\} \quad (6)$$

This specification is valid also in the presence of multiple streams.

We shall model the effect of finite source temperature by specifying the incoming distribution $f(0, u, t)$, $u > 0$ as

$$f(0, u, t) = n_0 \left(\frac{m}{2\pi kT} \right)^{\frac{1}{2}} \exp \left(-\frac{m(u - V)^2}{2kT} \right), \quad (7)$$

where T , assumed to be small, plays the role of an effective beam temperature.

For small enough T , we shall assume that we have a regime of steady behaviour, in analogy to the cold beam case. For steady states, particle energy is conserved and the solution to (1) is of the form (Davidson 1974)

$$f(x, u, t) = n_0 \left(\frac{m}{2\pi kT} \right)^{\frac{1}{2}} \exp \left(-\frac{m[V - (u^2 + (2e/m)(\phi(x) - \phi_0))^{\frac{1}{2}}]^2}{2kT} \right). \quad (8)$$

Then, combining (2), (3), (4) and (8) we find that the potential satisfies the equation

$$\phi_{xx} + \frac{n_0}{\epsilon_0} \left(\frac{m}{2\pi kT} \right)^{\frac{1}{2}} \int_{u_m}^{\infty} \exp \left(-\frac{m[V - (u^2 + (2e/m)(\phi(x) - \phi_0))^{\frac{1}{2}}]^2}{2kT} \right) du = 0, \quad (9)$$

where u_m , the velocity cut-off, is equal to

$$u_m(x) = \pm [(2e/m)(\phi_m - \phi(x))]^{\frac{1}{2}}. \quad (10)$$

Here ϕ_m is the potential minimum, the (+) sign applies to the right and the (-) to the left of the position $x = \xi$ of the potential minimum as shown in Appendix A. We define

$$I(\phi; T) = \int_{u_m}^{\infty} \exp \left(-\frac{m[V - (u^2 + (2e/m)(\phi(x) - \phi_0))^{\frac{1}{2}}]^2}{2kT} \right) du. \quad (11)$$

So (9) can be written as

$$\phi_{xx} + \frac{n_0 e}{\epsilon_0} \left(\frac{m}{2\pi kT} \right)^{\frac{1}{2}} I(\phi; T) = 0. \quad (12)$$

By introducing the variable

$$s = (u^2 + (2e/m)(\phi(x) - \phi_0))^{\frac{1}{2}} - V \quad (13)$$

the integral $I(\phi; T)$ in (12) can be rewritten as

$$I(\phi; T) = \int_{u=|u_m|}^{\infty} \exp\left(-\frac{ms^2}{2kT}\right) du + 2H(\xi - x) \int_{u=0}^{|u_m|} \exp\left(-\frac{ms^2}{2kT}\right) du. \quad (14)$$

The first of the integrals in (14) represents the transmitted flow, while the second is due to particles without sufficient energy to cross the potential minimum and which are therefore reflected and return to the anode. Equation (12) with the integral term given by (14) is very hard to solve for a finite temperature T , but for small T we can approximate the integrals in (14) by Laplace's method. As is well known (Erdelyi 1956), in approximating integrals of this type with a strong maximum at an interior point, the dominant contribution comes from the neighbourhood of this point. Thus, in (14) we can approximate $I(\phi; T)$ by

$$I(\phi; T) \simeq \int_{-\infty}^{\infty} \exp\left(-\frac{ms^2}{2kT}\right) ds + O\left(\exp\left(-\frac{ms_m^2}{2kT}\right)\right) \quad (14a)$$

where $-s_m = V - ((2e/m)(\phi_m - \phi_0))^{\frac{1}{2}}$. Then provided V is large enough so that the mean energy of the beam is never of order $O(kT)$, the correction term goes to zero faster than any power of T as $T \rightarrow 0$, and therefore is negligible to the order that we carry the calculations. Nevertheless, it gives us an estimate of the domain of validity of the subsequent discussion, for which we need

$$\exp(-ms_m^2/2kT) \ll kT. \quad (14b)$$

Therefore we write

$$I(\phi; \epsilon) \simeq \int_{-\infty}^{\infty} \exp\left(-\frac{ms^2}{2kT}\right) ds = \int_{-\infty}^{\infty} \frac{(s+V) \exp(-ms^2/2kT)}{((s+V)^2 - (2e/m)(\phi(x) - \phi_0))^{\frac{1}{2}}} ds, \quad (15)$$

and we get for the potential the approximate equation

$$\phi_{xx} + \frac{n_0}{\epsilon_0} \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} -\frac{(s+V) \exp(-ms^2/2kT)}{((s+V)^2 - (2e/m)(\phi(x) - \phi_0))^{\frac{1}{2}}} ds = 0. \quad (16)$$

This can be integrated once to give

$$\frac{1}{2}\phi_x^2 + J(\phi; T) = J(\phi_m; T) \quad (17)$$

where

$$J(\phi; T) = -\frac{n_0 m}{\epsilon_0 e} \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} (s+V) \times \left((s+V)^2 - \left(\frac{2e}{m}\right)(\phi(x) - \phi_0)\right)^{\frac{1}{2}} \exp(-ms^2/2kT) ds. \quad (18)$$

Integrating once more we find

$$x = \xi \pm 2^{-\frac{1}{2}} \int_{\phi_m}^{\phi} \frac{d\phi}{(J(\phi_m; T) - J(\phi; T))^{\frac{1}{2}}} \equiv \xi \pm S(\phi_m, \phi; T). \quad (19)$$

Imposing the boundary conditions at $x = 0$ and $x = l$ we are led to the two equations determining ξ and ϕ_m :

$$0 = \xi - S(\phi_m, \phi_0; T), \quad l = \xi + S(\phi_m, 0; T). \quad (20)$$

Eliminating ξ we find

$$l = S(\phi_m, \phi_0; T) + S(\phi_m, 0; T) \quad (21)$$

which must be analysed in order to determine, among other things, the desired steady state for the potential ϕ (and hence the distribution function f) and the SCL for small temperature $T > 0$.

To get an approximate expression for ϕ_m as $T \rightarrow 0$ we note that $J(\phi; T)$ can be approximated by Laplace's method if we expand the non-exponential part of the integrand in a Taylor series about $s = 0$ and integrate term by term. We find, after some algebra that

$$\left(-\frac{\epsilon_0 e}{n_0 m}\right) J(\phi; T) \simeq VR(\phi) + \frac{\epsilon V}{2} \left(\frac{R^2(\phi) - (e/m)(\phi - \phi_0)}{R^3}\right) + O(\epsilon^2) \quad (22)$$

where $\epsilon = (2kT/m) \ll 1$ and $R(\phi) = (V^2 - (2e/m)(\phi(x) - \phi_0))^{1/2}$. Using (22) we now approximate S :

$$\begin{aligned} S(\phi_m, \phi; T) &= 2^{-1/2} \int_{\phi_m}^{\phi} \frac{d\phi}{(J(\phi_m; T) - J(\phi; T))^{1/2}} \simeq \left(\frac{\epsilon_0 e}{2n_0 m V}\right)^{1/2} \\ &\times \int_{\phi_m}^{\phi} \frac{d\phi}{(R - R_m)^{1/2}} \left(1 - \frac{\epsilon}{4} \left(\frac{(r^2 - \Phi)R_m^3 - (R_m^2 - \Phi_m)R^3}{R^3 R_m^3 (R - R_m)}\right) + O(\epsilon^2)\right) \end{aligned} \quad (23)$$

where we set $R = R(\phi)$, $R_m = R(\phi_m)$, $\Phi = (e/m)(\phi(x) - \phi_0)$, $\Phi_m = (e/m)(\phi_m - \phi_0)$ and since $d\phi = -(m/e)RdR$,

$$\begin{aligned} S(\phi_m, \phi; T) &\simeq \left(\frac{2\epsilon_0 m}{9en_0 V}\right)^{1/2} (R(\phi) - R(\phi_m))^{1/2} (R(\phi) + 2R(\phi_m)) + \epsilon \left(\frac{\epsilon_0 m}{2en_0 V}\right)^{1/2} \\ &\times \left((R - R_m)^{1/2} \left(\frac{3}{4R_m} - \frac{V^2}{4R_m^3} - \frac{V^2}{8RR_m^3}\right) - \frac{3V^2}{8R_m^{3/2}} \sec^{-1} \frac{R^{1/2}}{R_m^{1/2}}\right) + O(\epsilon^2). \end{aligned} \quad (24)$$

This can be substituted in (21) to find ϕ_m which in turn will allow us to determine ξ , the position of the potential minimum from (20) and, finally, we can combine all this information in (19) to get the desired relation between x and ϕ .

For simplicity we demonstrate this for the unbiased case, $\phi_0 = 0$. Letting

$$R(0) = R(\phi_0) = V, \quad s = R_m/V, \quad \hat{\epsilon} = \epsilon/V^2,$$

and introducing

$$a = 9en_0 l^2 / 8\epsilon_0 m V^2,$$

(21) becomes

$$a^{1/2} \simeq (1-s)^{1/2} \left((1+2s) + \frac{3}{2}\hat{\epsilon} \left(\frac{3}{4s} - \frac{1}{4s^2} - \frac{1}{8s^2} - \frac{3 \sec^{-1} s^{-1/2}}{8s^{3/2}(1-s)^{1/2}} \right) + O(\hat{\epsilon}^2) \right). \quad (25)$$

We note that to leading order ($\epsilon = 0$), (24) reduces to the usual expression for cold beams. By including the $O(\hat{\epsilon})$ corrections we can find the first-order correction to the SCL. This is the value of a for which (24) has a double root. We find

$$a = a_0 + \hat{\epsilon}a_1 + \dots, \quad s = s_0 + \hat{\epsilon}s_1 + \dots,$$

and substitution yields

$$a_0 = 2, \quad s_0 = \frac{1}{2},$$

Eliminating ξ we find

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and substitution yields

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(A 3) can have either sign. Assuming that the potential achieves a unique minimum $\rho = \rho_m$ at some location $x = \xi$, we find that particles that arrive at ξ with zero velocity must start at $x = 0$ with velocity $u_m(0)$ such that

$$\frac{1}{2}u_m^2(0) + \frac{e}{m}\rho_0 = \frac{1}{2}u_m^2(x) + \frac{e}{m}\rho(x) = \frac{e}{m}\rho_m.$$

At each point $x < \xi$ particles with velocities less than u_m will be reflected before reaching $x = \xi$ while faster particles will be transmitted across the potential barrier. To the left of the potential minimum therefore, we shall have particles with velocities larger than $((2e/m)(\rho_m - \rho(x)))^{\frac{1}{2}}$ composing the flow that will be transmitted, and particles with velocities in the range

$$|u| < ((2e/m)(\rho_m - \rho(x)))^{\frac{1}{2}}$$

composing the counterstreaming flow (figure 1). To the right of the potential minimum we have only particles with velocities larger than u_m , so that the potential minimum has filtered out of the flow particles whose energies were too small to traverse it.

REFERENCES

- DAVIDSON, R. C. 1974 *Theory of Non-neutral Plasmas*. Benjamin.
 COUTSIAS, E. A. & SULLIVAN, D. J. 1983 *Phys. Rev. A*, **27**, 1535.
 ERDELYI, A. 1956 *Asymptotic Expansions*. Dover.
 GENONI, T. C. & PROCTOR, W. A. 1980 *J. Plasma Phys.* **23**, 129.
 MILLER, R. B. 1982 *An Introduction to the Physics of Intense Charged Particle Beams*. Plenum.
 READ, M. E. & NATION, J. A. 1975 *J. Plasma Phys.* **13**, 127.
 VORONIN, V. S., ZOZULYA, YU. T. & LEBEDEV, A. N. 1972 *Soviet Phys. Tech. Phys.* **17**, 432.