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DISORDER, RENORMALIZABILITY, THETA FUNCTIONS AND CORNU SPIRALS

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Turning and turning in the widening gyre
The falcon cannot hear the falconer;
Things fall apart; the center cannot hold....

W.B. Yeats

The partial sums of the trigonometrical series $S=\sum_{-\infty}^{\infty} e^{i\pi\alpha n^p}$ for α small mod 2 lead to distributions of points in the complex plane $\mathbb{C}$ composed of Cornu-like spirals. For $p \neq 2$ and $p>1$ the number $\mathcal{N}_l$ of points in the lth spiral is $\mathcal{O}(l^e)+1$, where $e=(2-p)/(p-1)$. If $p=2$, then $\mathcal{N}_l=[\frac{1}{2}+(l+\frac{1}{2})/\alpha]-[\frac{1}{2}+(l-\frac{1}{2})/\alpha]$, where $[x]$, denotes the greatest integer in x. Thus $\mathcal{N}_l$ increases (decreases) with n if $p>2$ $(1<p<2)$. We thus have two types of disorderly behavior for $p>1$, $p\neq 2$ (and p not an integer if α is rational). For $p>2$ the number of points per spiral decreases to 1 and the points are distributed in a seemingly random fashion; while for $1<p<2$ the number of points per spiral increases to $+\infty$ with $l\to\infty$, and the orientation of the spirals changes in seemingly random fashion with l. There is order if p is an integer: if α is rational the pattern is a pseudo-periodic arrangement of spirals which, depending on α, also may be composed of spirals; if α is a quadratic irrational and $p=2$ (i.e. α can be represented by a periodic continued fraction), then the pattern is renormalizable. In the second case the numbers s_l of points between the mid-points (points of inflection) of successive spirals form a Beatty sequence [16]. The proof of renormalizability depends upon Hardy and Littlewood's approximate functional equation for the theta function [6]. Similar behavior is exhibited by n-dimensional generalizations of the sum S above.

1. Introduction

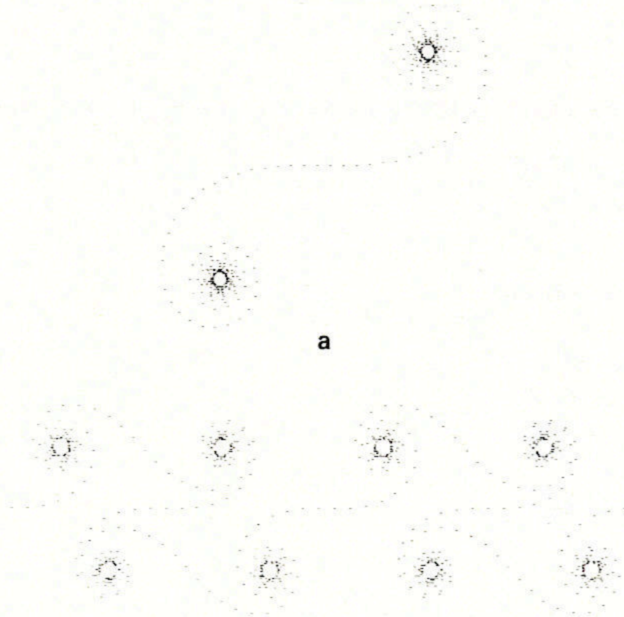
We discuss some phenomena exhibited by sums

$$S_N=\sum_{0}^{N} \exp\left[i\pi(\alpha n^p+\beta n)\right] \tag{1.1}$$

of unit vectors in $\mathbb{C}$ as $N\to\infty$. The phenomena exhibited by (1) for various choices of α, β and p have implications for numerical computations and disorder in dynamical systems and are beautiful as well. The sums (1) arise in several contexts, for example, as Riemann sums approximating oscillatory integrals typified by

$$I_p(z)=\int_0^z \exp(i\pi s^p/2)\,ds. \tag{1.2}$$

If $p=2$ $(\beta=0)$, (2) is the Fresnel integral, and $I_2(\infty)=e^{\pi i/4}$. The graph in $\mathbb{C}$ of (2) is then a Cornu spiral (C-spiral) [8] (fig. 1a). For a discussion of the C-spiral in relation to geometrical optics, see ref. 5. The approximation of (2) by Riemann sums is reliable provided the stepsize is



a

b

Fig. 1. (a) A Cornu spiral: $\Sigma_0^{1002} \exp(i\pi n^2/501)$; b) a periodic lattice of simple Cornu spirals: $\Sigma_0^{834} \exp(4i\pi n^2/417)$. In these figures one point is plotted corresponding to the value of each partial sum.

smaller than the period of the integrand. Since for $p > 1$ this period decreases as u increases, the approximation of (2) by Riemann sums of a given stepsize breaks down for z large. There is a $z = z_{\max}$ for which there is optimum agreement between (2) and its approximating Riemann sum, and the sum forms a spiral very close to the true C-spiral which is the graph of (2). But, for $z \to \infty$ and fixed stepsize the Riemann sums diverge. At first glance, it is surprising that, depending on the stepsize, the Riemann sums form new spirals, infinitely many spirals of spirals... for $z = \infty$.

If $p = 2$ and $\beta = 0$, the elementary spirals are all of the same size as the initial spiral, while for $p \neq 2$ their size varies with n, decreasing (increasing) for $p > 2$ $(1 < p < 2)$. Also, at first sight, on the screen of a Macintosh computer, these spirals seem to wander randomly over $\mathbb{C}$. Closer investigation reveals this "wandering" has structure. Indeed, the self-similar shapes of the spirals of spirals, ..., (fig. 2) suggest that the series (1) for $p = 2$ and $\beta = 0$ might be summable in blocks. For special values of α we prove this is approximately so in section 3 below. If we think of the concentrations of points in the dense windings of the elementary spirals as points, then the infinite sum collapses to a new one in which the building blocks are vectors of fixed length joining these points (the new points were spirals of elementary points in the original sum). The varying size of

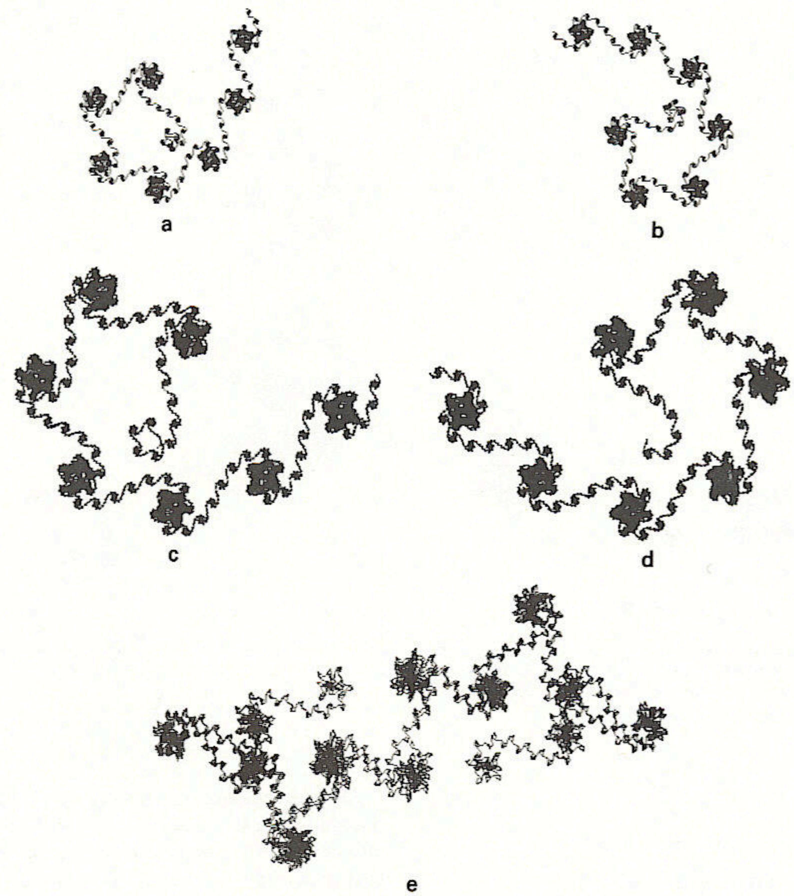


Fig. 2. Parts of $\Sigma_0^\infty \exp(\mathrm{i}\alpha\pi n^2)$, done in 32 bit precision TML Pascal. Shown in (a) are full or parts of kth level self-similar spirals of points with $\alpha = 13 - \sqrt{168}$ for $k = 1,2,3$; n varies from 1 to 5000. Shown in (c) is a continuation of a) for $n \leq 12500$ showing part of a 4th level self-similar spiral. Shown in (b) and (d) are anti-self-similar spirals with $\alpha = \sqrt{170} - 13$ for the same ranges of n. In these and the remaining figures successive terms in the sequence of partial sums are connected by straight line segments of unit length. (e) Part of the lattice formed by replacing α by 4α in figs. (b), (d) with $\alpha = -13 + \sqrt{170}$.

spirals for $p \neq 2$ makes a similar procedure much harder to implement; consequently, results for $p \neq 2$ lack the generality of those for $p = 2$. Nevertheless, for rational values of $p \neq 2$ and special values of $\alpha(\beta = 0)$, we are able to produce intervals of n in which spirals of spirals are formed; see fig. 3.

Some of the behavior just discussed was first observed by us in numerical experiments performed on a Macintosh computer to study the real

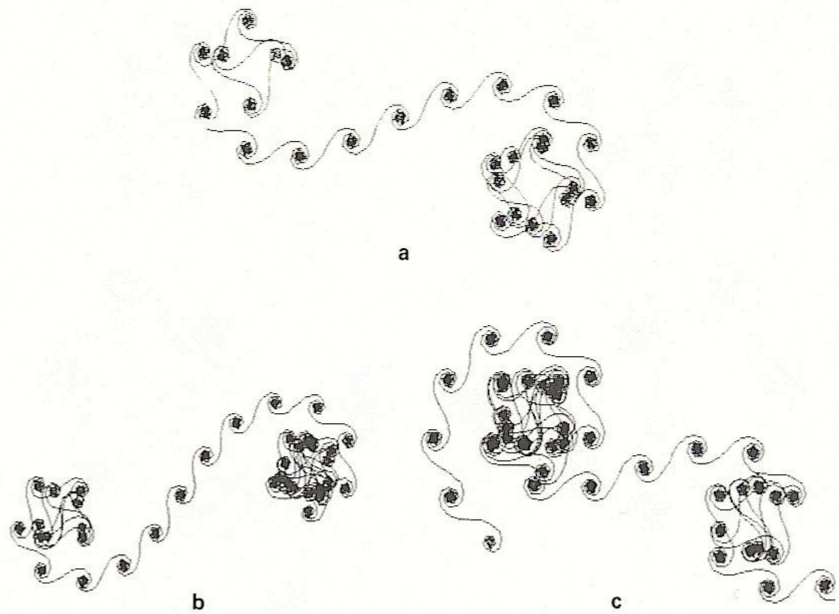


Fig. 3. A portion of the graph of $\{z_n\}$ showing spirals of spirals of points, all with $n_c = 10^6$ for: a) $n = 997750$ to $n = 1000350$, with $p = 1.5$ and $\alpha = 8(30 - \sqrt{899})/3$; b) $n = 999250$ to $n = 1000800$ with $p = 2.5$ and $\alpha = 8(20 - \sqrt{399})/15000$; c) $n = 999500$ to $n = 1000550$ with $p = 3$ and $\alpha = 24 \times 10^6(20 - \sqrt{399})$.

map

$$\begin{aligned} x_{n+1} &= x_n + \cos\left[\pi(\alpha n^p + \beta n)\right], \\ y_{n+1} &= y_n + \sin\left[\pi(\alpha n^p + \beta n)\right]. \end{aligned} \tag{1.3}$$

This map was shown to N.D.K. in 1983 by E. Bombieri.

Remark. We are indebted to a referee for the following observation. Since there is no coupling among the real and imaginary parts of $z_n = x_n + \mathrm{i}\, y_n$, the map (3) may be thought of as a periodically driven, one-dimensional mapping with polynomially increasing time: let $\beta = 0$ for simplicity, let the time be Θ, and let $x_{n+1} = x_n + \cos(\alpha\pi\Theta_n)$, $\Theta_{n+1} = (\Theta_n^{1/p} + 1)^p$.

Although the map (3) is, in the sense remarked one-dimensional, introducing a second dimension allowed us to use geometric ideas, especially curvature of the graph or pattern formed. This led us to concrete predictions, such as a renormalization transformation, which would otherwise have been difficult to discover. A sum of cosines of slowly varying argument is easier to visualize and investigate when unfolded as a sum of unit vectors in $\mathbb{C}$.

Computer graphics played an essential role in this work, both in verifying (or not) our conjectures and in suggesting usually unexpected new lines of investigation. The case $p = 2$ is fairly well understood from the number theorist's point of view [1, 6, 10–13, 16–18]; also see [9]. If $p = 2$, (1) relates to Gauss sums and to properties of Jacobian theta functions and their multidimensional generalizations [2, 6] near and on the real axis. However, complete, number-theoretic results appear to be unknown for $p \neq 2$.

Our main results use the concept of discrete curvature to establish a renormalization transformation, both for $p = 2$ and $p \neq 2$. This renormalization led us to the cases in which we found self-similarity in the graph of (1.1). For $p = 2$ and

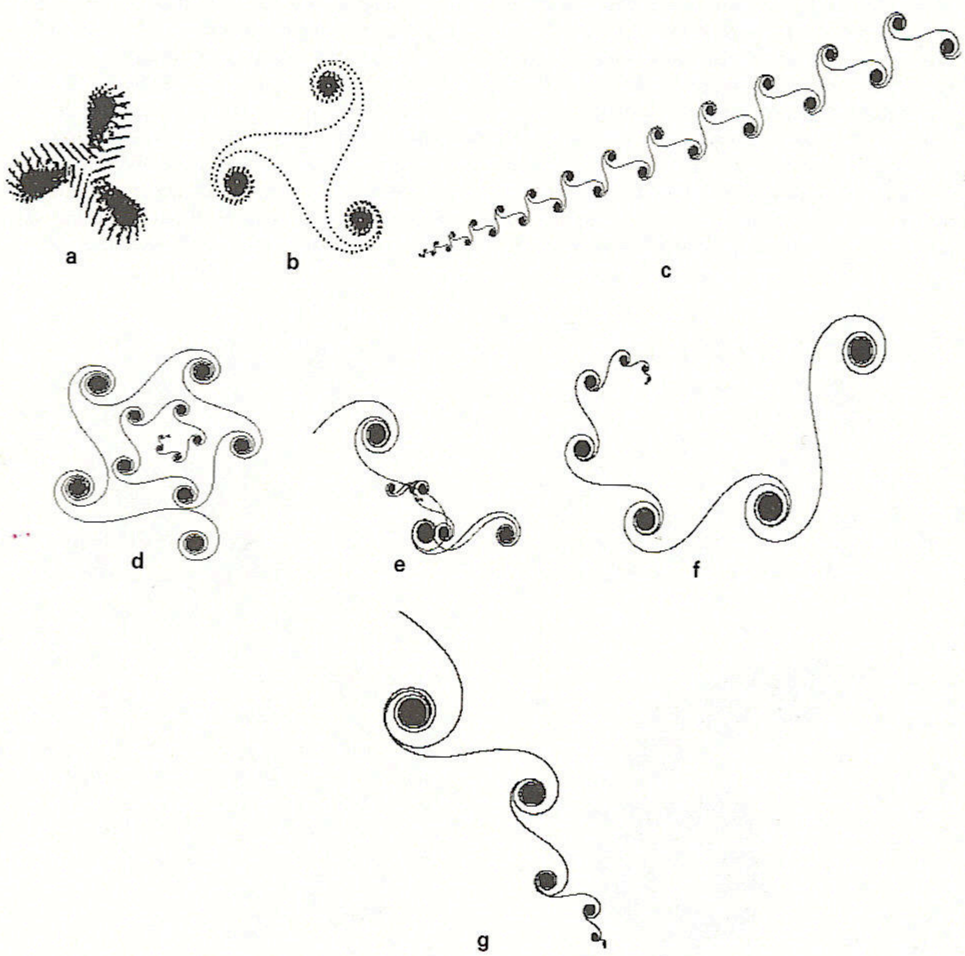


Fig. 4. The parameters p, α, and N, the number of vectors added in $\Sigma_1^N \exp(i\alpha\pi n^p)$ and the parameters p', α' of the renormalized sum $\Sigma_1^N \exp(i\alpha'\pi n^{p'})$ are in (a)–(g), respectively: $p = \frac{3}{2}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}$; $\alpha = \frac{4}{3}, \frac{4}{3}, \frac{6}{4}, \frac{8}{5}, \frac{10}{6}, \frac{12}{7}, \frac{14}{8}$; $N = 3000, 1.0 \times 10^5$ through 1.03×10^5, $\approx 2.5 \times 10^4, 5.0 \times 10^4, 1.0 \times 10^5, 9.99 \times 10^5, 2.227 \times 10^6$; $p' = 3, 3, 4, 5, 6, 7, 8$; $\alpha' = -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{4}, -\frac{2}{5}, -\frac{2}{6}, -\frac{2}{7}, -\frac{2}{8}$.

$\beta = 0$ we use a delicate result of Hardy and Littlewood, Theorem 2.128 in [6], the approximate functional equation for the theta function, to prove that the renormalization is correct up to an error that is at most a constant multiple of $\alpha^{-1/2}$. We thus predict that for $\alpha = m - (m^2 - 1)^{1/2}$ (m a positive integer), $p = 2$ and $\beta = 0$, the spirals are arranged in a self-similar sequence of ascending scales, and we establish relevant scaling laws. The self-similar pattern is easiest to observe numerically for $p = 2$, but for $p \neq 2$ we establish generalizations as well as some representative graphs; see figs. 3 and 4. If $\alpha = (m^2 + 1)^{1/2} - m$, $p = 2$, and $\beta = 0$, then (1) also yields spirals of spirals of... points, but the spirals at successive levels are oppositely oriented (α is replaced by $-\alpha$). Further, the renormalization of (1) for $p = 2$ can be extended to a multi-dimensional generalization of (1); see fig. 5. Use of discrete curvature also enables us to define the number of points $\mathcal{N}_l(\sigma_l)$ per spiral. It then follows for $p = 2$ that the sequence $\{\sigma_l\}$ is a Beatty sequence [1, 13, 16] for each α that yields a self-similar, or anit-self-similar, pattern; namely, if $\alpha = m - (m^2 - 1)^{1/2}$ (or $(m^2 + 1)^{1/2} - m$), the sequence of integers $\{\sigma_l\}$ is a Beatty sequence, of the integers $2m$ and $2m - 1$ (or $2m$ and $2m + 1$). D.H. Lehmer [10] in 1976 investigated the sums S_n in the case where $\alpha = 1/N$ for various classes of integers N, and he provided pictures of various C-spirals that result. He was perhaps the first mathematician to realize that there is geometric content in the sequences $\{S_N\}$.

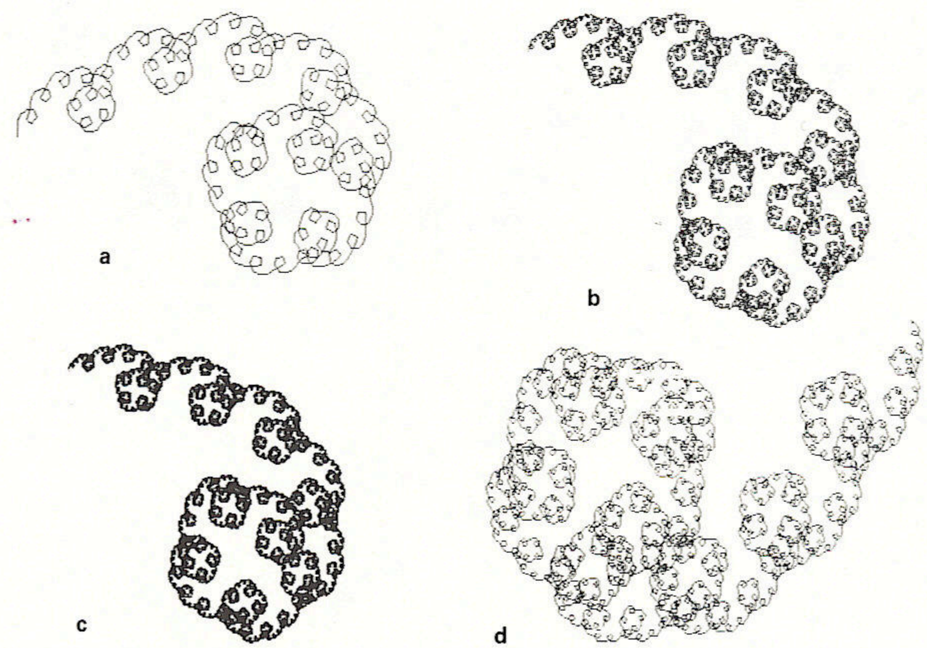


Fig. 5. (a)–(c) The graph of $\Sigma_{n_5=0}^{8}\Sigma_{n_4=0}^{8}\Sigma_{n3=0}^{8}\Sigma_{n2=0}^{8}\Sigma_{n_1=0}^{8}\exp(i\pi\alpha n^2)$ with $\alpha = 15 - (224)^{1/2}$: five orders of self-similar spirals. The largest spiral in (a) is of order 3, the largest in (b) is of order 4, and the largest in (c) is of order 5. (d) The graph of $\Sigma_{N_4=0}^{7}\Sigma_{N_3=0}^{7}\Sigma_{n_2=0}^{7}\Sigma_{n_1=-1}^{7}\exp(i\pi\alpha n^2)$ with $\alpha = 13 - (168)^{1/2}$.

While for $p=2$ order in the graph of (1) is global if it is present, for $p \neq 2$ order is only local (in n) and, in some cases intermittent if α is irrational. A question of interest from the point of view of dynamical systems, and which still needs to be pursued, is to define average numbers that describe the distribution of points in $\mathbb{C}$ so as to characterize the transition from an ordered to a disordered pattern. M. Mendès-France [1] has given a good review of what is known in this direction.

This paper is arranged as follows. In section 2 we discuss the geometric ideas that lead to our prediction of renormalizability and scaling laws. In section 3 we discuss (1) for $p=2$, deriving self-similarlity results and some results special to this case, such as the rule generating the number of points per spiral in the renormalizable case and some generalizations to multiple sums. In section 4 we present results for $p \neq 2$. In our closing section 5 we summarize our results, present a discussion, and state some conjectures.

2. Discrete spirals

We consider vectorial addition in $\mathbb{C}$ of the collection of points $\{S_N\}$ $(N \in g)$ with

$$S_N = \sum_0^N e^{i\pi\alpha n^p} \equiv \sum_0^N z_n, \tag{2.1}$$

where $z_n = \exp(i\pi\alpha n^p)$ is a unit vector in the complex plane, $p>1$, and $0<\alpha \ll 1$ (later we shall generalize to $0<|\alpha| \bmod 2 \ll 1$). We let

$$\phi_n = \pi\alpha n^p \text{ and } \Delta\phi_n = \phi_{n+1} - \phi_n \quad (n=0,1,2,\dots). \tag{2.2}$$

Definitions. (i) The *local discrete radius of curvature* R_n of the pattern generated by the points S_n is the radius of the circle passing through the three consecutive points S_{n-1}, S_n, S_{n+1}; namely,

$$R_n = \tfrac{1}{2}|\csc(\Delta\phi_n/2)|. \tag{2.3}$$

ii) For $n = n_{2l}$, z_n is the *mid-vector* (*point of inflection*) of the lth *C*-spiral of the pattern S_∞ if (see fig. 6b)

$$\Delta\phi_{n_{2l}-1} \le 2l\pi < \Delta\phi_{n_{2l}}. \tag{2.4}$$

iii) Similarly, for $n = n_{2l+1}$, z is the *end-vector* (*cusp*) of the lth C-spiral and for $n = n_{2l+1}+1$, z_n is the *initial-vector* (*cusp*) of the $(l+1)^{st}$ C-spiral if

$$\Delta\phi_{-1+n_{2l+1}} \le (2l+1)\pi < \Delta\phi_{n_{2l+1}}; \tag{2.5}$$

see fig. 6a.

Combining the inequalities (2.4-5) and using the binomial theorem, we find that approximately, for l large,

$$n_l \approx \left[(l/p\alpha)^{1/(p-1)}\right], \tag{2.6}$$

where the square brackets denote the greatest in-

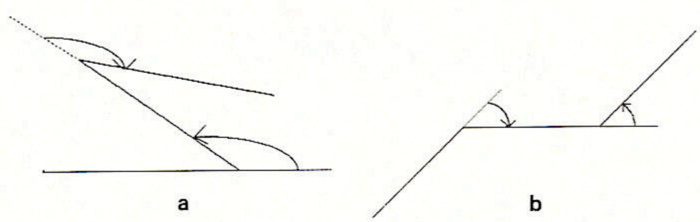


Fig. 6. a) An end (beginning) point or cusp; b) a point of inflection.

teger function. From the behavior of R_n as n increases we obtain the following theorem:

Theorem. For $0 < \alpha \ll 1 \bmod 2$ the pattern formed by the points

$$\{S_{n_{2l-1}}, \ldots, S_{n_{2l}}, \ldots, S_{n_{2l+1}}\}$$

is a double spiral, resembling a Cornu spiral from diffraction theory [5, 8]. The number of points $\mathcal{N}_l$ in the lth C-spiral is

$$\mathcal{N}_l = n_{2l+1} - n_{2l-1}. \tag{2.7}$$

For l large and $p > 1$ ($p \neq 2$),

$$\mathcal{N}_l = 1 + (p-1)^{-1}(2l^{2-p}/p\alpha)^{1/(p-1)} \times \{1 + \mathcal{O}(1/l)\}. \tag{2.8}$$

For $p = 2$,

$$n_k = [\tfrac{1}{2} + k/(2\alpha)]$$

and (2.9)

$$\mathcal{N}_l = [\tfrac{1}{2} + (l + \tfrac{1}{2})/\alpha] - [\tfrac{1}{2} + (l - \tfrac{1}{2})/\alpha] \approx 1/\alpha.$$

It follows from the above theorem that for $1 < p < 2$ ($p > 2$) the number of points per C-spiral decreases (increases) like $\mathcal{O}(l^e)$, where $e = (2-p)/(p-1)$. This can be observed in figs. 4 and 7. For $p = 2$ the relation of $\mathcal{N}_l$ to the size of a C-spiral and its application to obtain a renormalization transformation are presented in the next section.

3. The quadratic case: $p = 2$

For $p = 2$ the sum (2.1) is related to the Jacobian theta function $\theta_3(v, \tau)$ defined as

$$\theta_3(v, \tau) = \sum_{-\infty}^{\infty} e^{i\pi\tau n^2 + 2\pi i n v} \quad (\mathrm{Im}(\tau) > 0). \tag{3.1}$$

In our problem, $\mathrm{Im}\,\tau = 0$, so that the infinite series expression for θ_3 diverges. Our study of the partial sums (2.1) can be thought of as a study of partial sums of the series for θ_3 at its natural boundary. The problem of the behavior of a trigonometric series at a natural boundary was first posed systematically by Fatou [4]. Hardy and Littlewood [6] were able to settle several questions concerning the behavior of the theta functions on the real τ-axis by deriving approximate functional equations satisfied by partial sums of their expressions in series. These approximate functional equations are extensions of the functional equation, known as Jacobi's imaginary transformation [2; pp. 72–80], relating $\theta_3(v, \tau)$ to $\theta_3(v/\tau, -1/\tau)$ for $\mathrm{Im}(\tau) > 0$. For $v = 0$ Jacobi's transformation be-

a b

Fig. 7. a) The number of points per spiral and their size decreases: $\Sigma_{25}^{1250} \exp(i\alpha\pi n^4)$ with $\alpha = 10^{-8}\sqrt{2}$. b) The number of points per spiral and their size increases: $\Sigma_0^{5730} \exp(i\alpha\pi n^{1.01})$ with $\alpha = 750$. The last 1200 points are in the largest spiral.

comes $(\tau/\mathrm{i})^{1/2}\theta_3(0,\tau)=\theta_3(0,-1/\tau)$ or

$$\sum_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\pi\tau n^2}=(\mathrm{i}/\tau)^{1/2}\sum_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\pi n^2/\tau}. \tag{3.2}$$

Obviously, (3.2) diverges if $\operatorname{Im}\tau=0$; one has instead the approximate formula [6, Thm. 2.128, p. 209], adapted to our case

$$\sum_{0}^{n} \mathrm{e}^{\mathrm{i}\pi\tau k^2}-(\mathrm{i}/\tau)^{1/2}\sum_{0}^{[n\tau]} \mathrm{e}^{-\mathrm{i}\pi k^2/\tau}=\mathcal{O}(\tau^{-1/2}). \tag{3.3}$$

Hardy and Littlewood used this remarkable result to get several estimates on the rate of divergence of the sum (2.1) for $p=2$. For example, if τ is irrational, then the sum grows with n like $n^{1/2}$. More precise estimates depend upon the growth rate of the coefficients in the continued fraction expansion of τ. Hardy and Littlewood also studied (1) for $p=2$ in relation to the continued fraction expansions of τ. Thus it appears strange that, although they studied these sums for τ a quadratic irrationality, they missed their renormalizability. Their results were improved and extended by van der Corput [17, 18] to a wide range of sums of the type $\Sigma g(n)\mathrm{e}^{2\pi\mathrm{i}f(n)}$. A short proof of (3.3) was given by Mordell [12].

Here we give a derivation of the approximate functional equation based upon the geometric properties of C-spirals. We begin by summing the terms in the block $\mathscr{B}_l$ of terms that correspond to the lth spiral,

$$\mathscr{B}_l=\sum_{k=1}^{n_{2l+1}-n_{2l-1}} z_{n_{2l-1}+k}. \tag{3.4}$$

Clearly,

$$S_{n_{2l-1}+m}=S_{n_{2l-1}}+\sum_{k=1}^{m} z_{n_{2l-1}+k}$$
$$(1\le m\le n_{2l+1}-n_{2l-1}). \tag{3.5}$$

It is convenient to express each term within the lth spiral in terms of the mid-vector $z_{n_{2l}}$,

$$\begin{aligned}\beta_l&=\sum_{k=1}^{n_{2l+1}-n_{2l-1}} Z_{k+n_{2l}-(n_{2l}-n_{2l-1})}\\&=\sum_{k=1}^{n_{2l+1}-n_{2l}} \exp\mathrm{i}\pi\alpha(n_{2l}+k)^2.\end{aligned} \tag{3.6}$$

If we define δ_l by $\delta_l=(l/\alpha)-[\frac{1}{2}+(l/\alpha)]$, then

$$-\tfrac{1}{2}\le\delta_l\equiv(l/\alpha)-\left[\left(-\tfrac{1}{2}+l\right)/\alpha\right]<\tfrac{1}{2}. \tag{3.7}$$

If we recall (2.8) $(n_{2l}=[\frac{1}{2}+(l/\alpha)])$, we may write the exponent in (3.6) as

$$\begin{aligned}\alpha(n_{2l}+k)^2&=\alpha\left(k+(n_{2l}-l/\alpha)+l/\alpha\right)^2\\&=\alpha\left(k+\tfrac{1}{2}+[l/\alpha]-l/\alpha\right)^2\\&\quad-l^2/\alpha+2l(k+n_{2l}),\end{aligned} \tag{3.8}$$

so that, finally,

$$\mathscr{B}_l=\mathrm{e}^{-\mathrm{i}\pi l^2/\alpha}\sum_{1+n_{2l-1}-n_{2l}}^{n_{2l+1}-n_{2l}} \exp\left[\mathrm{i}\pi\alpha(k-\delta_l)^2\right]. \tag{3.9}$$

To estimate the sum in (3.8), we approximate the integral (1.2) by a Riemann sum

$$\begin{aligned}I_2(z)&=\int_0^z \exp(\mathrm{i}\pi s^2/2)\,\mathrm{d}s\\&=\tfrac{1}{2}\int_{-z}^{z} \exp(\mathrm{i}\pi s^2/2)\,8rds\\&=\tfrac{1}{2}\Delta s\sum_{1-M}^{M} \exp\left[\mathrm{i}\pi(k\Delta s-\xi)^2+\mathcal{O}(\Delta s)\right],\end{aligned} \tag{3.10}$$

where Δs is the stepsize, $M=z/\Delta s$, and $-\frac{1}{2}\Delta s\le\xi<\frac{1}{2}\Delta s$. By letting $\frac{1}{2}\Delta s=\alpha^{1/2}(\ll 1)$, $\xi=\delta_l/\alpha$ and $M=n_{2l+1}-n_{2l}\approx n_{2l}-n_{2l-1}\approx\frac{1}{2}\alpha$, we arrive at the estimate

$$\sum_{1+n_{2l-1}-n_{2l}}^{n_{2l+1}-n_{2l}} \exp\left[\mathrm{i}\pi\alpha(k-\delta_l)^2\right]$$
$$\approx \alpha^{-1/2}\int_{(1+n_{2l-1}-n_{2l})\alpha^{1/2}}^{(n_{2l+1}-n_{2l})\alpha^{1/2}} \exp(\mathrm{i}\pi s^2)\,\mathrm{d}s$$
$$= \alpha^{-1/2}\int_{-\frac{1}{2}\alpha^{-1/2}}^{\frac{1}{2}\alpha^{-1/2}} \exp(\mathrm{i}\pi s^2)\,\mathrm{d}s + \mathcal{O}(1). \quad (3.11)$$

But the last integral converges to $e^{i\pi/4}$ as $\alpha \to 0$. Thus, by (3.9) and (3.11), we conclude that

$$\alpha^{1/2}\mathrm{e}^{-\pi\mathrm{i}/4}\sum_{n=-1}^{n_{2l+1}} \exp(\mathrm{i}\pi\alpha n^2)$$
$$\approx \sum_{n=0}^{l} \exp(-\mathrm{i}\pi n^2/\alpha). \quad (3.12)$$

The result of Hardy and Littlewood [6] guarantees that the error in (3.12) is $\mathcal{O}(1)$.

The block summation in (3.12) embodies the intuitive idea that the clusters (elementary C-spirals) of (2.1) may be thought of as "points" of a new sum of vectors joining these "points". These vectors have length approximately $\alpha^{-1/2}$ (the "size" of a C-spiral), and are rotated counter-clockwise by $\pi/4$ from the mid-vector of their corresponding C-spiral. The new sum is of the same type as the original one except that α is replaced by $-1/\alpha$; that is, summation by blocks left the form of the sum invariant but "renormalised" the parameter α and scaled the new sum by the factor $\alpha^{-1/2}e^{\pi i/4}$.

It is natural to ask under what conditions will the renormalized sum exhibit behavior similar to the original sum (i.e., C-spirals and renormalizability or self-similarity). Since the arguments we have presented rely essentially on $\alpha \bmod 2$ being small, while for $\alpha \bmod 2$ not small there is apparent disorder in the distribution of the S_N, it follows that the desired condition for our analysis of renormalization to hold is

$$|-1/\alpha| \bmod 2 \ll 1. \quad (3.13)$$

We stress that the condition $\alpha \bmod 2$ be small is only needed if we insist on the appearance of C-spirals in the pattern generated by a sequence $\{S_N\}$, since the discussion in section 5 below shows, when it is applied to the case $p=2$, that the condition $\alpha \bmod 2$ small is not necessary for self-similarity and order.

Returning to the renormalization map

$$(-1,1)\setminus 0^{\pm}, \quad \alpha \mapsto -1/\alpha \bmod 2, \quad (3.14)$$

we look for conditions on α so that (3.13) is satisfied. The mapping (3.14) associates to every number $\alpha = \alpha_0$ in $(-1,1)$ the sequence $\{\alpha_i\}$ of iterates of the map. If α is rational, the sequence terminates with a 0 or with an infinite sequence of 1's. If α is irrational, then so are all the α_i. Indeed, if α has the continued fraction expansion

$$\alpha \equiv \alpha_0 2n_0 + (1/2n_1 + (1/2n_2 + (1/n_3 + \cdots$$
$$\equiv (2n_0, 2n_1, 2n_2, \dots)$$
$$(-1<\alpha<1;\ \text{each } n_i > 0), \quad (3.15)$$

then the ith iterate α_i is given by

$$\alpha_i = (-1)^i(2n_i, 2n_{i+1}, 2n_{i+2}, \dots).$$

Our renormalization argument requires all $\alpha_i \bmod 2$ to be sufficiently small. It follows that all numbers α whose continued fraction expansions are composed of sufficiently large even integers will lead to infinitely renormalizable patterns $\{S_N\}$ in $\mathbb{C}$. If α is also a quadratic irrational, namely, if its continued fraction becomes periodic with period k eventually, then the pattern generated by $\{S_N\}$ in $\mathbb{C}$ is eventually self-similar of order k. The special case $n = n_0 = -n_1 = n_2 = -n_3, \dots,$ which implies

$$\alpha = \pm n - (n^2-1)^{1/2}, \quad (3.16)$$

leads to a pattern that is self-similar; see fig. 2a. The special case $n = n_0 = n_1 = n_2 = \dots,$ which implies

$$\alpha = (n^2+1)^{1/2} \pm n, \tag{3.17}$$

leads to a pattern that is anti-self-similar, i.e., a pattern whose spirals are inverted by a reflection at each iteration; see fig. 2b.

The similarity that is observed is precisely this: Let V_{l+1} be the vector from the origin to the center of the first spiral of order $l+1$ formed by the spirals of order l, and let the length of v_l be $\|v_l\|$. Then v_{l+1} makes an angle of $\pi/4$ (clockwise) to v_l and $\alpha^{1/2}\|v_{l+1}\| = \|v_l\|$. If we rotate the plane by $-\pi/4$ and contract by $\alpha^{1/2}$ with respect to the origin then the images of the points of inflection of the spirals of order l forming the first spiral of order $l+1$ almost coincide with the points of inflection of the spirals of order $l-1$ that form the first spiral of order l. In the case of anti-self-similarity v_{2lb} makes an angle of $-\pi/4$ with v_{2l-1}; v_{2l+1} makes an angle of $\pi/4$ with v_{2l}. The self-similarity is never exact (α is irrational), but it improves as $\alpha \to 0$.

To cover cases of other irrational numbers, we allow negative integers in their continued fraction expansions. Since, it can be shown that every irrational number has a unique continued fraction expansion of the form (3.15) if we allow negative integers n_i, the statements made above cover any irrational in $(-1,1)$. Thus, the numbers

$$\alpha = (2n, -2n, 2n, -2n, \dots) = 2n - 1/\alpha,$$

correspond to patterns generated by $\{S_N\}$ in $\mathbb{C}$ that are self-similar. An example is shown in fig. 2a.

The case of anti-self-similarity (3.16) is the case of quadratic irrationalities considered by Stolarsky [16]. As he shows, in this case the differences

$$s_k = [(k+1)/\alpha + \gamma] - [k/\alpha + \gamma] \qquad (k = 0, \pm 1, \pm 2, \dots)$$

form a (renormalizable) Beatty sequence. Namely, if $\alpha = (n^2+1)^{1/2} - n$ and $\gamma = \frac{1}{2}$, then it can be shown that $\{s_l\}$ $(=\{\sigma_l\})$ is composed of the integers $2n$ and $2n+1$ and is left invariant by the nonuniform substitution

$$2n\{(2n-1)-\text{terms}\}, 2n+1 \to 2n \quad \text{and}$$
$$2n\{(2n-\text{terms})\}, 2n+1 \to 2n+1$$

and its inverse. For example, if $n=1$, then $\{s_l\}$ is the sequence

$$\dots, 3,\ 2,\ 3,\ 2,\ 2,\ 3,\ 2,\ 3,\ 2,\ 3,\ 2,\ 2,\ 3,\ 2,\ 3, \cdots \quad (l \geq 2).$$
$$\to \quad 2, \quad 3, \quad 2, \quad 2, \quad 3, \quad 2, \cdots .$$

If $\alpha = n - (n^2-1)^{1/2}$ and $\gamma = \frac{1}{2}$, then $\{s_l\}$ is an ordering of the integers $2n$ and $2n-1$ that remains invariant under the mapping

$$2n\{(2n-1)-\text{terms}\}, 2n-1 \to 2n \quad \text{and}$$
$$2n\{(2n-\text{terms}\}, 2n-1 \to 2n-1$$

and its inverse. For example, if $n=2$,

$$\dots, 3,\ 4,\ 4,\ 4{,}3,\ 4,\ 4,\ 4{,}3,\ 4,\ 4,\ 4{,}3,\ 4,\ 4,\ 4,\ 4{,}3, \dots$$
$$\to \quad \dots, 4, \quad 4, \quad 4, \quad 3, \cdots .$$

Since the number of points σ_l between the midpoints of the $(l+1)$st and the lth C-spiral generated by $\{S_N\}$ is $[\frac{1}{2} + 2(l+1)/(2\alpha)] - [\frac{1}{2} + 2l/(2\alpha)] = [\frac{1}{2} + (l+1)/\alpha] - [\frac{1}{2} + l/\alpha] = s_l$, we see that self-similar sequences are associated with our self-similar and anti-self-similar patterns. Finally, we remark that Beatty sequences were employed by de Bruijn [1] in his study of Penrose's five-fold tilings of the plane [15].

We close this section with a generalization of some of the previous results to multi-indexed sums. Jacobi's θ_3 has been generalized and used to study the number of ways that an even integer can be represented as a sum of squares [2, chap. XI]. This enables us to generalize (1.1) and, heuristically, its renormalization for $p=2$. Let $Q(x) = Q(x_1, \dots, x_m) = \sum_{k,l=1}^{m} \alpha_{kl} x_k x_l$ be a positive-definite quadratic form associated with the symmetric, positive-definite, real matrix $(\alpha_{k,l})$ with determinant D. Let $(\alpha'_{k,l})$ be the inverse of the matrix $(\alpha_{k,l})$, and let $Q'(x)$ be the quadratic form associated with $(\alpha'_{k,l})$. Define

$$\theta(\tau, Q) = \sum_{n_1,\dots,n_m=1}^{\infty} \exp(\mathrm{i}\pi\tau Q(n_1,\dots n_m)) \qquad (\mathrm{Im}(\tau) > 0).$$

Then $\theta(\tau, Q) = [\theta_3(0, \tau)]^m$, and $\theta(-1/\tau, Q) = (\tau/i)^{m/2} D^{-1/2} \theta(\tau, Q')$. For $\tau = \alpha = m - (m^2-1)^{1/2}$ and $(\alpha_{k,l})$ the identity matrix we obtain the formal identity $\theta(-1/\alpha, Q) = (\alpha/i)^{m/2}\theta(\alpha, Q)$. This identity might possibly be justified as an approximation for partial sums as was done above in the case of (2.1). Iterations of the map $z_{n+1} = z_n + \exp(i\pi\alpha n^2)$, where now n is a multi-index with components $n_1, \dots, n_m$ and $n+1$ means add 1 to some component of n, yield graphs exhibiting selfsimilarity or anti-self-similarity for α a quadradic irrationality; see fig. 5.

4. The case $p > 1$, $p \neq 2$

We now give a derivation, in close analogy to that given above for $p = z$, of a functional equation for sums of the form

$$S_{a,b} = \sum_{a \le n \le b} \exp(\mathrm{i}\pi\alpha n^p) \quad (p \neq 2, p > 1). \quad (4.1)$$

Our result (4.6–7) below in which $p \mapsto p' = p/(p-1)$ agrees with the approximate functional equation given by van der Corput [18] for general sums of the form $\sum_{a \le n \le b} g(n) \mathrm{e}^{2\pi i f(n)}$ (when it is specialized to the sums (4.1) we consider). Since van der Corput's proof is long, intricate and extremely difficult to motivate, we believe that our simple argument demonstrates the power of geometric thinking in this context. We also apply the method of stationary phase to reduce our functional equation, if the range of n in (4.1) is further limited, to the case $p \mapsto 2$ so that we obtain C-spirals of C-spirals of points for these sums. The self-similarity obtained is only approximate, and we were able to find it only up through level 2: the C-spirals of C-spirals of points do not form C-spirals.

Motivated by the geometry seen on the Macintosh screen we again separate the sum (4.1) into blocks $\mathscr{B}_l$ corresponding to individual C-spirals,

$$\mathscr{B}_l = \sum_{1+n_{2l-1} n_{2l}}^{n_{2l+1} - n_{2l-1}} \exp[\mathrm{i}\pi\alpha(n_{2l} + k)p]. \quad (4.2)$$

Since, by (2.6), if l is large,

$$n_l \approx \left[(l/p\alpha)^{1/(p-1)}\right].$$

Then, assuming for the moment that $|n_{2l} - (2l/p\alpha)^{1/(p-1)}| < 1$, we have

$$\begin{aligned} \alpha(n_{2l} + k)^p &= \alpha\Big[(2l/p\alpha)^{1/(p-1)} \\ &\quad + \left(k + n_{2l} - (2l/p\alpha)^{1/(p-1)}\right)\Big]^p \\ &\approx \alpha(2l/p\alpha)^{p/(p-1)} + \alpha p(2l/(p\alpha)) \\ &\quad \times \left[k + n_{2l} - (2l/p\alpha)^{1/(p-1)}\right] \\ &\quad + \tfrac{1}{2}\alpha p(p-1)(2l/p\alpha)^{(p-2)/(p-1)}(k - \delta_l)^2 \\ &\quad + \dots, \end{aligned} \quad (4.3)$$

where we keep only the terms in the binomial expansion (4.3) involving the highest powers of the (large) number

$$L = (2l/p\alpha)^{1/(p-1)} \quad (4.4)$$

and in which

$$\delta_l = L - n_{2l} \quad (4.5)$$

with $|\delta_l| < 1$. Let

$$\gamma = \left(\tfrac{1}{2}\alpha p\right)^{-1/(p-1)} \quad (4.6)$$

We obtain from (4.3), after performing some algebraic manipulation, the approximation

$$\alpha(n_{2l} + k)^p \approx \gamma\left\{-2[(p-1)/p]\, l^{p/(p-1)} + (p-1)\, l^{(p-2)/(p-1)}(k - \delta_l)^2 + \dots\right\}.$$

Thus,

$$\mathscr{B}_l \approx \exp\{-2\pi i[(p-1)/p]\gamma\} \times \sum_{1+n_{2l-1}-n_{2l}}^{n_{2l+1}-n_{2l-1}} \exp\left[i\pi(p-1)(1/\gamma) \times l^{(p-2)/(p-1)}(k-\delta_l)^2\right]. \tag{4.7}$$

Using (3.12) to approximate the sum in (4.7), we obtain the approximate functional equation

$$\sum_{1+n_{2l_0+1}}^{n_{2l_1+1}} \exp[i\pi\alpha n^p] \approx \sum_{l=l_0}^{l_1} \mathscr{B}_l, \tag{4.8}$$

where

$$\mathscr{B}_l \approx \left\{[\gamma/(p-1)]\,l^{(2-p)/(p-1)}\right\}^{1/2} \times e^{i\pi/4}\exp\{-i\pi\alpha' l^{p'}\}, \tag{4.9}$$

$$\alpha' = -2[(p-1)/p]\gamma, \quad p' = p/(p-1). \tag{4.10}$$

Note that one must take care and understand that (4.8) holds only for large $l_0 < l_1$. We also observe that our block summation process has led to a new sum, the right-hand side of (4.8), in which α and p have been replaced by α' and p', respectively, while a change of phase by $\pi/4$ and a scale factor r_l have been introduced, where

$$r_l = \left\{[\gamma/(p-1)]\,l^{(2-p)/(p-1)}\right\}^{1/2}. \tag{4.11}$$

The search for approximately self-similar patterns is complicated in this case. Since α' is not necessarily a small number, the renormalized sum will exhibit regular behavior if p' is an integer and α' is rational. In this case the graph of the right-hand side of (4.8) is a portion of a pseudo-periodic graph of C-spirals. The graphs produced by (4.1) will have spirals that grow in size, according to (4.11), if $1 < p < 2$ and which decrease in size for $p \geq 2$. A series of patterns produced in this case for rational p's yielding $p' = 3, 4, \ldots, 8$, $\gamma = 1$, and special α's for which α' is rational are shown in fig. 4.

To find selfsimilar behavior in sums of the form (4.1) it turns out to make sense to look for self-similar patterns locally in n, that is, for n in the neighborhood of some fixed value n_c. We do this by utilizing some properties of Diophantine approximations and applying the method of stationary phase. Let n_c be a large integer and consider a sum

$$S = \sum_{n_c-a}^{n_c+b} \exp(i\pi\alpha n^p) \approx \exp(i\pi\alpha n_c^p) \sum_{n_c-a}^{n_c+b} \exp\left\{i\pi\alpha\left[p n_c^{p-1}k + \tfrac{1}{2}p(p-1)n_c^{p-2}k^2\right]\right\}. \tag{4.12}$$

In (4.12) the first neglected term in the exponent is of order $k^3 n_c^{p-3}$. Now let

$$\alpha p n_c^{p-1} = 2\|\tfrac{1}{2}\alpha p n_c^{p-1}\| + r,$$

where $\|\cdot\|$ denotes the nearest integer function. Then, after we complete the square in the exponent of the terms in the sum (4.12) and neglect even multiples of π, it becomes

$$S \approx \exp(i\pi[\alpha n_c^p - \delta']) \times \sum_{n_c-a}^{n_c+b} \exp\left\{i\pi[(p-1)/n_c] \times \left[\tfrac{1}{2}\alpha p n_c^{p-1}\right](k+\delta)^2\right\}, \tag{4.13}$$

where

$$\delta' = r^2/\left(2\alpha p(p-1)n_c^{p-2}\right), \quad \delta = r/\left(\alpha p(p-1)n_c^{p-2}\right). \tag{4.14}$$

The sum in (4.13) will behave like a quadratic sum

$$\sum_{n_c+a}^{n_c+b} \exp\{i\pi\alpha' k^2\}, \tag{4.15}$$

provided δ can be made sufficiently small. This will be the case provided n_c^{p-1} is sufficiently close to the denominator of a convergent to $\frac{1}{2}\alpha p$ in its continued fraction expansion. For, then, by the best approximation property, one can always find a fixed constant c such that

$$\left|\tfrac{1}{2}\alpha p - m/n_c^{p-1}\right| < c/n_c^{p-1}, \quad (4.16)$$

which implies that $r \approx c/n_c^{p-1}$. This in turn implies that

$$\delta \approx c/\left(\tfrac{1}{2}\alpha p(p-1)n_c^{2p-3}\right). \quad (4.17)$$

Since in (4.17), $\frac{1}{2}\alpha p$, although it might be small, is fixed, and c is a fixed constant (like $\frac{1}{5}$, if we use Hurwitz's theorem [7]), δ can be made arbitrarily small by choosing n_c large enough provided $p > \frac{3}{2}$. However, in terms of observing patterns arising from (4.14) this formula is not useful for $p < 2$ except for special values of α. If $\alpha = m - (m^2-1)^{1/2}$ (m a positive integer) is small, then $-1/\alpha$ is also small and equal to $\alpha \bmod 2$ so that δ is small. For, if we choose $\alpha = K[m-(m^2-1)^{1/2}]$ and choose K so that the coefficient of $(k+\delta)^2$ is exactly $\mathrm{i}\pi[m-(m^2-1)^{1/2}]$, then (4.12) should produce some level of self-similarity for a and b small enough, provided the neglected terms in (4.12) are small as well. In fig. 3 we show three second order spirals for p's $\neq 2$, produced by using the α's determined from (4.14) as described in this section.

The estimate (4.17) and the formula (4.14) also have an interesting implication in the case $p = 2$: (4.14) with δ as in (4.18) implies that the pattern centered at n_c (meaning n is close to n_c) will be arbitrarily close to the pattern centered at $n = 0$ if n_c is a convergent denominator for α.

Note. While revising this paper for publication, we became aware of current work of A.P. Mulhaupt [13] in which he explores this aspect of the behavior of the patterns generated by sums (1.1) in the case $p = 2$ in connection with Beatty sequences.

For some $p \neq 2$ we can also use the functional equation (4.8) as follows. Let $p > 2$ be an integer, and let

$$\tfrac{1}{2}\alpha p = \left[0;\, n_0^{p-1};\, n_1^{p-1}; \dots\right] \quad (n_1 \in \mathrm{Z}^+). \quad (4.18)$$

Choose n_0 not too large and choose n_1 very large. Let $n_c = n_0 n_1$. Then the second convergent's denominator $q = n_c^{p-1} + 1 \approx n_c^{p-1}$, so that by (4.14), we get (4.15) with $\alpha' = [(p-1)/n_c][n_1^{p-1} + \mathcal{O}(n_0^{1-p})]$. In this case α' is very nearly equal to $(p-1)n_1^{p-1}/n_c$, and we expect ordered behavior for n close to n_c. In the general case, we can only expect order for some range of n for $\frac{1}{2}\alpha p$'s whose continued fraction expansions contain intergers that are in some sense close to numbers of the form n^{p-1}. This leads to several interesting questions of Diophantine approximation which we do not pursue further here.

5. Discussion

Many of the ideas presented above can be found in the work of Hardy and Littlewood [6], van der Corput [17, 18], Lehmer [8] and Koksma [9]. Here the central questions investigated are new: Is there order and disorder in the graph of

$$S_N = \sum_0^N \exp\left[\mathrm{i}\pi\alpha n^p\right] \quad (5.1)$$

for various α and p? Is there self-similarity exhibited in the graph of S_N for some (α, p)? We found that for $1 < p < 2$ ($p > 2$), and any $\alpha > 0$, no matter how large (no matter how small), the graph of the sum S_N as N increases eventually contains C-spirals (initially contains C-spirals), with each succeeding spiral containing more (fewer) points according to the law $\mathcal{O}(l^e)$, where $e = (2-p)/(p-1)$. In the case $p = 2$, we found that for any α the map consists of C-spirals, of seemingly random orientation if α is irrational, except for special quadratic α's for which we found self-similarity and anti-self-similarity. For related α's but $1 < p \leq 3$, we were able to show intervals of n exist in which the graph of S_n formed spirals of spirals of points nearly similar to the spirals of points. The key tool we use in our investigation, not used by the number theorists, and which leads to all our new results, is the local discrete curvature of the graph of S_N. In particular, it led to our summing S_N by blocks. This tool may be of use in the theory of trigonometrical series in other contexts. If $\alpha = -1/\alpha \bmod 2$, and the sum (5.1) is taken over values of n corresponding to a single C-spiral (between consecutive changes in the local discrete curvature of the graph generated by (5.1)), then we conjecture that error in the approximate functional equation (3.12) approaches zero as $\alpha \to 0$, which would make it nearly a direct generalization of the formula for Gauss sums in this case. As far as we know, the results of Hardy and Littlewood and others for an approximate functional equation for finite sums associated with θ_3 have not been generalized to the case of multisums. The self-similarity we have found on the Macintosh indicates that such formulae might be found.

We also conjecture that if $\alpha = -1/\alpha \bmod 2$ and one approximates α by a finite portion of its continued fraction expansion, then there exists similarity up to a corresponding level in the graph generated by (5.1), but this similarity is exact. If such a magic α is replaced by $2^m\alpha$, then (5.1) yields a lattice of points building spirals building spirals of spirals; see fig. 2c. Further, there is a wide variety of cases for which (1.1) is periodic. For example, if $\alpha = s/q$, $\beta = r/q$, and $s \geq 2$, q and r are positive integers, with $(s, q) = s$, $(r, q) = 1$, then the graph generated by (1.1) is periodic with period at least as small as $2q$ or yields a periodic lattice; see fig. 1b. The case $s = 1$ and $r = q = 2m$ is associated with generalized Gauss sums [2, p. 144].

Finally, we observe that the S_n of (1.1) can also be interpreted to be special solutions of Schrödinger's equation with $x \in [0, 1]$, $t > 0$ and periodic boundary conditions. If $u_{xx} = \mathrm{i}u_t$, then $\exp \mathrm{i}(n\pi x + n^2\pi^2 t)$ is a solution; and setting $\alpha = \pi^2 t$ and $\beta = \pi x$, we see that S_n is a sum of such solutions. Indeed S_∞ is the (formal) solution of this problem satisfying the initial condition $u(x, 0) = \delta(x)$; that is, S_∞ is formally the causal Green's function for Schrödinger's equation. For a map like (1.1), but involving quartic, quadratic, and linear powers of n, S_∞ is formally the Green's function for the linear part of the Kuramoto–Velarde operator $u_t + 4u_{xxxx} + \alpha(u_{xx} + \frac{1}{2}(u_x)^2 + (uu_x)_x)$.

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