

Governing Equations and the Problem Statement

Computational fluid dynamics requires an observance of conservation of mass, conservation of momentum, and energy balance. The flow under investigation is 2D, viscid, and incompressible. Experiments for this type of flow have been conducted over several decades, and the analysis has produced a rich field of nonlinear dynamics. The 2D geometry for this problem is a rotating disk having a large aspect ratio. The geometry of the disk is:

The aspect ratio is defined as $\Gamma = \frac{r_0}{h}$ with $7 \leq \Gamma \leq 20$.

For this problem, the fluid inside the radius a is rotating as though it were a rigid body. The fluid outside radius a is not moving at all. Thus a shear layer develops at a where the difference in velocity is realized. The dynamics of interest for this study occur at the shear layer.

To model the flow we start with conservation of mass. Since the fluid is incompressible, there is no divergence in the flow (no sources or sinks). This assumption simplifies the mass equation to divergence free. The Navier-Stokes' equations (conservation of momentum) retain their viscosity terms but are simplified due to the 2D nature of the flow. The resulting equations for this problem are:

conservation of mass (CM): $\nabla \cdot u = 0$

conservation of momentum (CMO): $\partial_t u + u \cdot \nabla u = -\frac{\nabla P}{\rho} + v\Delta u + v\partial_z^2 u$

u is velocity

P is pressure

ρ is density

v is kinematic viscosity

h is the cell height of the 2D disk

u^* is initial velocity

The last term in the CMO is a vertical component that is modeled in this problem using an Ekman spiral. The Ekman spiral results from the friction on the top and bottom of the 2D disk. When the disk rotates, fluid is transported vertically through the Ekman layer. Due to the low height of the disk, vertical flow is negligible and the effect is accurately modeled with the Ekman friction in lieu of the vertical gradient.

Two tools are available to help simplify the above equations and rewrite them in a more appealing form. The first is the stream-function resulting from the 2D nature of the flow.

The stream-function, ψ , can be used to give an alternate definition for the velocity, $u = \partial_\theta \psi \hat{e}_r - \partial_r \psi \hat{e}_\theta$. The second tool is the definition for vorticity, $\omega = \nabla \times u$. These adjustments in combination with the Ekman friction give the final form for the CM and CMO equations:

$$\text{CM: } \Delta\psi = -\omega$$

$$\text{CMO: } \partial_t \omega + J(\omega, \psi) = v\Delta\omega + \frac{8v}{h^2}(\omega^* - \omega)$$

$$J \text{ is the Jacobian - } J(\omega, \psi) = \frac{1}{r}(\partial_r \omega \partial_\theta \psi - \partial_r \psi \partial_\theta \omega)$$

These equations are referred to as the stream-function/vorticity (S/V) formulation of the Navier-Stokes' equation with Ekman friction.

Flow is modeled using a Chebyshev-Fourier (C-F) expansion. In the radial direction the flow is modeled with Chebyshev polynomials while in the azimuthal direction it is modeled with a Fourier expansion. Thus the stream-function and vorticity are modeled in the following way (this is the stream-function expansion):

$$\psi(r, \theta) = \sum_{n=0}^N \sum_{m=0}^M \psi_{nm} T_m(r) e^{in\theta} \quad -1 \leq r \leq 1 \quad 0 \leq \theta \leq 2\pi$$

$T_m(r)$ is the m^{th} Chebyshev polynomial

ψ_{nm} is the stream-function coefficient for the n^{th} Fourier mode and the m^{th}

Chebyshev mode

Note the Fourier modes are $0 \rightarrow N$ and the Chebyshev modes are $0 \rightarrow M$. Also, the disk is covered twice in this expansion. With $-1 \leq r \leq 1$ instead of $0 \leq r \leq 1$ each point on the disk is the same at (r, θ) and $(-r, \theta \pm \pi)$. This double covering builds some regularity into the solution. Consider a function $\zeta(r, \theta) = f(r)e^{in\theta}$. Then we know $f(r)e^{in\theta} = f(-r)e^{in(\theta \pm \pi)}$ and this simplifies to $f(r) = (-1)^n f(-r)$. The result means $\zeta(r, \theta)$ is even for even Fourier modes, n , and is odd for odd Fourier modes.

Stability Analysis of the Homogeneous Flow

The homogeneous analysis is characterized via the fact that the flow has no variations in the θ direction. This means the problem can be decoupled in the Fourier expansion so as to consider only a single Fourier mode when applying a perturbation to the problem. Furthermore, the bifurcations are determined for steady flow after equilibrium has been reached. To analyze

this problem the S/V equations are made dimensionless, and the resulting equations are:

$$\partial_t \omega + J(\omega, \psi) = \frac{1}{Re} \left[\Delta \omega + \frac{8}{h^2} (\tilde{\omega}^* - \omega) \right] \text{ and } \Delta \psi = -\omega.$$

Re is the Reynolds number, $Re = \frac{a\Omega\beta}{\nu}$

Ω is the angular velocity of the rigidly rotating flow inside radius a

β is the characteristic length scale

$\tilde{\omega}^*$ is the dimensionless, initial vorticity field

Next consider the steady state problem and expand the Laplacian operator as $\Delta \equiv \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$. The resulting system of equations is:

$$\frac{1}{r}(\partial_r \omega \partial_\theta \psi - \partial_r \psi \partial_\theta \omega) = \frac{1}{Re} \left(\partial_r^2 \omega + \frac{1}{r}\partial_r \omega + \frac{1}{r^2}\partial_\theta^2 \omega + \frac{8}{h^2}(\tilde{\omega}^* - \omega) \right) \quad (1)$$

$$\partial_r^2 \psi + \frac{1}{r}\partial_r \psi + \frac{1}{r^2}\partial_\theta^2 \psi = -\omega \quad (2)$$

Once the system is determined, we apply a perturbation. This perturbation is an adjustment to the homogeneous flow and is linear in nature. The homogeneous flow is given by ψ_0 and ω_0 as the initial stream-function and vorticity, respectively. These quantities vary only in the radial direction and the perturbation is also considered in the radial direction. Using the azimuthal independence of the flow, each Fourier mode is considered independently. The linear asymptotic expansion of the flow yields the appropriate perturbation. Thus the resulting homogeneous flow and perturbation have the form:

$$\omega = \omega_0(r) + \varepsilon \omega_1(r) e^{\lambda t + in\theta} \text{ and } \psi = \psi_0(r) + \varepsilon \psi_1(r) e^{\lambda t + in\theta}$$

n is the number of the Fourier mode

λ is the eigenvalues for the n^{th} mode

After allowing the substitution of $\gamma = \lambda t + in\theta$, the asymptotic expansions are substituted into equations (1) and (2). The time dependency of the perturbation modifies equation (1) to include the time dependent term, $\partial_t \omega$. The resulting system is:

$$\begin{aligned} \varepsilon \omega_1 \lambda e^\gamma + \frac{1}{r}(\partial_r \omega_0(\varepsilon in \psi_1 e^\gamma) - \partial_r \psi_0(\varepsilon in \omega_1 e^\gamma)) = \\ \frac{1}{Re} \left(\partial_r^2 \omega_0 + \varepsilon \partial_r^2 \omega_1 e^\gamma + \frac{1}{r}(\partial_r \omega_0 + \varepsilon \partial_r \omega_1 e^\gamma) - \frac{1}{r^2} \varepsilon n^2 \omega_1 e^\gamma + \frac{8}{h^2}(\tilde{\omega}^* - \omega_0 - \varepsilon \omega_1 e^\gamma) \right) \end{aligned} \quad (3)$$

$$\partial_r^2 \psi_0 + \varepsilon \partial_r^2 \psi_1 e^\gamma + \frac{1}{r}(\partial_r \psi_0 + \varepsilon \partial_r \psi_1 e^\gamma) - \frac{1}{r^2} \varepsilon n^2 \psi_1 e^\gamma = -\omega_0 - \varepsilon \omega_1 e^\gamma \quad (4)$$

From this point, the system is divided into 2 parts. The order one, $O(1)$,

terms are grouped separately from the order epsilon, $O(\varepsilon)$, terms. The $O(1)$ system provides information about the homogeneous flow without perturbations. The initial vorticity is used to determine the homogeneous streamfunction and vorticity, ψ_0 and ω_0 . These equations are:

$$\partial_r^2 \omega_0 + \frac{1}{r} \partial_r \omega_0 - \frac{8}{h^2} \omega_0 = -\frac{8}{h^2} \tilde{\omega}^* \quad \text{and} \quad \partial_r^2 \psi_0 + \frac{1}{r} \partial_r \psi_0 = -\omega_0 \quad (5)$$

The $O(\varepsilon)$ equations pertain to the linear perturbation of the homogeneous problem. The equations can be solved to give the eigenvalues for various Reynold's numbers. A positive eigenvalue corresponds to an instability in the homogeneous flow. The $O(\varepsilon)$ system is:

$$\begin{aligned} \omega_1 \lambda + \frac{in}{r} (\psi_1 \partial_r \omega_0 - \omega_1 \partial_r \psi_0) &= \frac{1}{Re} (\partial_r^2 \omega_1 + \frac{1}{r} \partial_r \omega_1 - \frac{n^2}{r^2} \omega_1 - \frac{8}{h^2} \omega_1) \\ \partial_r^2 \psi_1 + \frac{1}{r} \varepsilon \partial_r \psi_1 - \frac{n^2}{r^2} \psi_1 &= -\omega_1 \end{aligned} \quad (4)$$

Using the results from solving system (5), the solution for system (4) can be found through the formulation of a generalized eigenvalue (G/Eig) problem. Looking at (4) the terms $\partial_r \omega_0$ and $\partial_r \psi_0$ are needed to solve the problem. Thus (5) is differentiated with respect to (w.r.t.) r in order to solve for these terms directly. Applying ∂_r to (5) gives:

$$(\partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2} - \frac{8}{h^2}) \partial_r \omega_0 = -\frac{8}{h^2} \partial_r \tilde{\omega}^* \quad \text{and} \quad (\partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2}) \partial_r \psi_0 = -\partial_r \omega_0 \quad (7)$$

To solve systems (4) and (7) we use a similar approach. Initially, to remove the singularities multiply the equations by r^2 . Next recall the functions ω_0 , ψ_0 , ω_1 , and ψ_1 are functions of r only and thus have only Chebyshev expansions (the Fourier expansion has been considered term by term though mode n). Each function has an expansion in Chebyshev polynomials e.g. $\omega_0 \equiv [\omega_{0_0} \ \omega_{0_1} \ \cdots \ \omega_{0_M}]^T$, $\psi_1 \equiv [\psi_{1_0} \ \psi_{1_1} \ \cdots \ \psi_{1_M}]^T$, etc. The j^{th} entry in the vector is the coefficient for the j^{th} Chebyshev polynomial, namely $T_j(r)$. Thus the functions can be represented as vectors in the Chebyshev basis, and multiplication by r and differentiation by r can be represented as matrices. The appropriate matrices are given here:

$$\mathbf{R} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \quad \text{for multiplication by } r$$

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 3 & 0 & 5 & \cdots & M \\ 0 & 0 & 4 & 0 & 8 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 6 & 0 & 10 & \ddots & 2M \\ 0 & 0 & 0 & 0 & 8 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & \ddots & 2M \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 2M \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ for differentiation by } r$$

The \mathbf{R} matrix is formed using convolution techniques described below. The \mathbf{D} matrix results from the following Chebyshev recurrence relation:

$$T'_n = n \left(2T_{n-1} + \frac{T'_{n-2}}{n-2} \right) \text{ starting with } T'_0 = 0 \text{ and } T'_1 = 1$$

T_n is the n^{th} Chebyshev polynomial

Applying the multiplication by r^2 and changing to the vector representation of the functions gives the following system for (7):

$$\begin{aligned} (\mathbf{R}^2 \mathbf{D}^2 + \mathbf{R} \mathbf{D} - \mathbf{I} - \frac{8}{h^2} \mathbf{R}^2) \partial_r \omega_0 &= -\frac{8}{h^2} \mathbf{R}^2 \partial_r \tilde{\omega}^* \\ (\mathbf{R}^2 \mathbf{D}^2 + \mathbf{R} \mathbf{D} - \mathbf{I}) \partial_r \psi_0 &= -\mathbf{R}^2 \partial_r \omega_0 \end{aligned} \quad (5)$$

System (5) consist of a set of coupled equations and is straight forward to solve since $\partial_r \tilde{\omega}^*$ is a given function. Solutions to (5) are the vectors $\partial_r \omega_0$ and $\partial_r \psi_0$. These vectors are used to solve the $O(\varepsilon)$ G/Eig problem.

At this point apply the multiplication by r^2 and change to vector representation of the functions to system (4). One requirement for changing to vector representation is the implementation of convolution matrices for point space multiplication. The terms $\psi_1 \partial_r \omega_0$ and $\omega_1 \partial_r \psi_0$ are multiplied in point space, and these expressions require a convolution when written in terms of Chebyshev polynomials. The convolution is written as a matrix vector product. Since ω_1 and ψ_1 are unknowns, they are left as vectors. This implies the $\partial_r \omega_0$ and $\partial_r \psi_0$ vectors need to be written as matrices. The resulting matrices are referred to as Chebyshev convolution matrices (C-conmtx). The \mathbf{R} matrix used previously is actually a C-conmtx for the vector $\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^{\mathbf{T}}$. In general a C-conmtx is built using the Chebyshev recurrence relation:

$$T_n T_m = \frac{1}{2} (T_{n+m} + T_{|n-m|})$$

To match the above recurrence relation, each T_j is written as a matrix \mathbf{T}_j :

$$\mathbf{T}_j = \begin{bmatrix} 0 & \cdots & 0 & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & 0 & \frac{1}{2} & 0 & & \vdots \\ 0 & \frac{1}{2} & \ddots & & 0 & \frac{1}{2} & \ddots & \vdots \\ 1 & 0 & & \ddots & & 0 & \ddots & 0 \\ 0 & \frac{1}{2} & 0 & & \ddots & & \ddots & \frac{1}{2} \\ \vdots & 0 & \frac{1}{2} & 0 & & & & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \frac{1}{2} & 0 & \cdots & 0 \end{bmatrix} \leftarrow j^{\text{th}} + 1 \text{ row}$$

In general the convolution of 2 Chebyshev vectors and is accomplished in the following manner:

$*$ = $\sum_{j=0}^M g_j \mathbf{T}_j$ where each \mathbf{T}_j is a matrix vector multiplication.

Let $\partial_r \omega_0 = \left[\delta_0 \quad \delta_1 \quad \delta_2 \quad \cdots \quad \delta_M \right]^{\mathbf{T}}$ then the convolution for $\partial_r \omega_0$ is given by the following matrix: $\overline{\partial_r \omega_0} = \delta_0 \mathbf{T}_0 + \delta_1 \mathbf{T}_1 + \cdots + \delta_M \mathbf{T}_M$.

**Note due to the positive slope diagonal in each \mathbf{T}_j matrix, the representation $\partial_r \omega_0 = \delta_0 \mathbf{T}_0 + \delta_1 \mathbf{T}_1 + \cdots + \delta_M \mathbf{T}_M$ is only accurate for the $0 \rightarrow \frac{M}{2}$ Chebyshev modes. Matrices larger than $\frac{M}{2} + 1 \times \frac{M}{2} + 1$ are missing meaningful data contributed by the $M + 1$ and higher modes. This means an M mode accurate C-conmtx requires $2M$ modes of data within the vector.

Making the adjustments to system (4) yields:

$$\begin{aligned} \mathbf{R}^2\omega_1\lambda + in\mathbf{R}(\overline{\partial_r\omega_0}\psi_1 - \overline{\partial_r\psi_0}\omega_1) &= \frac{1}{Re}(\mathbf{R}^2\mathbf{D}^2\omega_1 + \mathbf{R}\mathbf{D}\omega_1 - n^2\mathbf{I}\omega_1 - \frac{8}{h^2}\mathbf{R}^2\omega_1) \\ \mathbf{R}^2\mathbf{D}^2\psi_1 + \mathbf{R}\mathbf{D}\psi_1 - n^2\mathbf{I}\psi_1 &= -\mathbf{R}^2\omega_1 \end{aligned} \quad (6)$$

$\overline{\zeta}$ is the C-conmtx for the vector ζ

Finally the system (6) is ready to rewrite as a G/Eig problem. The form of the problem is $\mathbf{A}\xi = \lambda\mathbf{C}\xi$. Let $\xi = \begin{bmatrix} \omega_1 \\ \psi_1 \end{bmatrix}$ and find the matrix representation for the problem. (6) can be written as:

$$\begin{bmatrix} \frac{1}{Re}(\mathbf{R}^2\mathbf{D}^2 + \mathbf{R}\mathbf{D} - n^2\mathbf{I} - \frac{8}{h^2}\mathbf{R}^2) + in\mathbf{R}\overline{\partial_r\psi_0} & -in\mathbf{R}\overline{\partial_r\omega_0} \\ \mathbf{R}^2 & \mathbf{R}^2\mathbf{D}^2 + \mathbf{R}\mathbf{D} - n^2\mathbf{I} \end{bmatrix} \xi = \lambda \begin{bmatrix} \mathbf{R}^2 & 0 \\ 0 & 0 \end{bmatrix} \xi$$

In order to solve this G/Eig problem boundary conditions (BCs) are required. The physical BCs for this flow problem are already satisfied through the $O(1)$ equations. The BCs for the $O(\varepsilon)$ equations are homogeneous since the perturbation has homogeneous end conditions. The right side BC occurs at r_o which has been normalized so $r_o = 1$. All Chebyshev polynomial are equal to 1 at $r_o = 1$. Furthermore, the left BC occur when $r_o = -1$, and the Chebyshev polynomials are equal to ± 1 here. This means the BCs are:

$$0 = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^{\mathbf{T}} @_{r_o = 1} \text{ and } 0 = \begin{bmatrix} 1 & -1 & 1 & -1 & \cdots & 1 \end{bmatrix}^{\mathbf{T}} @_{r_o = -1}$$

These boundary conditions are applied by removing the least important equations in the G/Eig problem, namely the last two rows of each equation in matrix form, and replacing these equations with the BCs.

After the BCs have been placed in the G/Eig problem, the stability of each Fourier mode can be determined by the resulting eigenvalues. An unstable Fourier mode indicates the flow has bifurcated to a new stable solution.

Stability Analysis of the Secondary Bifurcations

In this next step we are looking for secondary bifurcations from a steady state, nonhomogeneous solution. The steady solution has r and θ components and is characterized by a specific number of vortices. Since the flow is now dependent upon azimuthal variations, the perturbations from steady flow are functions of r , θ , and t . These perturbations are still linear since we are looking only for instabilities. Here are the forms for the vorticity and stream-function steady solutions with perturbations:

$$\omega(r, \theta, t) = \omega_0(r, \theta) + \varepsilon\omega_1(r, \theta)e^{\lambda t} \text{ and } \psi(r, \theta, t) = \psi_0(r, \theta) + \varepsilon\psi_1(r, \theta)e^{\lambda t}$$

Unlike the homogeneous case the steady flow, ω_0 and ψ_0 , are functions of θ and are obtained from the numerical simulation. Of interest for this perturbation analysis are ω_1 , ψ_1 , and λ . Note λ is the eigenvalue and ω_1/ψ_1 form the eigenvector for this problem. As with the homogeneous problem the G/Eig is $\mathbf{A}\xi = \lambda\mathbf{C}\xi$. Substituting these expansions into (1) and (2) gives:

$$\begin{aligned} \varepsilon\lambda\omega_1e^{\lambda t} + \frac{1}{r} \left(\partial_r\omega_0\partial_\theta\psi_0 - \partial_r\psi_0\partial_\theta\omega_0 + \varepsilon\partial_r\omega_1\partial_\theta\psi_0e^{\lambda t} \right) - \\ \frac{1}{r} \left(\varepsilon\partial_r\psi_0\partial_\theta\omega_1e^{\lambda t} + \varepsilon\partial_r\omega_0\partial_\theta\psi_1e^{\lambda t} - \varepsilon\partial_r\psi_1\partial_\theta\omega_0e^{\lambda t} \right) = \\ \frac{1}{Re} \left[\Delta\omega_0 + \varepsilon\Delta\omega_1e^{\lambda t} + \frac{8}{h^2}(\tilde{\omega}^* - \omega_0 - \varepsilon\omega_1e^{\lambda t}) \right] \end{aligned} \quad (7)$$

$$\Delta\psi_0 + \varepsilon\Delta\psi_1e^{\lambda t} = -\omega_0 - \omega_1e^{\lambda t} \quad (8)$$

As before the system (7) and (8) has components of $O(1)$ and $O(\varepsilon)$. However, this time ω_0 and ψ_0 are already known so the $O(1)$ equations are used only if we want to find $\partial_r\omega_0$ and $\partial_r\psi_0$ directly instead of using $\mathbf{D}\omega_0$ and $\mathbf{D}\psi_0$. The $O(1)$ system is:

$$\begin{aligned} \frac{1}{r} (\partial_r\omega_0\partial_\theta\psi_0 - \partial_r\psi_0\partial_\theta\omega_0) &= \frac{1}{Re} \left[\Delta\omega_0 + \frac{8}{h^2}(\tilde{\omega}^* - \omega_0) \right] \\ \Delta\psi_0 &= -\omega_0 \end{aligned} \quad (9)$$

The $O(\varepsilon)$ system is:

$$\begin{aligned} \lambda\omega_1e^{\lambda t} + \frac{1}{r} \left(\partial_r\omega_1\partial_\theta\psi_0e^{\lambda t} - \partial_r\psi_0\partial_\theta\omega_1e^{\lambda t} + \partial_r\omega_0\partial_\theta\psi_1e^{\lambda t} - \partial_r\psi_1\partial_\theta\omega_0e^{\lambda t} \right) = \\ \frac{1}{Re} \left[\Delta\omega_1e^{\lambda t} - \frac{8}{h^2}\omega_1e^{\lambda t} \right] \\ \Delta\psi_1e^{\lambda t} = -\omega_1e^{\lambda t} \end{aligned} \quad (13)$$

Considering (13) once again the problem is converted to Chebyshev/Fourier vector form. This form differs from the homogeneous problem since multiple Fourier modes are now present. The following expansion is generic and is used for both the stream-function and vorticity:

$$\zeta(r, \theta) = \sum_{n=0}^N \sum_{m=0}^M \zeta_{nm} T_m(r) e^{in\theta} \quad -1 \leq r \leq 1 \quad 0 \leq \theta \leq 2\pi$$

$T_m(r)$ is the m^{th} Chebyshev polynomial

ζ_{nm} is the ζ coefficient for the n^{th} Fourier mode and the m^{th} Chebyshev mode

An alternate form is used when only considering the outside, Fourier, expansion:

$$\zeta(r, \theta) = \sum_{n=0}^N \zeta_n(r) e^{in\theta}$$

$\zeta_n(r)$ is the Chebyshev expansion for the n^{th} Fourier mode

This means the outer vector is Fourier and the inner vector is Chebyshev. The Fourier coefficients are given by mode in terms of cosines and sine using the conversion from exponential functions to trigonometric functions. Thus a Fourier vector has the form:

$$\zeta(r, \theta) = \left[\zeta_{\cos 0}(r) \quad \zeta_{\cos 1}(r) \quad \cdots \quad \zeta_{\cos N}(r) \quad 0 \quad \zeta_{\sin 1}(r) \quad \zeta_{\sin 2}(r) \quad \cdots \quad \zeta_{\sin N}(r) \right]^{\mathbf{T}}$$

$\zeta_{\cos j}(r)/\zeta_{\sin j}(r)$ have Chebyshev expansions in vector form for each j

Thus each element in the Fourier vector is a Chebyshev vector having the same form as the one explained for the homogeneous problem. Each vector $\omega_0, \psi_0, \omega_1,$ and ψ_1 in the full problem is $2(N+1)(M+1) \times 1$. The sine zero coefficients (all zeros) is kept to maintain symmetry in the problem.

Once the form of the vectors is described we consider multiplication and differentiation by r . Since these operations are independent of θ , the matrices for multiplication and differentiation by r have block diagonal form. These matrices are shown here:

$$\hat{\mathbf{R}} = \begin{bmatrix} \mathbf{R} & 0 & \cdots & 0 \\ 0 & \mathbf{R} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{R} \end{bmatrix} \text{ for multiplication by } r$$

$$\hat{\mathbf{D}} = \begin{bmatrix} \mathbf{D} & 0 & \cdots & 0 \\ 0 & \mathbf{D} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{D} \end{bmatrix} \text{ for multiplication by } r$$

\mathbf{R} and \mathbf{D} are given for the homogeneous problem

In addition to the Chebyshev operations, $\hat{\mathbf{R}} \equiv r$ and $\hat{\mathbf{D}} \equiv \partial_r$, the full problem now includes multiple Fourier modes and requires a matrix for differentiation by θ . A new matrix is defined for this purpose, $\hat{\mathbf{Q}} \equiv \partial_\theta$. This matrix is formed by applying differentiation to sine and cosine functions:

$$\hat{\mathbf{Q}} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & \vdots & \vdots & -\mathbf{I} & & & \vdots \\ \vdots & & \ddots & & \vdots & \vdots & & -2\mathbf{I} & & \vdots \\ \vdots & & & \ddots & \vdots & \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & -N\mathbf{I} \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \mathbf{I} & & & \vdots & \vdots & \ddots & & & \vdots \\ \vdots & & 2\mathbf{I} & & \vdots & \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 & \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & N\mathbf{I} & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \quad \text{for differentia-}$$

tion by θ

\mathbf{I} is an $M + 1 \times M + 1$ identity matrix

After the tools are in place, rewrite system (13) in matrix vector form. Recall system (9) simply shows the equality of the ω_0 and ψ_0 solutions which have already been obtained via numerical simulations. The equations of interest for the secondary bifurcations are the $O(\varepsilon)$ equations (13). These equations are listed here after applying a multiplication by r^2 and the above definitions:

$$\begin{aligned} \lambda \hat{\mathbf{R}}^2 \omega_1 + \hat{\mathbf{R}} \hat{\mathbf{Q}} \psi_0 \hat{\mathbf{D}} \omega_1 - \hat{\mathbf{R}} \hat{\mathbf{D}} \psi_0 \hat{\mathbf{Q}} \omega_1 + \hat{\mathbf{R}} \hat{\mathbf{D}} \omega_0 \hat{\mathbf{Q}} \psi_1 - \hat{\mathbf{R}} \hat{\mathbf{Q}} \omega_0 \hat{\mathbf{D}} \psi_1 = \\ \frac{1}{Re} \left[\hat{\mathbf{R}}^2 \hat{\mathbf{D}}^2 \omega_1 + \hat{\mathbf{R}} \hat{\mathbf{D}} \omega_1 + \hat{\mathbf{Q}}^2 \omega_1 - \hat{\mathbf{R}}^2 \frac{8}{h^2} \omega_1 \right] \end{aligned} \quad (10)$$

$$\hat{\mathbf{R}}^2 \hat{\mathbf{D}}^2 \psi_1 + \hat{\mathbf{R}} \hat{\mathbf{D}} \psi_1 + \hat{\mathbf{Q}}^2 \psi_1 = -\hat{\mathbf{R}}^2 \omega$$

The matrix vector form is similar to the homogeneous problem except for the $\hat{\mathbf{Q}}$ matrix. The existence of multiple Fourier modes increases the number of nonlinear terms (those terms involving both $\hat{\mathbf{D}}$ and $\hat{\mathbf{Q}}$). Furthermore, these nonlinear terms are convolutions in Fourier space for the outer vectors. Convolution in Fourier space using the cosine/sine basis is discussed below. An additional expansion is used to clearly represent the nonlinear terms. Following are the Fourier expansions of the perturbation vectors:

$$\begin{aligned} \omega_1(r, \theta) &= \sum_N \varphi_n(r) \cos n\theta + \rho_n(r) \sin n\theta \\ \psi_1(r, \theta) &= \sum_N \eta_n(r) \cos n\theta + \mu_n(r) \sin n\theta \end{aligned}$$

$\varphi_n(r)$, $\rho_n(r)$, $\eta_n(r)$, and $\mu_n(r)$ are all Chebyshev vectors

Using these expansions and the Fourier vector notation, it's time to investigate the nonlinear terms in (10). The nonlinear terms pose the greatest difficulty, and these terms are completely represented using $\hat{\mathbf{R}}\hat{\mathbf{Q}}\psi_0\hat{\mathbf{D}}\omega_1$ and $-\hat{\mathbf{R}}\hat{\mathbf{D}}\psi_0\hat{\mathbf{Q}}\omega_1$ as examples. (Note that the ω_1 is substituted with a ψ_1 for the other two nonlinear terms, and the methods discussed here are independent of ω_1 and ψ_1). First look at the vectors involved in this calculation:

$$\hat{\mathbf{Q}}\psi_0 = \left[0 \quad -\tilde{\mu}_1 \quad -2\tilde{\mu}_2 \quad \cdots \quad -N\tilde{\mu}_N \quad 0 \quad \tilde{\eta}_1 \quad 2\tilde{\eta}_2 \quad \cdots \quad N\tilde{\eta}_N \right]^T$$

$$\hat{\mathbf{D}}\omega_1 = \left[\hat{\mathbf{D}}\varphi_0 \quad \hat{\mathbf{D}}\varphi_1 \quad \hat{\mathbf{D}}\varphi_2 \quad \cdots \quad \hat{\mathbf{D}}\rho_N \quad 0 \quad \hat{\mathbf{D}}\rho_1 \quad \hat{\mathbf{D}}\rho_2 \quad \cdots \quad \hat{\mathbf{D}}\rho_N \right]^T$$

For the zero modes, ω_0 and ψ_0 , the cosine/sine representations in vector form are:

$$\omega_0 \equiv \left[\tilde{\varphi}_0 \quad \tilde{\varphi}_1 \quad \tilde{\varphi}_2 \quad \cdots \quad \tilde{\varphi}_N \quad 0 \quad \tilde{\rho}_1 \quad \tilde{\rho}_2 \quad \cdots \quad \tilde{\rho}_N \right]^T$$

$$\psi_0 \equiv \left[\tilde{\eta}_0 \quad \tilde{\eta}_1 \quad \tilde{\eta}_2 \quad \cdots \quad \tilde{\eta}_N \quad 0 \quad \tilde{\mu}_1 \quad \tilde{\mu}_2 \quad \cdots \quad \tilde{\mu}_N \right]^T$$

Continuing the representation of the nonlinear terms in vector form requires multiplication by $\hat{\mathbf{R}}$ and gives:

$$\hat{\mathbf{R}}\hat{\mathbf{Q}}\psi_0 = \left[0 \quad -\hat{\mathbf{R}}\tilde{\mu}_1 \quad -2\hat{\mathbf{R}}\tilde{\mu}_2 \quad \cdots \quad -N\hat{\mathbf{R}}\tilde{\mu}_N \quad 0 \quad \hat{\mathbf{R}}\tilde{\eta}_1 \quad 2\hat{\mathbf{R}}\tilde{\eta}_2 \quad \cdots \quad N\hat{\mathbf{R}}\tilde{\eta}_N \right]^T$$

$\hat{\mathbf{R}}\hat{\mathbf{Q}}\psi_0\hat{\mathbf{D}}\omega_1$ is actually the convolution of these vectors in Fourier space so we write $[\hat{\mathbf{R}}\hat{\mathbf{Q}}\psi_0] *_F [\hat{\mathbf{D}}\omega_1]$. This is easily accomplished when the Fourier modes are represented in exponential form. However, the conversion to the trigonometric expansion is non trivial.

The Fourier convolution matrix (F-conmtx) resulting from an exponential basis uses the additive nature of exponents. Consider two expansions $\sum_{-N}^N c_n e^{in\theta}$ and $\sum_{-N}^N \gamma_n e^{in\theta}$ multiplied together then the convolution matrix and matrix vector multiplication is given by:

$$\begin{bmatrix} c_0 & c_{-1} & c_{-2} & \cdots & c_{-N} & 0 & \cdots & \cdots & 0 \\ c_1 & c_0 & c_{-1} & \cdots & c_{-N+1} & c_{-N} & 0 & \cdots & 0 \\ \vdots & & & & & & \ddots & \ddots & \vdots \\ c_{N-1} & \cdots & \cdots & c_0 & \cdots & \cdots & \cdots & c_{-N} & 0 \\ c_N & c_{N-1} & \cdots & c_1 & c_0 & c_{-1} & \cdots & c_{-N+1} & c_{-N} \\ 0 & c_N & \cdots & \cdots & \cdots & c_0 & \cdots & \cdots & c_{-N+1} \\ \vdots & \ddots & \ddots & & & & & & \vdots \\ 0 & \cdots & 0 & c_N & c_{N-1} & \cdots & c_1 & c_0 & c_{-1} \\ 0 & \cdots & \cdots & 0 & c_N & \cdots & c_2 & c_1 & c_0 \end{bmatrix} \begin{bmatrix} \gamma_{-N} \\ \gamma_{-N+1} \\ \vdots \\ \gamma_{-1} \\ \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{N-1} \\ \gamma_N \end{bmatrix}$$

Thus each coefficient of the convolution is a vector dot product given by:

$$\sum_{k=0}^{2N+n} c_{N+n-k} \gamma_{k-N} \quad -N \leq n \leq 0$$

$$\sum_{k=0}^{2N-n} c_{N-k} \gamma_{k-N+n} \quad 0 > n \geq -N$$

where n is the Fourier mode

The vector resulting from this multiplication is the Fourier mode space equivalent to point space multiplication of the original functions – just as it is with C-conmtx.

Converting from the exponential expansion to the trigonometric expansion is the next task. Using the standard form for the Fourier trigonometric expansion, $a_0 + \sum_1^N a_n \cos n\theta + b_n \sin n\theta$, look at the modification of $\sum_{-N}^N c_n e^{in\theta}$. There are three cases to investigate:

- 1) $n = 0$ - resulting in $a_0 = c_0$
- 2) $1 \geq n \geq N$ - then $\sum_1^N c_n e^{in\theta} = \sum_1^N c_n (\cos n\theta + i \sin n\theta)$
- 3) $-N \leq n \leq -1$ (but make the following change of variables: $n = -n$ so the sum is $1 \rightarrow N$ as in case 2) - then $\sum_1^N c_{-n} e^{-in\theta} = \sum_1^N c_{-n} (\cos n\theta - i \sin n\theta)$.

Combining the above cases leads to:

$$\sum_{-N}^N c_n e^{in\theta} = c_0 + \sum_1^N [(c_n + c_{-n}) \cos n\theta + i (c_n - c_{-n}) \sin n\theta]$$

The results are $a_0 = c_0$, $a_n = c_n + c_{-n}$, and $b_n = i (c_n - c_{-n})$. Rewriting and solving for c_n and c_{-n} gives $c_n = \frac{a_n - ib_n}{2}$ and $c_{-n} = \frac{a_n + ib_n}{2}$. Similarly then we can write $\gamma_n = \frac{\alpha_n - i\beta_n}{2}$ and $\gamma_{-n} = \frac{\alpha_n + i\beta_n}{2}$. Using these conversions between the exponential and trigonometric Fourier series in combination with the previous F-conmtx gives the trigonometric convolution matrix and matrix vector product:

$$\begin{bmatrix} a_0 & \frac{1}{2}a_1 & \frac{1}{2}a_2 & \cdots & \frac{1}{2}a_N & 0 & \frac{1}{2}b_1 & \frac{1}{2}b_2 \\ a_1 & a_0 + \frac{1}{2}a_2 & \frac{1}{2}(a_1 + a_3) & \cdots & \frac{1}{2}(a_{N-1} + a_{N+1}) & 0 & \frac{1}{2}b_2 & \frac{1}{2}(b_1 + b_3) \\ a_2 & \frac{1}{2}(a_1 + a_3) & a_0 + \frac{1}{2}a_4 & \cdots & \frac{1}{2}(a_{N-2} + a_{N+2}) & 0 & \frac{1}{2}(b_3 - b_1) & \frac{1}{2}b_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_N & \frac{1}{2}(a_{N-1} + a_{N+1}) & \frac{1}{2}(a_{N-2} + a_{N+2}) & \cdots & a_0 + \frac{1}{2}a_{2N} & 0 & \frac{1}{2}(b_{N+1} - b_{N-1}) & \frac{1}{2}(b_{N+2} - b_{N-2}) \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & \cdots \\ b_1 & \frac{1}{2}b_2 & \frac{1}{2}(b_3 - b_1) & \cdots & \frac{1}{2}(b_{N+1} - b_{N-1}) & 0 & a_0 - \frac{1}{2}a_2 & \frac{1}{2}(a_1 - a_3) \\ b_2 & \frac{1}{2}(b_1 + b_3) & \frac{1}{2}b_4 & \cdots & \frac{1}{2}(b_{N+2} - b_{N-2}) & 0 & \frac{1}{2}(a_1 - a_3) & a_0 - \frac{1}{2}a_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_N & \frac{1}{2}(b_{N-1} + b_{N+1}) & \frac{1}{2}(b_{N-2} + b_{N+2}) & \cdots & \frac{1}{2}b_{2N} & 0 & \frac{1}{2}(a_{N-1} - a_{N+1}) & \frac{1}{2}(a_{N-2} - a_{N+2}) \end{bmatrix}$$

Note the F-conmtx is made of 4 distinct blocks. Each block is $N+1 \times N+1$ but requires a vector with $2N$ entries. This is an advantage since $2N$ modes can be investigated while using a matrix of half the size.

Returning to the problem at hand we see the F-conmtx given by $[\hat{\mathbf{R}}\hat{\mathbf{Q}}\psi_0]_{*F}$ $[\hat{\mathbf{D}}\omega_1]$ can be represented in the form shown above. The convolution matrix is written as $\mathbf{F} = \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix}$ and the pieces of this matrix are:

$$\mathbf{F}_{11} = \begin{bmatrix} 0 & -\frac{1}{2}\hat{\mathbf{R}}\tilde{\mu}_1 & -\hat{\mathbf{R}}\tilde{\mu}_2 & \cdots & -\frac{N}{2}\hat{\mathbf{R}}\tilde{\mu}_N \\ -\hat{\mathbf{R}}\tilde{\mu}_1 & -\hat{\mathbf{R}}\tilde{\mu}_2 & -\hat{\mathbf{R}}\left(\frac{1}{2}\tilde{\mu}_1 + \frac{3}{2}\tilde{\mu}_3\right) & \cdots & -\hat{\mathbf{R}}\left(\frac{(N-1)}{2}\tilde{\mu}_2\right) \\ -2\hat{\mathbf{R}}\tilde{\mu}_2 & -\hat{\mathbf{R}}\left(\frac{1}{2}\tilde{\mu}_1 + \frac{3}{2}\tilde{\mu}_3\right) & -2\hat{\mathbf{R}}\tilde{\mu}_4 & \cdots & -\hat{\mathbf{R}}\left(\frac{(N-2)}{2}\tilde{\mu}_3\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -N\hat{\mathbf{R}}\tilde{\mu}_N & -\hat{\mathbf{R}}\left(\frac{(N-1)}{2}\tilde{\mu}_{N-1} + \frac{(N+1)}{2}\tilde{\mu}_{N+1}\right) & -\hat{\mathbf{R}}\left(\frac{(N-2)}{2}\tilde{\mu}_{N-2} + \frac{(N+2)}{2}\tilde{\mu}_{N+2}\right) & \cdots & -N\hat{\mathbf{R}}\tilde{\mu}_{2N} \end{bmatrix}$$

$$\mathbf{F}_{21} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \hat{\mathbf{R}}\tilde{\eta}_1 & \hat{\mathbf{R}}\tilde{\eta}_2 & \hat{\mathbf{R}}\left(\frac{3}{2}\tilde{\eta}_3 - \frac{1}{2}\tilde{\eta}_1\right) & \cdots & \hat{\mathbf{R}}\left(\frac{(N+1)}{2}\tilde{\eta}_{N+1} - \frac{(N-1)}{2}\tilde{\eta}_1\right) \\ 2\hat{\mathbf{R}}\tilde{\eta}_2 & \hat{\mathbf{R}}\left(\frac{1}{2}\tilde{\eta}_1 + \frac{3}{2}\tilde{\eta}_3\right) & 2\hat{\mathbf{R}}\tilde{\eta}_4 & \cdots & \hat{\mathbf{R}}\left(\frac{(N+2)}{2}\tilde{\eta}_{N+2} - \frac{(N-2)}{2}\tilde{\eta}_1\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ N\hat{\mathbf{R}}\tilde{\eta}_N & \hat{\mathbf{R}}\left(\frac{(N-1)}{2}\tilde{\eta}_{N-1} + \frac{(N+1)}{2}\tilde{\eta}_{N+1}\right) & \hat{\mathbf{R}}\left(\frac{(N-2)}{2}\tilde{\eta}_{N-2} + \frac{(N+2)}{2}\tilde{\eta}_{N+2}\right) & \cdots & N\hat{\mathbf{R}}\tilde{\eta}_{2N} \end{bmatrix}$$

$$\mathbf{F}_{22} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \hat{\mathbf{R}}\tilde{\mu}_2 & \hat{\mathbf{R}}\left(\frac{3}{2}\tilde{\mu}_3 - \frac{1}{2}\tilde{\mu}_1\right) & \cdots & \hat{\mathbf{R}}\left(\frac{(N+1)}{2}\tilde{\mu}_{N+1} - \frac{(N-1)}{2}\tilde{\mu}_1\right) \\ 0 & \hat{\mathbf{R}}\left(\frac{3}{2}\tilde{\mu}_3 - \frac{1}{2}\tilde{\mu}_1\right) & 2\hat{\mathbf{R}}\tilde{\mu}_4 & \cdots & \hat{\mathbf{R}}\left(\frac{(N+2)}{2}\tilde{\mu}_{N+2} - \frac{(N-2)}{2}\tilde{\mu}_1\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \hat{\mathbf{R}}\left(\frac{(N+1)}{2}\tilde{\mu}_{N+1} - \frac{(N-1)}{2}\tilde{\mu}_{N-1}\right) & \hat{\mathbf{R}}\left(\frac{(N+2)}{2}\tilde{\mu}_{N+2} - \frac{(N-2)}{2}\tilde{\mu}_{N-2}\right) & \cdots & N\hat{\mathbf{R}}\tilde{\mu}_{2N} \end{bmatrix}$$

Finally, each element of the \mathbf{F} matrix is the C-conmtx for the vector occurring in that element. E.g. $\hat{\mathbf{R}}\tilde{\eta}_2$ is the C-conmtx associated with the $\hat{\mathbf{R}}\tilde{\eta}_2$ vector. Recall $\tilde{\eta}_2(r) = \sum_M \tilde{\eta}_{m,2} T_m(r)$, and the C-conmtx with $g_{\tilde{\eta}_2} = \mathbf{R}\tilde{\eta}_2$ is given by $\mathbf{Y}_{\tilde{\eta}_2} = \sum_0^{2M} \mathbf{T}_m g_{\tilde{\eta}_2}(m)$. The $\mathbf{Y}_{\tilde{\eta}_2}$ is cropped to $M+1 \times M+1$ as to contain the full data for a $2M$ vector. The C-conmtx is $M+1 \times M+1$ but requires $2M$ entries in the vector just as with the Fourier convolution. This is an advantage for the size of the matrix. Thus each element of the matrix representing $[\hat{\mathbf{R}}\hat{\mathbf{Q}}\psi_0]_{*F} [\hat{\mathbf{D}}]$ has the form $\mathbf{Y}_\xi \mathbf{D}$, a simple matrix multiplication of the C-conmtx \mathbf{Y}_ξ and the Chebyshev derivative matrix \mathbf{D} .

Looking at the other form of nonlinear terms, $-\hat{\mathbf{R}}\hat{\mathbf{D}}\psi_0\hat{\mathbf{Q}}\omega_1$, is nothing more than a quick series of substitutions. Writing the point space multiplication in Fourier vectors gives the convolution $-\left[\hat{\mathbf{R}}\hat{\mathbf{D}}\psi_0\right] *_{\text{F}} \left[\hat{\mathbf{Q}}\omega_1\right]$. This F-conmtx is formed as described above with each element given by \mathbf{Y}_ξ , the C-conmtx for the vector $-\hat{\mathbf{R}}\hat{\mathbf{D}}\xi$ where $\xi \in \psi_0 \equiv \left[\tilde{\eta}_0 \ \tilde{\eta}_1 \ \tilde{\eta}_2 \ \cdots \ \tilde{\eta}_N \ 0 \ \tilde{\mu}_1 \ \tilde{\mu}_2 \ \cdots \ \tilde{\mu}_N \right]^{\mathbf{T}}$. Finally, the entire F-conmtx is multiplied by the $\hat{\mathbf{Q}}$ matrix. Recall the $\hat{\mathbf{Q}}$ matrix acts on the whole Fourier vector or convolution matrix while the $\hat{\mathbf{R}}$ and $\hat{\mathbf{D}}$ matrices act on individual elements, Chebyshev expansions, in a Fourier vector or convolution matrix. The remaining nonlinear terms in the system (10) are calculated using the ideas outlined in the previous paragraph of this section.

The resulting G/Eig eigenvalue problem, $\mathbf{A}\xi = \lambda\mathbf{C}\xi$, with $\xi = \begin{bmatrix} \omega_1 \\ \psi_1 \end{bmatrix}$ has the form:

$$\begin{bmatrix} \frac{1}{Re} \left(\hat{\mathbf{R}}^2 \hat{\mathbf{D}}^2 + \hat{\mathbf{R}}\hat{\mathbf{D}} + \hat{\mathbf{Q}}^2 - \hat{\mathbf{R}}^2 \frac{8}{h^2} \right) - \hat{\mathbf{R}}\hat{\mathbf{Q}}\psi_0\hat{\mathbf{D}} + \hat{\mathbf{R}}\hat{\mathbf{D}}\psi_0\hat{\mathbf{Q}} & \hat{\mathbf{R}}\hat{\mathbf{Q}}\omega_0\hat{\mathbf{D}} - \hat{\mathbf{R}}\hat{\mathbf{D}}\omega_0\hat{\mathbf{Q}} \\ \hat{\mathbf{R}}^2 & \hat{\mathbf{R}}^2 \hat{\mathbf{D}}^2 + \hat{\mathbf{R}}\hat{\mathbf{D}} + \hat{\mathbf{Q}}^2 \end{bmatrix} \xi = \lambda \begin{bmatrix} \hat{\mathbf{R}}^2 & 0 \\ 0 & 0 \end{bmatrix} \xi$$

Each nonlinear term ($-\hat{\mathbf{R}}\hat{\mathbf{Q}}\psi_0\hat{\mathbf{D}}$, $\hat{\mathbf{R}}\hat{\mathbf{D}}\psi_0\hat{\mathbf{Q}}$, $\hat{\mathbf{R}}\hat{\mathbf{Q}}\omega_0\hat{\mathbf{D}}$, and $-\hat{\mathbf{R}}\hat{\mathbf{D}}\omega_0\hat{\mathbf{Q}}$) is an F-conmtx

Boundary conditions (BCs) are the final requirement prior to solving this problem. As before with the homogeneous problem, the BCs for the $O(\varepsilon)$ equations are homogeneous since the perturbation has zero end conditions. The right side BC occurs at $r_o = 1$. All Chebyshev polynomial are equal to 1 at $r_o = 1$. Furthermore, the left BC occur when $r_o = -1$, and the Chebyshev polynomials are equal to ± 1 here. This means the BCs are:

$$0 = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^{\mathbf{T}} @_{r_o = 1} \text{ and } 0 = \begin{bmatrix} 1 & -1 & 1 & -1 & \cdots & 1 \end{bmatrix}^{\mathbf{T}} @_{r_o = -1}$$

With the full problem the BCs must be applied to every Fourier mode simultaneously. The conditions are applied as before by removing the least important equations in the G/Eig problem, namely the last two rows of each equation in matrix form, and replacing these equations with the BCs. However, this must be done to all Fourier modes so each $M + 1 \times M + 1$ diagonal block must have the BCs applied by replacing the last 2 rows.

The primary concern for this problem is the fact that \mathbf{A} and \mathbf{C} are $2(M + 1)2(N + 1) \times 2(M + 1)2(N + 1)$. This severely limits the size of the Chebyshev-Fourier expansion that can be considered. However, the C-conmtx uses information of size $2M$ while the F-conmtx uses information

of size $2N$. This implies a problem initially containing $0 \rightarrow M$ Chebyshev modes and $0 \rightarrow N$ Fourier modes produces a G/Eig problem of size $(M + 2)(N + 1) \times (M + 2)(N + 1)$. An eigenvalues problem containing 20 million elements is the most we can hope for currently. This restriction limits the product of modes, $(M + 2)(N + 1)$, to about 4400. The exact effects of this limitation, and the nature of the limitation are still undetermined.

Improving Computational Efficiency

In order to reduce the size of the matrix for computation, we consider the even/odd nature of the system. Due to the double coverage of the computational domain (discussed in the problem statement), even/odd structure is built into the Chebyshev-Fourier (C-F) expansion. The vorticity and streamfunction, ω and ψ , vectors are built from matrices with the following form:

$$\begin{bmatrix} * & 0 & 0 & 0 & * & * & 0 & 0 & \cdots & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * & \cdots & 0 & 0 \\ * & 0 & 0 & 0 & * & * & 0 & 0 & \cdots & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * & \cdots & 0 & 0 \\ * & 0 & 0 & 0 & * & * & 0 & 0 & \cdots & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * & \cdots & 0 & 0 \\ * & 0 & 0 & 0 & * & * & 0 & 0 & \cdots & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ * & 0 & 0 & 0 & * & * & 0 & 0 & \cdots & * & * \end{bmatrix}$$

where the rows are Chebyshev modes beginning with 0 and going to M and the columns are grouped in pairs for each Fourier mode beginning with 0 and going to N . The first column in a Fourier mode pair is the cosine coefficient and the second is the sine coefficient. Thus because the Fourier mode vectors alternate even/odd the full problem vectors have the form:

$$\omega/\psi \equiv \left[* \ 0 \ * \ 0 \ * \ \cdots \ * \ 0 \ 0 \ \cdots \ 0 \ 0 \ * \ 0 \ * \ 0 \ \cdots \ 0 \right]^{\mathbf{T}}.$$

**Note the vectors always have the above form when M is even and N is odd.

Now define E \equiv evenmode and O \equiv oddmode. The vectors can also be seen as

$$\omega/\psi \equiv \left[\text{E} \ \text{O} \ \text{E} \ \text{O} \ \cdots \ \text{O} \ 0 \ \cdots \ 0 \ \text{O} \ \text{E} \ \text{O} \ \text{E} \ \cdots \ \text{O} \right]^{\mathbf{T}}$$

Furthermore, the $\hat{\mathbf{R}}$ and $\hat{\mathbf{D}}$ matrices also contain even/odd structure. The Chebyshev polynomials alternate between being even and odd. Using the definitions given above T_j has the same parity as j . Thus all the terms in the full G/Eig problem are even. The result of this structure is a reduction of the G/Eig problem size by a factor of 4. Since ω/ψ alternate even/odd and the problem is even, every other row and every other column of the G/Eig matrix is not required for finding eigenvalues. This allows us to consider problems where $(M + 2)(N + 1)$ is about 8800.

Summary

The generalized eigenvalue problem for the rotating disk is a rich problem of nonlinear dynamics. The problem and solution outlined above addresses only a single aspect of the dynamics involved with this problem. The solution formulation using standard expansion methods for Chebyshev and Fourier series results in an eigenvalue problem that can be solved. This problem consists of two coupled equations in the variables of vorticity and streamfunction. The primary question revolves around the Chebyshev Fourier resolution required for accurate eigenvalues. This question is currently open. However, the solution to the homogeneous case with independent Fourier modes, and the solution to the full problem using the homogeneous flow as the steady state yield identical results. While this result is encouraging, the number of Fourier modes required to solve the nonhomogeneous problem is unknown.