Elementary Constructions in Spatial Constraint Solving

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Graphics Vs. Constraints

- Traditional connections:
  - Constraint-based model creation (CAD)
  - Constraint-based scene creation (assembly)
- Other connections:
  - Constrained motion (actors, shadows, …)

What is a GC Problem?

- A set of geometric elements in some space
  - Points, lines, arcs, spheres, cylinders, …
- A set of constraints on them
  - Distance, angle, tangency, incidence, …
- Solution:
  - Coordinate assignment such that the constraints are satisfied, or notification that this cannot be done.

Task Structure

Problem preparation
- Underconstrained, fixed, etc.
- Certain transformations, reasoning

Decomposition of large problems
- Degree of freedom analysis
- Graph analysis

Equation solving
- Numerical techniques
- Algebraic techniques

2D Constraint Solving

- Fairly mature technology –
  - Efficient, robust and competent solvers
    - Triangle decomposition of problems or other methods
      - Points, lines, circular arcs
      - Distance, angle, tangency, perpendicularity, etc.
      - Under- and overconstrained cases
      - Variety of extensions
    - Other techniques also succeed

What Helps the Planar Case

1. Small vocabulary already useful
2. Small catalogue of algebraic systems
3. Algebraic systems easy

Example: Apollonius' Problem

- Given 3 circles, find a circle tangent to all of them:
  - Degree 8 system – but it factors into univariate quadratic equations by a suitable coordinate transformation

3D Solvers and Issues

- Points and planes
- Lines as well as points and planes
- Graph decomposition is OK
  - Hoffmann, Lomonosov, Sitharam. JSC 2001
- But equation solving is tricky:
  - Sequential case involving lines
  - Simultaneous cases
  - No compact subset that has good applicability

Consequences

- Spatial constraint solvers are fairly limited in ability:
  - Technology limitations impair application concepts
  - Limited application concepts fail to make the case for better technology

Problem Subtypes

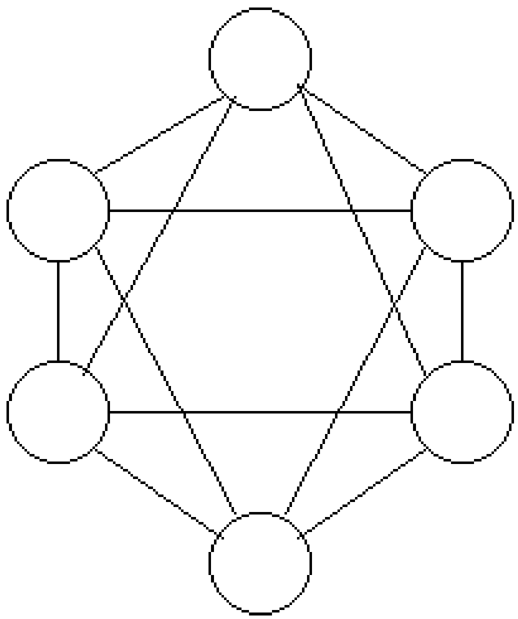
- Sequential:
  - Place a single geometric element by constraints on other, known elements
- Simultaneous:
  - Place a group of geometric elements simultaneously
- In 2D, sequential problems are easy, but in 3D…

Equation Solving Techniques

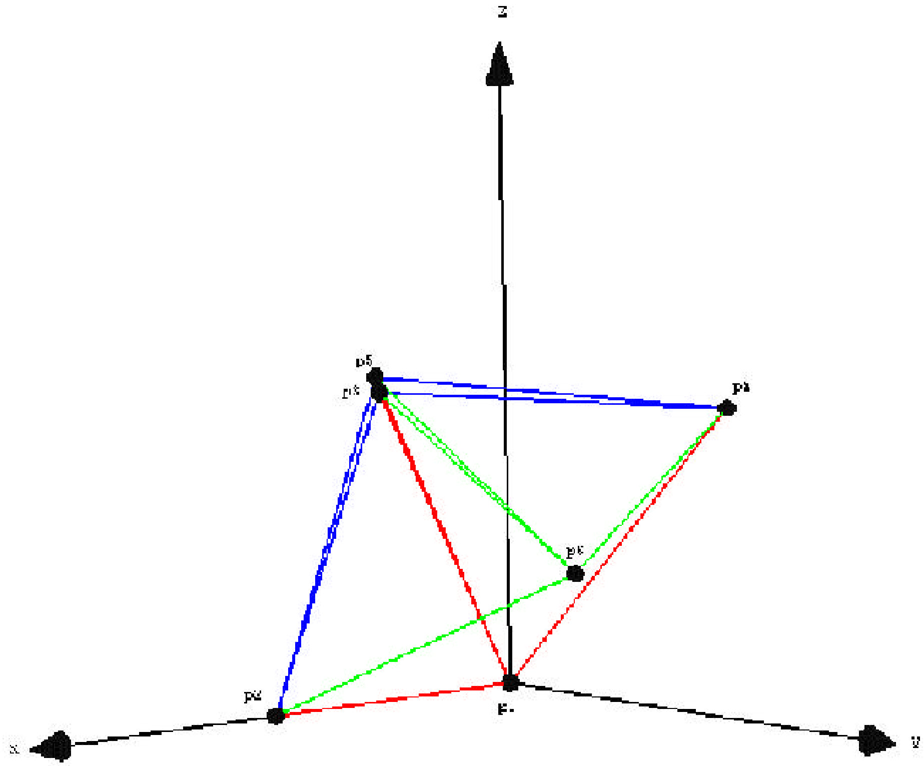
1. Geometric reasoning plus elimination
2. Systematic algebraic manipulation
3. Parametric computation
4. Geometric analysis (of sequential line constructions)

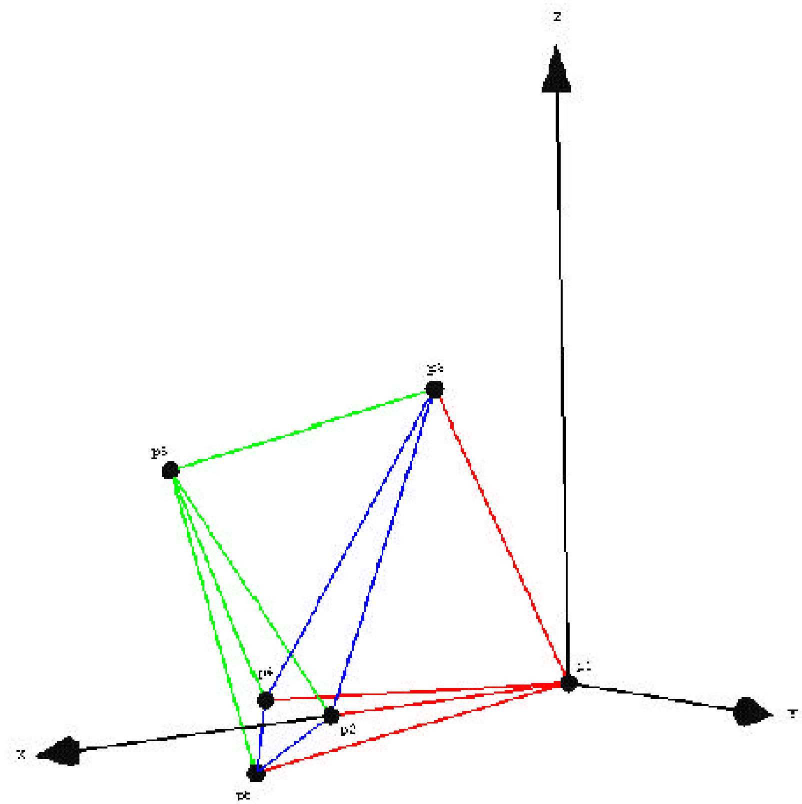
Octahedral Problems

- 6 points/planes, 12 constraints:



6p Example



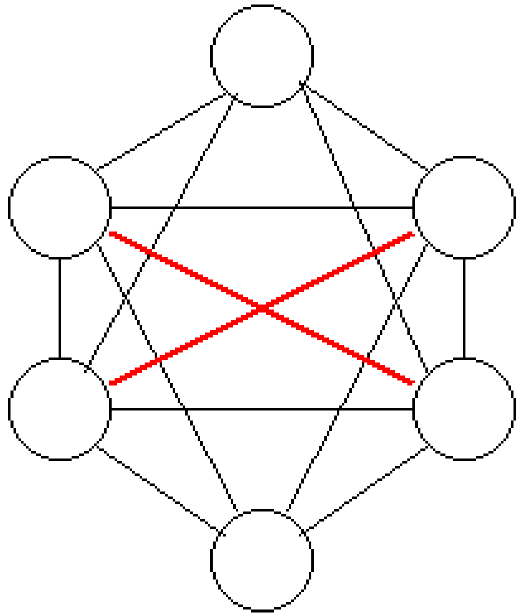


Early Solutions (Vermeer)

- Mixture of geometric reasoning and algebraic simplification using resultants
- Univariate polynomial of degree 16 for 6p – tight bound

Michelucci's Solution

- Formulate the Cayley-Menger determinant for 2 subsets of 5 entities
- Yields two degree 4 equations in 2 unknowns
- Extensions for planes

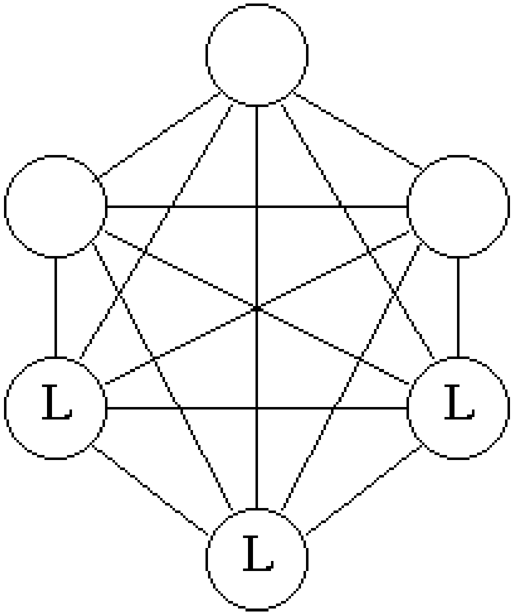


Systematic Framework (Durand)

- Process for 6p:
  1. Gaussian elimination
  2. Univariate equation solving
  3. Bilinear and biquadratic equation parameterization
- 3 quartic equations in 3 variables (6p). BKK bound is 16.
- Homotopy tracking for 16 paths.

Simultaneous 3p3L

- Complete graph K_6



Systematic Solution (Durand)

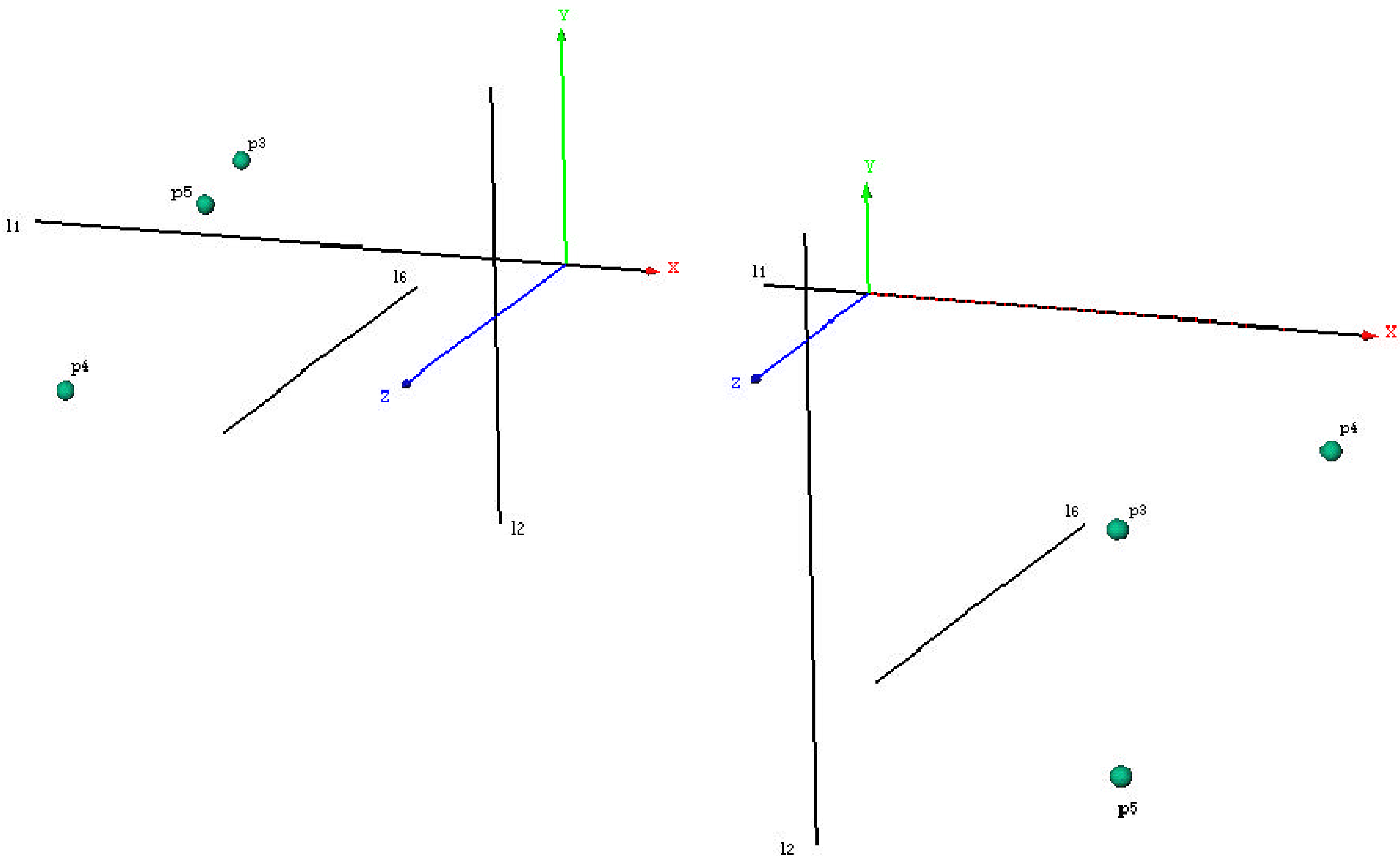
- Initially 21 equations, process as before
  1. Gaussian elimination
  2. Univariate equation solving
  3. Bilinear and biquadratic equation parameterization
- 6 equations in 6 variables, but total degree is $24^3\,8^3$

Durand cont'd

- Homotopy techniques applied to special case of orthogonal lines (~4100 paths):

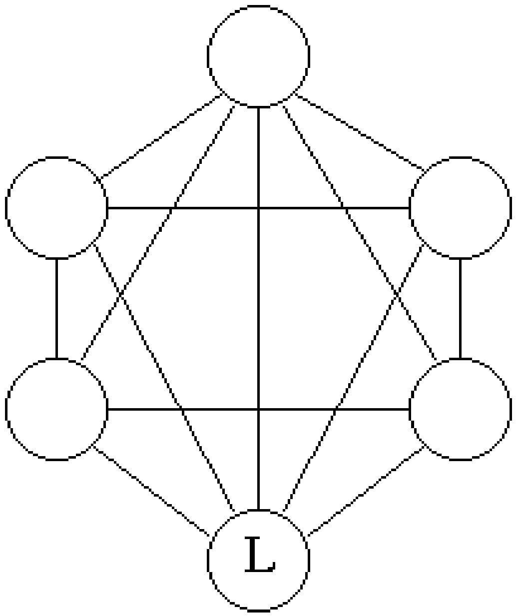
Real	48
Complex	895
At Infinity	3031
Failure	122

3p3L Example (Ortho Lines)



Simultaneous 5p1L

- Place 5 points and 1 line from distance constraints between the line and every point and between the points, in a square pyramid



5p1L Problem, Systematic

- Systematic algebraic treatment yields a system of degree 512
- Coordinate system choices
  - Heuristic: Choosing the line in a standard position tends to yield simpler equations

5p1L, Adding Reasoning

- Approach:
  - Line on x-axis, 1 point on z-axis
  - One point placed as function of $z(t) = t$
  - Other points yield constraint equations
- Result:
  - System $(4^2, 3^4, 2^2)$ not resolving square roots.
  - No significant algebraic simplifications

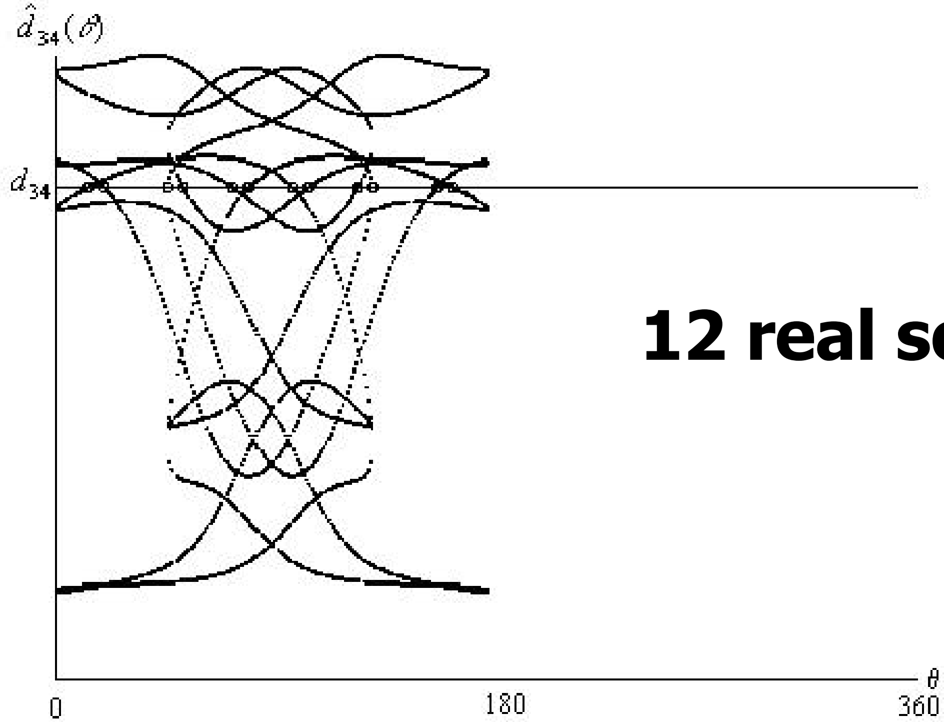
5p1L, Computation (Yuan)

- The parameterized equations are numerically quite tractable
  - Trace the curve of the "missing dimension" numerically
  - Intersect with the nominal value

Example Problem

r1	5.1286		d51	5.4039
r2	3.4797		d52	4.9275
r3	5.1201		d53	6.5569
r4	4.4887		d54	5.0478
r5	0.8548			
d12	2.4992		d34	9.1500
d23	9.5569		d41	7.1859

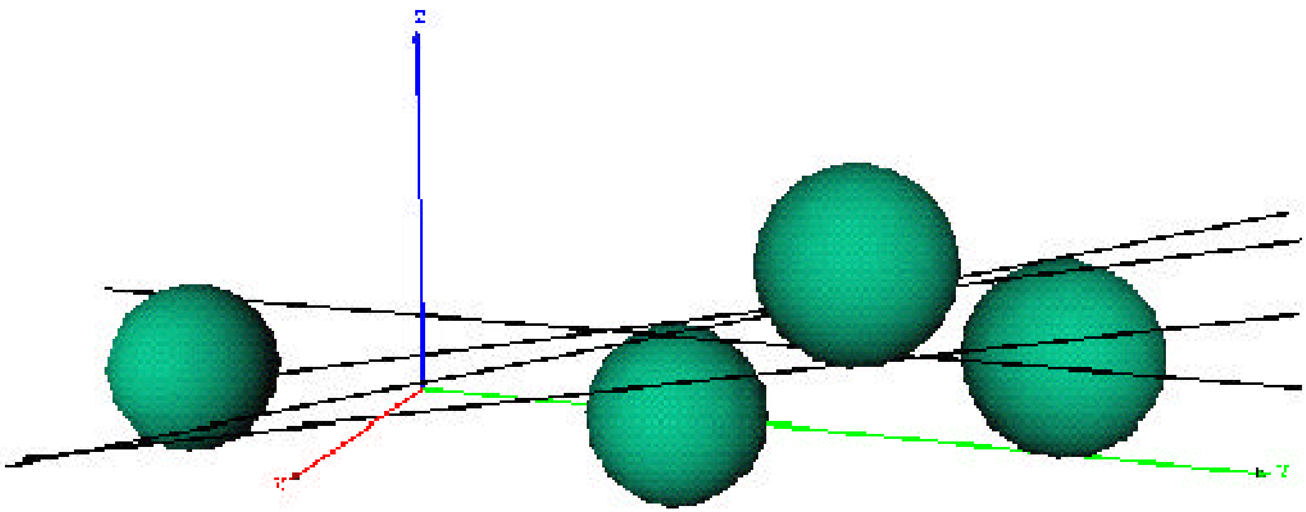
Resulting Curve



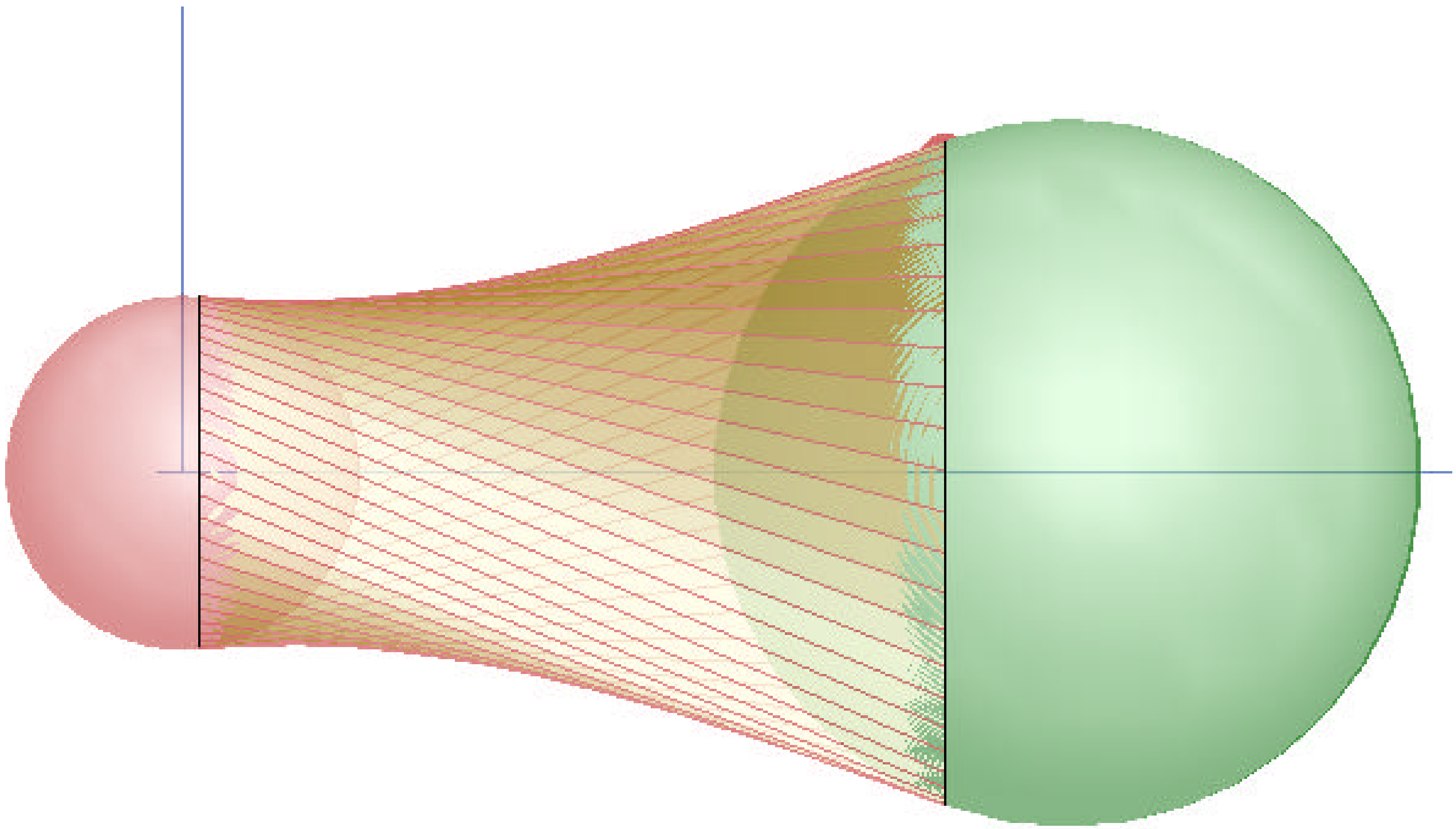
12 real solutions found

Sequential: L-pppp

- Given 4 fixed points and 4 distances, place a line



Nota Bene



L-pppp Solved

Coordinate system choice

$$L : (x, y, z; u, v, w)$$

$$S_1 : (0,0,0,r_1)$$

$$S_2 : (a,0,0,r_2)$$

$$S_3 : (b,c,0,r_3)$$

$$S_4 : (d,e,f,r_4)$$

Constraint Equations on L

$$x^2 + y^2 + z^2 = r_0^2$$

$$(x-a)^2 + y^2 + z^2 - (au)^2 = r_1^2$$

$$(x-b)^2 + (y-c)^2 + z^2 - (bu+cv)^2 = r_2^2$$

$$(x-d)^2 + (y-e)^2 + (z-f)^2 - (du+ev+fw)^2 = r_3^2$$

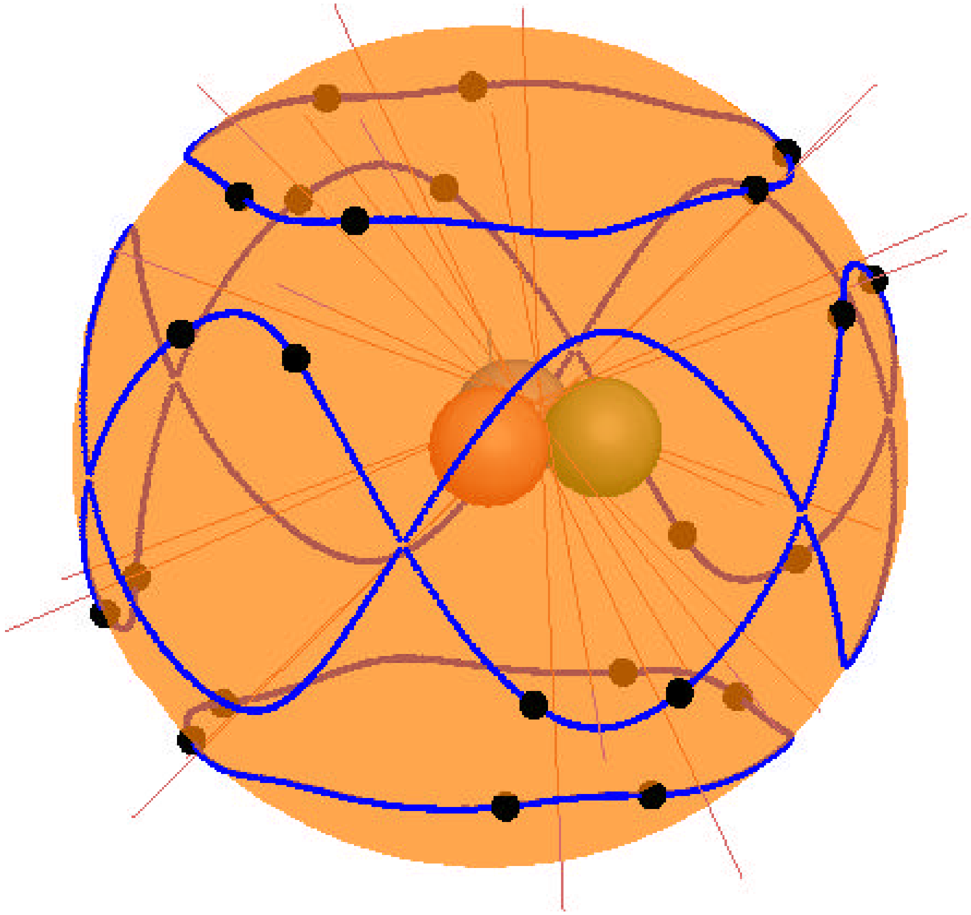
$$xu + yv + zw = 0$$

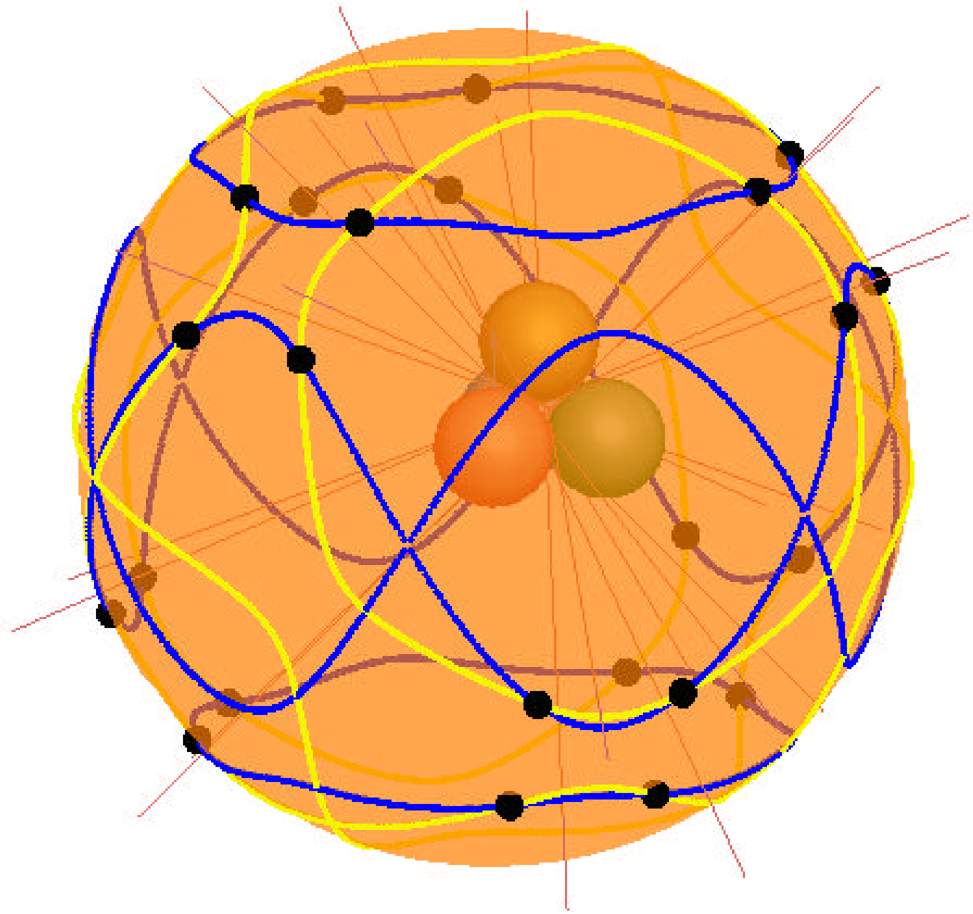
$$u^2 + v^2 + w^2 = 1$$

Algebraic Simplifications

- Use equations (2), (3) and (4) to solve for x, y, and z
- Resulting system has three equations of degree 4, 3, and 2 (Bezout bound 24)
- But if (u,v,w) solves the system, then so does (-u,-v,-w)…

Structure of surface of line tangents to 3 spheres on Gauss sphere

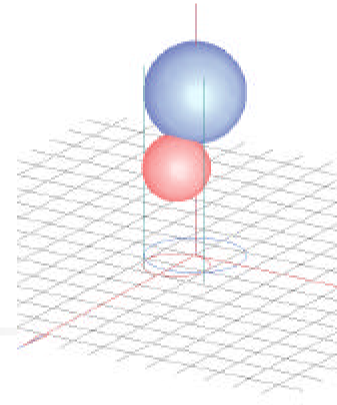




L-Lppp Problem

- Construct a line from another line and up to 3 points
- Subcases, by LL constraints:
  - L-Lpp: The lines are parallel; clearly 2 solutions maximum
  - L-Lpp: A distance is required; need good understanding of a kinematic curve
  - L-Lppp: No distance is required (includes perpendicular); intersect 3 of the L-Lpp curves

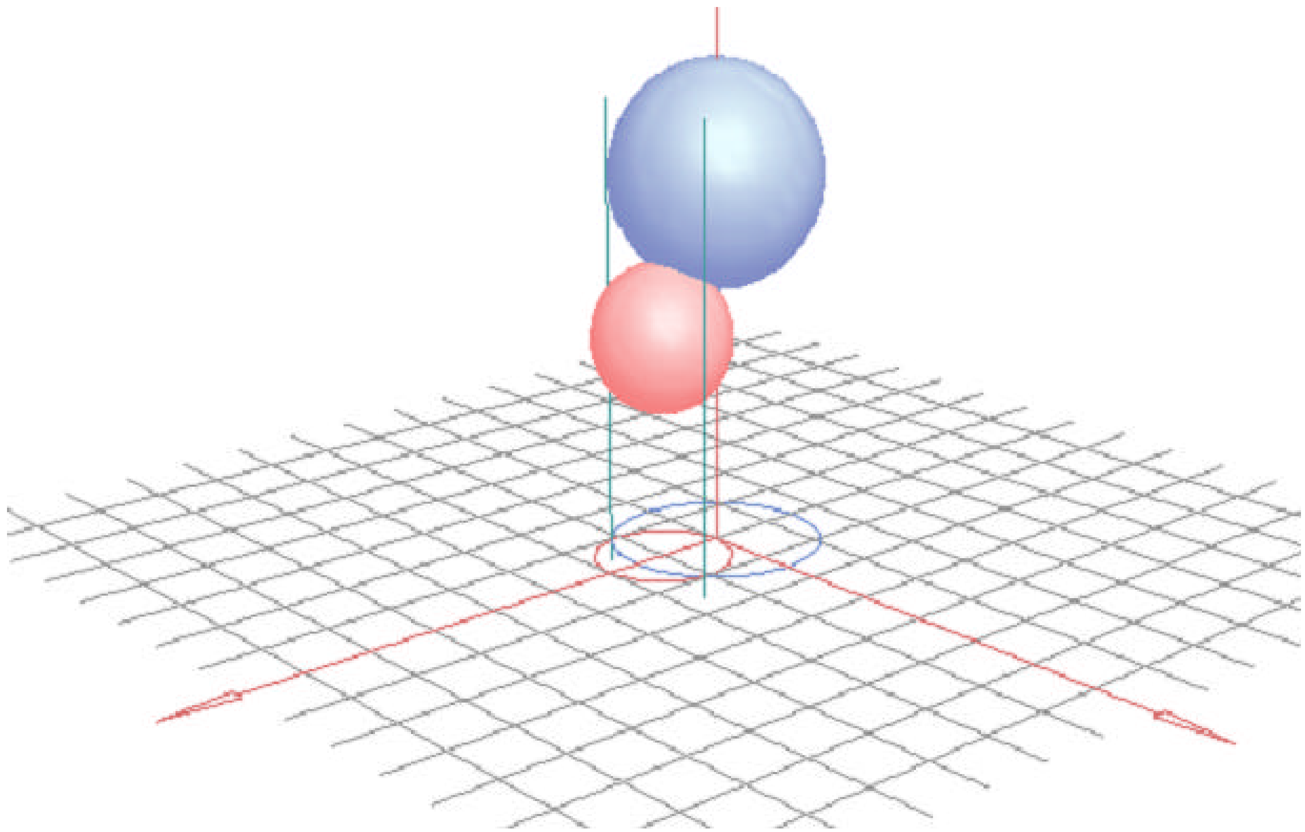
Subcase LL Parallel



- Take a plane perpendicular to the fixed line
- Sphere silhouettes intersect in up to two points
- Up to two solutions

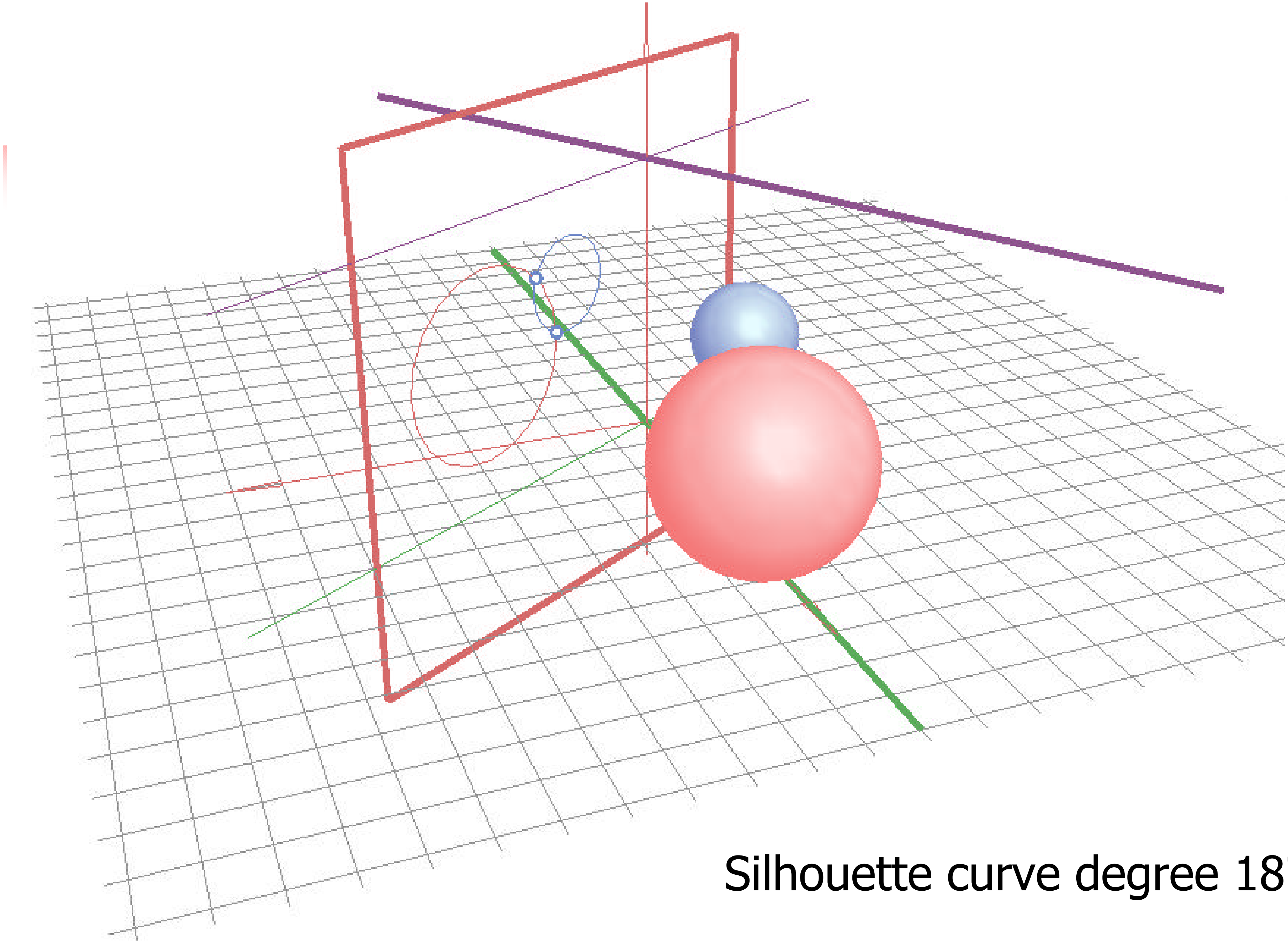
Main Tool

- Given 2 spheres and a direction, find the two tangents in that direction



Subcase LL Distance

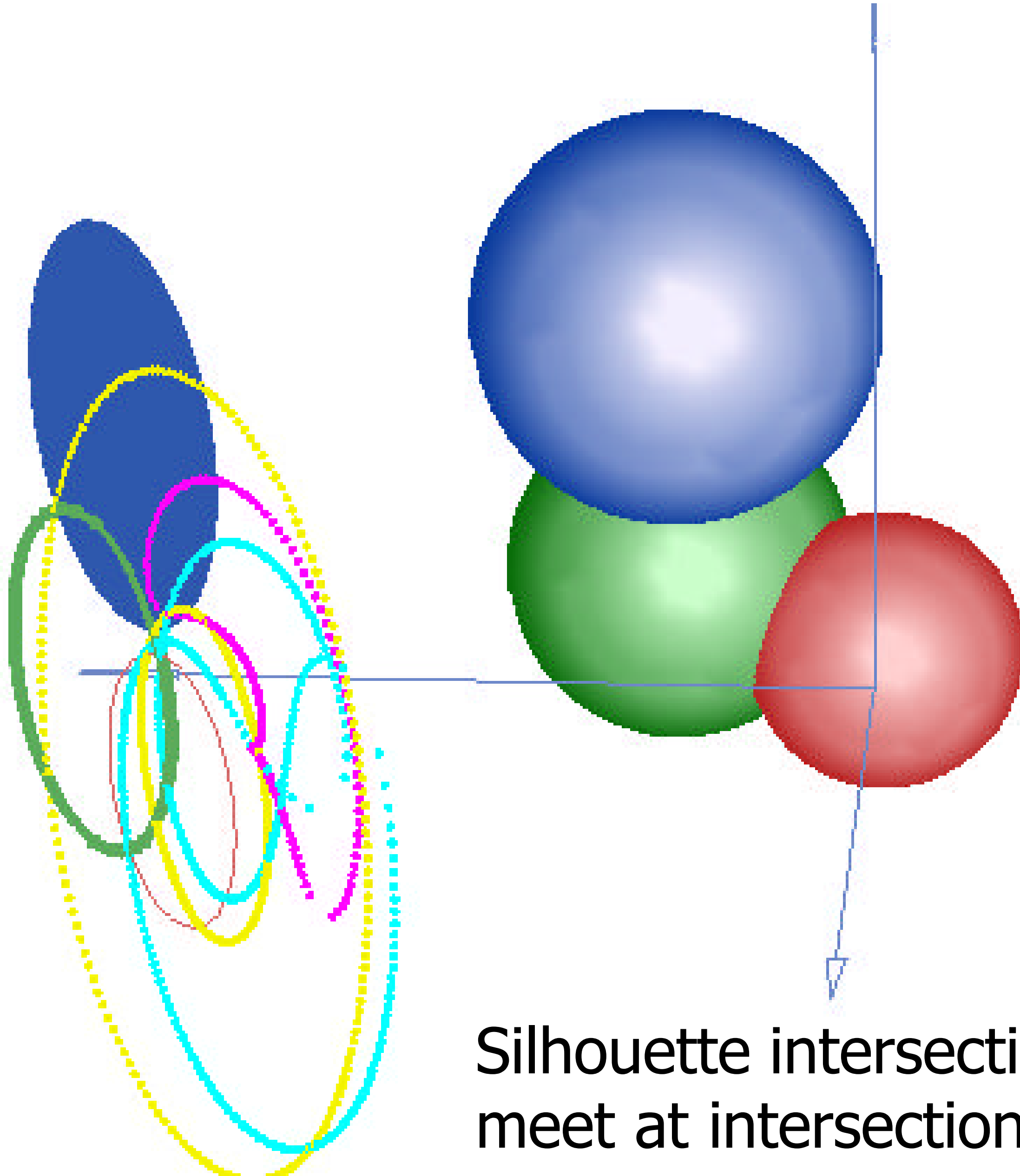
- Only 2 spheres needed
- Fix plane at complement angle to fixed line
- Rotate the 2 spheres around the fixed axis yielding silhouette intersection curves
- Intersect with horizontal line



Silhouette curve degree 18?

No LL Distance

- Additional constraint from a third sphere (point with distance)
- Intersect the silhouette intersection pairs
- No degree estimates



Silhouette intersection curves meet at intersections of sought tangents