A Hamiltonian formulation of the dynamics of spatial mechanisms using Lie Groups and Screw Theory

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Abstract

In this paper two main topics are treated. In the first part we give a synthetic presentation of the geometry of rigid body motion in a projective geometrical framework. An important issue is the geometric approach to the identification of twists and wrenches in a Lie group approach and their relation to screws. In the second part we give a formulation of the dynamics of multibody systems in terms of *implicit* port controlled Hamiltonian system defined with respect to Dirac structures.

1 Introduction

The geometry of rigid body motion was one of the main topics in the developments of geometry in the nineteenth century (see the remarquable historical perpective by R. Ziegler [47]). One of

the comprehensive theories was proposed by Sir R.S. Ball in the *theory of screw* exposed in his famous book [3]. In the following century this subject remained atopic of active research as well for the kinematic and static as well as for the dynamical models of multibody systems especially in the robotics community. It is characterized by a variety of subtle geometric approaches. One of them is the theory of Lie groups [17] which turned out to be very useful for the study of motion and robotics [32]. The theory of screws has the advantage that it is based on the geometry of lines and is therefore geometrically attractive, giving intuitive insight in the mechanisms [22] [37] [19]. On the other hand, Lie groups are analytically very handy. A formal relation between the two is shown in Sec.2.

Concerning the formulation of the dynamical models, additonal structure has to be added. For nstance in the Lagrangian formulation, the symplectic canonical form on the space of generalized coordinates and velocities or in the standard hamiltonian approach the corresponding symplectic Poisson bracket. In section Sec.3 we shall briefly indicate how another geometrical structure, called *Dirac structure* [9] may be used to formulate the dynamics of multibody systems in terms of an implicit port controlled Hamiltonian system [29].

2   Rigid body motions

Rigid bodies motions have been studied in the past using different techniques like skrew theory [3] and Lie Groups [32]. Screw theory uses the projective extension of the Eucledian three dimensional space. Since in this work we relate the two approaches, we first start with a formal explanation relating the usual conception of an Eucledian space to its projective extension.

In what follows, it is easy to grasp the idea for the lower two dimensional case considering a cinema with an infinitely extended screen (the two dimensional Eucledian space). A point on the screen is actually a beam passing throw the projector (the origin of the vector space of dimension three which embeds the screen). The plane parallel to the screen and passing through the projector is called the improper hyperplane.

2.1   Projective geometry and Eucledian Spaces

An extention to the conception we have of the three dimensional Eucledian spaces can be found in projective geometry[1]. To talk about the 3D Eucledian world in a projective setting, we need three ingredients:

- A real vector space $\mathcal{V}^4$ called the *supporting vector space* of dimension 4 from which we exclude the origin.

- An equivalence relation on $\mathcal{V}^4 - \{0\}$: $v_1 \sim v_2 \Leftrightarrow \exists \alpha \in \mathbb{R} \neq 0$ s.t. $v_1 = \alpha v_2$.

- A *polarity P*, which is a 2 covariant, symmetric tensor defined on $\mathcal{V}^4$ which in the sequel will be taken semipositive defined and of rank 1.

The basic transformations between points of projective spaces are defined as injective linear transformations between the supporting vector spaces. These transformations must be injective to prevent that the subspace corresponding to the kernel of the transformation is mapped to the 0 element of the codomain which is NOT a valid element of the projective space[2]. These kinds of transformations are called *homographies* or *collineations* and in our case are mappings from $\mathcal{V}^4$ to

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[1]Caley in 1859, by introducing the concept of an "absolute", showed that projective geometry is the most general.
[2]It will be shown later that to have a proper definition of the projective space, the 0 element must be excluded.

$\mathcal{V}^4$. Maps from $\mathcal{V}^4$ to the dual $\mathcal{V}^{4*}$ are instead called *correlations*. A symmetric correlation as P is called *polarity*.

2.1.1 Improper hyperplane

Using the polarity P, it is possible to consider the vectors p^i belonging to the quadric defined by the polarity P which is called *the absolute*:

$$P_{ij}p^i p^j = 0 \qquad p^i \in \mathcal{V}^4.$$

The absolute, is a three dimensional subspace of $\mathcal{V}^4$ which is called the *improper hyperplane* and it is indicated with $\mathcal{I}^3 \subset \mathcal{V}^4$. This hyperplane represents the "points at infinity".

The improper hyperplane splits $\mathcal{V}^4$ in two disjointed semi-spaces which we will call respectively *positive* semi-space and indicate it with $\mathcal{I}^+$ and *negative* semi-space and indicate it with $\mathcal{I}^-$.

The three dimensional projective space is defined as the quotient space of $\mathcal{V}^4$ excluding the origin with respect to the defined equivalent relation:

$$\mathcal{P}^3 := \frac{\mathcal{V}^4 - \{0\}}{\sim}$$

In a purely projective setting, without considering the polarity, all points are of the same type. Considering the polarity, we can make a distinction between finite points and infinite points. Infinite points are those whose representative in $\mathcal{V}^4$ belong to $\mathcal{I}^3$ and finite points are the others. We indicate with $\mathcal{P}_F$ the finite points and with $\mathcal{P}_\infty$ the points at infinity. Clearly we have that $\mathcal{P} = \mathcal{P}_F \cup \mathcal{P}_\infty$. Furthermore, for each finite point $p \in \mathcal{P}_F$ ($P_{ij}p^i p^j \neq 0$), there are representatives $v \in \mathcal{V}^4$ belonging either to $\mathcal{I}^+$ or to $\mathcal{I}^-$. We can define the sign function σ for elements $v \in \mathcal{V}^4$:

$$\sigma(v) := \begin{cases} +1 & v \in \mathcal{I}^+ \\ 0 & v \in \mathcal{I}^3 \\ -1 & v \in \mathcal{I}^- \end{cases} \tag{1}$$

2.1.2 Adjoint polarity

Associated with the polarity P, one defines its adjoint[3]Q which is a 2 contravariant, symmetric, semipositive tensor of rank 3. Once a proper base $\{e_x, e_y, e_z, e_0\}$ for $\mathcal{V}^4$ is chosen, the representations of P and Q become:

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

In the same coordinates, a vector of $\mathcal{V}^4$ has the form $(x, y, z, \alpha)^T$ and if $\alpha = 0$ the vector belongs to $\mathcal{I}^3$.

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[3]If instead of 0 on the diagonal elements of P and Q we substitute ϵ, we see that $QP = PQ = \epsilon I$ where I is the identity matrix.

2.1.3 Points and Free vectors

It is possible to associate to each pair of finite vectors in $\mathcal{P}_F$ a unique element of $\mathcal{I}$:

$$f : \mathcal{P}_F \times \mathcal{P}_F \rightarrow \mathcal{I} \; ; \; (p, q) \mapsto \frac{p}{||p||\sigma(p)} - \frac{q}{||q||\sigma(q)} \tag{3}$$

where p, q are any representatives and $|| \cdot ||$ represents the P-norm. It is possible to see that the previous operation is indeed independent of the representatives of the points and therefore well defined. Note that the difference of two points can be calculated using the vector structure of $\mathcal{V}^4$. Furthermore, the improper hyperplane without the equivalence relation and with the origin of $\mathcal{V}^4$, gets the meaning of the vector space of free-vectors. In the usual coordinates, this means that if

$$p = \begin{pmatrix} \alpha_p x_p \\ \alpha_p y_p \\ \alpha_p z_p \\ \alpha_p \end{pmatrix} \text{ and } q = \begin{pmatrix} \alpha_q x_q \\ \alpha_q y_q \\ \alpha_q z_q \\ \alpha_q \end{pmatrix}$$

we have $||p||\sigma(p) = \alpha_p$ and $||q||\sigma(q) = \alpha_q$ and therefore

$$f(p, q) = \begin{pmatrix} x_p - x_q \\ y_p - y_q \\ z_p - z_q \\ 0 \end{pmatrix}$$

Note that indeed the last component is equal to zero which confirms the fact that the element belongs to $\mathcal{I}$. It is usual to use the notation:

$$p - q := f(p, q)$$

for obvious reasons, but the normalisation of Eq.3 before the subtraction is essential to make the operation intrinsically defined.

2.1.4 Lines

A line $l \subset \mathcal{P}^3$ in a projective context is nothing else than a one dimensional subspace which corresponds to a two dimensional subspace $L \subset \mathcal{V}^4$ (without the origin) of the supporting vector space $\mathcal{V}^4$. In a projective context it is also possible to talk about lines at infinity when $L \subset \mathcal{I}^3$. As a consequence, a line can be described both as the subspace spanned by two points of $\mathcal{V}^4$ or as the intersection of two hyperplanes of $\mathcal{V}^4$. As in every $2n$ dimensional vector space an n dimensional subspace is self-dual, so is a line in $\mathcal{P}^3$ a self dual entity.

It is possible to show that using the previous coordinates, given two distinct points x, y belonging to a line where

$$x = \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} \bar{y} \\ 1 \end{pmatrix} \text{ where } \bar{x}, \bar{y} \in \mathbb{R}^3$$

the corresponding line can be identified homogeneously to a vector of the form

$$\alpha \begin{pmatrix} \bar{y} - \bar{x} \\ \bar{x} \wedge \bar{y} \end{pmatrix} \text{ where } \alpha \in \mathbb{R} - \{0\}. \tag{4}$$

Eq.4 can be clearly written as:

$$\begin{pmatrix} \bar{y} - \bar{x} \\ \bar{x} \wedge (\bar{y} - \bar{x}) \end{pmatrix} = \begin{pmatrix} \bar{u} \\ \bar{r} \wedge \bar{u} \end{pmatrix}. \tag{5}$$

It is easy to see that choosing instead of $\bar{x}$ and $\bar{y}$ other two distinct points on the same line in the usual geometrical sence, the vector l defined as in Eq.4 is the same if a proportional multiplication constant is used. This implies that the space of lines has somehow a projective nature by itself, but it should be noticed that the linear combination of two lines as expressed in Eq.4 is in general NOT a line.

2.1.5 Euclidean product on the improper hyperplane

It is now possible to define a proper non-singular, internal product for $\mathcal{I}^3$ which gives rise to the scalar product which characterises a proper Euclidean Space. In the previous canonical coordinates, an element $v_i \in \mathcal{I}^3$ is characterised by the last component equal to zero. We can associate to $v_i \in \mathcal{I}^3$ the subspace of hyperplanes $\mathfrak{H}_{v_i}$ for which v_i is what is called the *polar* with respect to Q:

$$h \in \mathfrak{H}_{v_i} \Leftrightarrow v_i = Qh$$

where Q is the adjoint of P. We can now define the scalar product of two vectors v_1, v_2 in $\mathcal{I}^3$ as:

$$\langle v_1, v_2 \rangle := \sqrt{h_1^T Q h_2} \qquad h_1 \in \mathfrak{H}_{v_1}, h_2 \in \mathfrak{H}_{v_2} \tag{6}$$

It is easy to see that the previous definition is well posed since it is independent on the elements h_1 and h_2 due to the structure of Q.

2.1.6 The Euclidean space

It is now possible to define the following Euclidean three dimensional space as the pair $(\mathcal{E}^3, \langle \rangle)$ such that $\mathcal{E}^3 := \mathcal{P}^3$, and $\langle , \rangle$ is the scalar product just defined on $\mathcal{E}_*^3 := \mathcal{I}^3$ which is treated as a vector space. As with $\mathcal{P}$, we indicate with $\mathcal{E}_F$ finite points and with $\mathcal{E}_\infty$ infinite points and again:

$$\mathcal{E} = \mathcal{E}_F \cup \mathcal{E}_\infty.$$

Proper collineations of the Euclidean space are defined as the ones which keep the polarity P, and therefore the improper hyperplane $\mathcal{I}^3$ invariant. It can be seen that in the previous coordinates a proper Euclidean collineation has the form:

$$c : \mathbb{R}^4 \to \mathbb{R}^4; q \mapsto \begin{pmatrix} \sigma R & p \\ 0 & 1 \end{pmatrix} q \tag{7}$$

where $R \in SO(3)$, the set of positively oriented orthonormal matrices, σ can be either equal to $+1$ for an orientation preserving transformation or -1 for a transformation changing orientation and $p \in \mathbb{R}^3$.

Looking at the expression of Eq.4 for a line, it is possible to find how these coordinates transform. Using Eq.7 and Eq.4, it is possible to see that the mapping of a line corresponds to the mapping of points as reported in Eq.7 becomes:

$$\begin{pmatrix} \bar{y}_2 - \bar{x}_2 \\ \bar{x}_2 \wedge \bar{y}_2 \end{pmatrix} = \begin{pmatrix} \sigma R & 0 \\ \sigma \tilde{p} R & R \end{pmatrix} \begin{pmatrix} \bar{y}_1 - \bar{x}_1 \\ \bar{x}_1 \wedge \bar{y}_1 \end{pmatrix} \tag{8}$$

where the operator tilde is such that $\tilde{p}$ is a skew-symmetric matrix and it is such that $\tilde{p}a = p \wedge a$ for each $a \in \mathbb{R}^3$.

Using the matrix rotation group $SO(3)$ and the intrinsically defined operation of Lie brackets [21], it is possible to orient $\mathcal{E}$ by defining a $3-$form Ω on $\mathcal{I}^3$. We suppose that the coordinates (e_x, e_y, e_z, e_0) chosen are such that $\Omega(e_x, e_y, e_z) = +1$ and $e_z = [e_x, e_y]$ where $[,]$ represents the Lie algebra commutator of $so(3)$. In these coordinates, the operation of vector product "$\wedge$" in $\mathbb{R}^3$ results the usual one.

2.2 Displacements of rigid bodies

2.2.1 Euclidean system

As shown in [24] and [39], to describe the relative motion of 3D rigid parts in a geometrically intrinsic way, it is necessary to consider a set of Euclidean spaces of dimension three. This set is called a *Euclidean system* in [38] and indicated with

$$S^m := \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m\} \tag{9}$$

The concept of an observer is very important and is coupled to the concept of space: an observer is identified with a Euclidian space in which he is rigidly connected together with all those objects which are always stationary in relation to him. A set of objects which are never changing position in relation to one another can be considered as a single entity from a kinematic point of view.

For these reasons and for more formal ones that should become clear in the sequel, we consider as many Euclidian spaces as there are bodies moving relative to one another. All these Euclidean spaces are such that they have the improper hyperplane of their projective extension in common. An object will be a constant subset of a Euclidian space together with a mass density function which associates a value to each of the body's points.

It is also possible to define, using the Killing form [38], an *Equivalence Inertial Relation* on the set of all possible Euclidean spaces. Elements of the same class can only translate with constant velocities with respect to one another.

One Euclidean class which is particularly important and is called *Inertial class* is the class where physical laws are invariant. We will usually indicate a particular representative of this class with $\mathcal{E}_0$.

In the projective setting, these Euclidean spaces can be seen as a set of four dimensional vector spaces with a common polarity P and a common improper hyperplane.

2.2.2 A rigid body

In the presented setting, a rigid body is is a subset of a Euclidean space $\mathcal{E}_i \in S^m$ together with a mass density function which associates a value to each of the body's points:

$$\rho_i : \mathcal{E}_i \to \mathbb{R}^+$$

The mass density function is relevant for dynamic considerations. For kinematic considerations, it is sufficient to define a proper body B_i as a constant subset of a Euclidian space $\mathcal{E}_i$ instead.

2.2.3 Coordinate functions

A coordinate Ψ_i for the space $\mathcal{E}_i$ is a 4-tuple of the form

$$\Psi_i = (o_i, \hat{x}_i, \hat{y}_i, \hat{z}_i) \qquad o_i \in \mathcal{E}_i, \hat{x}_i, \hat{y}_i, \hat{z}_i \in \mathcal{E}_{*i}$$

where $\hat{x}_i, \hat{y}_i, \hat{z}_i$ should be linear independent. If furthermore $\hat{x}_i, \hat{y}_i, \hat{z}_i$ are orthonormal, we call the coordinates *orthonormal*. Eventually we call the coordinates positively oriented iff $\Omega(\hat{x}_i, \hat{y}_i, \hat{z}_i) = +1$ where Ω is the form orientating the Euclidean space. If not otherwise specified, we will always consider positively oriented, orthonormal coordiantes systems. We can now define a base for $\mathcal{V}_i^4$ dependent on Ψ_i using the four linear independent vectors:

$$\hat{x}_i, \hat{y}_i, \hat{z}_i, \frac{o_i}{||o_i||\sigma(o_i)} \in \mathcal{V}^4$$

of $\mathcal{V}^4$.

The map which associate a representative of a point of $\mathcal{E}$ to its numerical representation using this base, is indicated with ψ_i:

$$\psi_i(p) = \big(p_x, p_y, p_z, \alpha_p\big) \Rightarrow p = p_x\hat{x}_i + p_y\hat{y}_i + p_z\hat{z}_i + \alpha\frac{o_i}{||o_i||\sigma(o_i)}$$

Clearly we have that:

$$\psi_i^{-1}(\cdot, \cdot, \cdot, 0) = \mathcal{I}^3 \qquad \forall i = 1, \ldots, m \tag{10}$$

and therefore this coordinates are characterised by the fact that the last component equal to zero represents a point at infinity.

2.2.4 Displacements of rigid bodies

We can associate the relative positions of bodies belonging to two different Euclidean spaces $\mathcal{E}_i$ and $\mathcal{E}_j$ to a positive isometry between $\mathcal{E}_i$ and $\mathcal{E}_j$ [24] [38]:

$$h_i^j : \mathcal{E}_i \to \mathcal{E}_j \quad ; \quad p_i \mapsto h_i^j(p_i)$$

such that h_i^j is an isometry:

$$||(h_i^j(p_i) - h_i^j(q_i))|| = ||(p_i - q_i)|| \,\forall p_i, q_i \in \mathcal{E}_i$$

and it is orientation preserving

$$(h_i^j(p_i) - h_i^j(o_i)) \wedge (h_i^j(q_i) - h_i^j(o_i)) = h_i^j((p_i - o_i) \wedge (q_i - o_i) + o_i) - h_i^j(o_i)$$

We indicate the set of positive isometries from $\mathcal{E}_i$ to $\mathcal{E}_j$ with $SE_i^j(3)$. In the case $i = j$, $SE_i^i(3)$ will be indicated with $SE_i(3)$ and is a Lie group [21].

Once we choose a reference relative position $r_i^j \in SE_i^j(3)$, using the commutation diagram shown in Fig.1, it is possible to associate with each element $h_i^j \in SE_i^j(3)$ an isometry in the Lie group $SE_i(3)$:

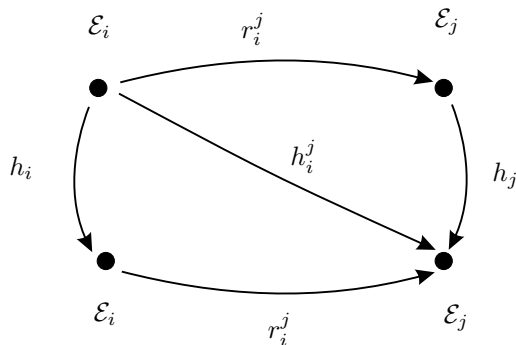$$h_i^{i,j} = r_i^{j^{-1}} \circ h_i^j \tag{11}$$

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Figure 1: The commutation diagram of the relation among elements of the Lie groups and the reference.

In an analoguous way, one may associate with each element $h_i^j \in SE_i^j(3)$ an isometry of $SE_j(3)$:

$$h_i^{j,j} = h_i^j \circ r_i^{j^{-1}} \tag{12}$$

Note that by NO means is this mapping intrinsic [39]: it depends on the reference choice r_i^j!

An element $h_i^j \in SE_i^j$ is an intrinsic, coordinate free representative of the relative position of $\mathcal{E}_i$ with respect to $\mathcal{E}_j$.

If we assign to each space $\mathcal{E}_i$ a coordinate ψ_i with the properties reported in Eq.10 and we assume that in a certain instant the relative position is $h_i^j \in SE_i^j$, a point $p \in \mathcal{E}_i$ corresponds uniquely to the point $h_i^j(p) \in \mathcal{E}_j$. The usual change of coordinate from Ψ_i to Ψ_j for a certain relative configuration is therefore:

$$c : \mathbb{R}^4 \to \mathbb{R}^4 : p^i \mapsto (\psi_j \ o \ h_i^j \ o \ \psi_i^{-1})(p^i)$$

If the coordinate systems Ψ_i and Ψ_j are positively oriented and orthonormal and since $h_i^j \in SE_i^j(3)$ is a positively oriented isometry, then the mapping c is linear and is equal to a matrix of the form:

$$H_i^j = \begin{pmatrix} R_i^j & p_i^j \\ 0 & 1 \end{pmatrix} \tag{13}$$

where $R_i^j \in SO(3)$, $p_i^j \in \mathbb{R}^3$ and therefore $H_i^j \in SE(3)$. Note that p_i^j characterises how the origin of Ψ_i is mapped to in frame Ψ_j.

It is important to notice that we can easily relate the restrictions that in the coordinates Ψ_i the polarity P gets the form of Eq.2 together with the Euclidean product defined in Eq.6, to the usual concept of orthonormal bases for the Euclidean spaces.

This can be seen considering the three vectors $\hat{x}_i := \psi_i^{-1}((1,0,0,0))$, $\hat{y}_i := \psi_i^{-1}((0,1,0,0))$ and $\hat{z}_i := \psi_i^{-1}((0,0,1,0))$ belonging to $\mathcal{E}_*^3$. It is easy to see that for the defined coordinates we have that $\hat{x}_i$, $\hat{y}_i$ and $\hat{z}_i$ are an orthonormal set since they have norm one and are orthogonal with respect to each other. Furthermore, the vector $\psi_i((0,0,0,1))$ belongs to the dual space of $\mathcal{I}^3$ and has P-norm equal to 1.

It is possible to see that we can define a bijective relation between elements of $\mathcal{E}_*^3$ and elements of $\mathcal{V}^4$ which have, using a coordinate ψ_i, the last component[4] equal to $\alpha > 0$. If we consider the

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[4]Note that both element with the last component equal to α and $-\alpha$ have P-norm equal to α

last component as a scaling factor respect to the P-norm, it is reasonable to give elements with the last component equal to one a special role.

For what said, using the coordinates ψ_i and considering elements with $\alpha = 1$ as special representative of the equivalent classes which are elements of $\mathcal{E}$, we have all the properties of an orthonormal base for the elements $\hat{x}_i, \hat{y}_i$ and $\hat{z}_i$.

2.3 Twists

2.3.1 Velocities and twists

So far we have considered relative positions without introducing motion. The latter is clearly a change of relative position. To be able to talk about change, we need to consider a time set I which will be an open interval of $\mathbb{R}$. We can then consider curves in $SE_i^j(n)$ parameterized by $t \in I$.

Definition 1 (Relative Motion). *We call a differentiable function of the following form[5]:*

$$h_i^j : I \to SE_i^j(n)$$

a relative motion of space i with respect to j

Since relative motions are differentiable, we can consider the velocity vector of the curve at h_i^j which we indicate as

$$\dot{h}_i^j \in T_{h_i^j} SE_i^j(n).$$

The vector $\dot{h}_i^j(t)$ is a bounded vector dependent on the current configuration $h_i^j(t)$. Using the identification of relative positions with isometries in $SE_j(3)$ reported in Eq.12, one can associate to a local velocity $\dot{h}_i^j(0)$ an element of $se_j(n)$, the Lie algebra of the Lie group $SE_j(3)$ of $\mathcal{E}_j$ called *twist in $j-$frame*. Analogously, using the identification reported in Eq.11 one can associate an element of $se_i(n)$, the Lie algebra corresponding to the Lie group $SE_i(n)$ of $\mathcal{E}_i$ called *twist in $i-$frame*. These elements may be defined in a geometrical way [21, 24, 38], but in this text we will define them in their numerical representation below. We denote with $t_i^{j,k} \in se_j(3)$ the twist which describes the motion of $\mathcal{E}_i$ with respect to $\mathcal{E}_k$ as an element of $se_j(3)$. It is shown in [39, 38] that special care should be considered in the definitions of intrinsic twists.

2.3.2 Numerical representation of twists

We have shown in the previous section that after we have chosen coordinates, we can relate a relative configuration $h_i^j \in SE_i^j(n)$ to a matrix belonging to $SE(3)$. Namely, we can associate the relative position $h_i^j \in SE_i^j(n)$ to the matrix H_i^j which corresponds to the change of coordinates from Ψ_i to Ψ_j.

Consider now a point $p \in \mathcal{E}_i$. If we take the numerical representation P^i of this point p using the coordinates Ψ_i, we clearly have that $\dot{P}^i = 0$ since p is a fixed point of $\mathcal{E}_i$. We can then write the numerical representation of this point in the coordinate Ψ_j fixed with $\mathcal{E}_j$. Using the coordinates change matrix, we obtain:

$$P^j = H_i^j P^i$$

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[5] Note the abuse of notation here: h_i^j has been used both as an element of $SE_i^j(n)$ and as a function from I to $SE_i^j(n)$.

$\tilde{T}$	$\dot{H}_i^j$		Notation
$\tilde{T} := H_j^i \dot{H}_i^j$	$\dot{H}_i^j = H_i^j \tilde{T}$	$\dot{P}^j = H_i^j (\tilde{T} P^i)$	$\tilde{T}_i^{i,j}$
$\tilde{T} := \dot{H}_i^j H_j^i$	$\dot{H}_i^j = \tilde{T} H_i^j$	$\dot{P}^j = \tilde{T}(H_i^j P^i)$	$\tilde{T}_i^{j,j}$

Table 1: The possible forms of twists

Differentiating over time, we have:

$$\dot{P}^j = \dot{H}_i^j P^i$$

where we used the fact that $\dot{P}^i = 0$ and where we consider the following representative for points:

$$P^k := \begin{pmatrix} p^k \\ 1 \end{pmatrix} \qquad k = i, j$$

Before proceeding, we need a result which is well known and useful for what follows.

Theorem 1. *Given a matrix $H \in SE(3)$ continuously depending on a scalar t, we have that:*

$$\dot{H}H^{-1}, H^{-1}\dot{H} \in se(3).$$

where

$$se(3) := \left\{ \begin{pmatrix} \tilde{\omega} & v \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{4 \times 4} \ s.t. \ \omega, v \in \mathbb{R}^3 \right\}$$

and $\tilde{\omega}$ is the skew-symmetric matrix such that for any vector x, $\tilde{\omega} x = \omega \wedge x$.

The set of matrices $se(3)$ which result from the product of a matrix belonging to $SE(3)$ times the derivative of its inverse (or vice versa) is called the Lie algebra of $SE(3)$. The elements belonging to $se(3)$ are the numerical representative of the twists in $i-$frame or in $j-$frame. We will use the following notation:

$$T := \begin{pmatrix} \omega \\ v \end{pmatrix} \Rightarrow \tilde{T} = \begin{pmatrix} \tilde{\omega} & v \\ 0 & 0 \end{pmatrix} \in se(3)$$

The vector T is the vector of Plücker coordinates of the twist $\tilde{T}$ and we will abusively also call a twist. We have therefore a vector representation of a twist and a matrix representation: respectively $T \in \mathbb{R}^6$ and $\tilde{T} \in \mathbb{R}^{4 \times 4}$. For the twist of $\mathcal{E}_i$ with respect to $\mathcal{E}_j$ we consider $\dot{P}^j = \dot{H}_i^j P^i$

and we have the two possibilities reported in Tab.1. Furthermore

$$\dot{P}^j = H_i^j(\tilde{T}P^i) \Rightarrow \begin{pmatrix} \dot{p}^j \\ 0 \end{pmatrix} = \begin{pmatrix} R_i^j & p_i^j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\omega} & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p^i \\ 1 \end{pmatrix} \Rightarrow \dot{p}^j = R_i^j(\omega \wedge p^i) + R_i^j v$$

and

$$\dot{P}^j = \tilde{T}(H_i^j P^i) \Rightarrow \begin{pmatrix} \dot{p}^j \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{\omega} & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R_i^j & p_i^j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^i \\ 1 \end{pmatrix} \Rightarrow \dot{p}^j = \omega \wedge (R_i^j p^i + p_i^j) + v$$

The rotational part is the usual angular rotation. The linear velocity part of the twist $T_j^{i,i}$, $v = p_j^i \wedge \omega_j^{i,i} + \dot{p}_j^i$ is the velocity of a point q rigidly connected to Ψ_j and passing through the origin of Ψ_i.

2.3.3 Changes of coordinates for twists: the adjoint representation

The twists in $i-$frame and in $j-$frame are related by a linear map called *adjoint mapping* or *adjoint representation of the Lie group $SE(3)$ on $se(3)$* [21, 38] and have a numerical representation relating the vector representations of the twists as follows:

$$T_i^{j,j} = Ad_{H_i^j} T_i^{i,j}$$

where

$$Ad_{H_i^j} := \begin{pmatrix} R_i^j & 0 \\ \tilde{p}_i^j R_i^j & R_i^j \end{pmatrix}.$$

The matrix $Ad_{H_i^j}$ is dependent on H_i^j and this implies that if the relative position of $\mathcal{E}_i$ with respect to $\mathcal{E}_j$ changes, H_i^j will change and as a consequence, the matrix $Ad_{H_i^j}$ will be time varing. We will need the time derivative of $Ad_{H_i^j}$ for expressing the dynamics of a rigid body.

$$\left(\dot{\overline{Ad_{H_i^j}}} \right) = Ad_{H_i^j} \, ad_{T_i^{i,j}}$$

where ad is called *the adjoint representation of the Lie algebra* [21]

$$ad_{T_i^{k,j}} := \begin{pmatrix} \tilde{\omega}_i^{k,j} & 0 \\ \tilde{v}_i^{k,j} & \tilde{\omega}_i^{k,j} \end{pmatrix}$$

Note that the matrix ad_T is a function of a twist T and not of a relative position H as Ad_H.

2.4   Wrenches and their identification with twists

In order to define properly the interaction of a rigid body with the rest of the mechanism, force variables, in the sense of the action of a set of forces on a body, have to be considered. The action of a set of forces on a body is called *wrench*. The wrench is a vector of $se^*(3)$, the dual space of the space of twists, which allows to express the power resulting from the action of the wrench W on the body undergoing a trajectory $H(t)$ with twist T, is the scalar product:

$$\text{Power} = \langle W, T \rangle$$

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where $\langle,\rangle$ denotes the duality product. Using Plücker coordinates for the wrenches (in the dual basis to the Plücker basis of twists) one obtain a vector representation of the wrenches:

$$W = \begin{pmatrix} m & f \end{pmatrix}$$

where m represents the torque and f the linear force, the power may be expressed:

$$\text{Power} = WT$$

2.4.1 Changes of coordinates for Wrenches

As the wrenches belong to the dual space of twists, changes of coordinates induces the adjoint change of wrenches. Hence in the numerical representation, the transformation of wrenches is given by the transposition of the adjoint mapping:

$$(W^i)^T = Ad^T_{H^j_i}(W^j)^T$$

It is important to note that if the mapping $Ad_{H^j_i}$ was mapping twists from Ψ_i to Ψ_j, the transposed maps wrenches in the opposite direction: from Ψ_j to Ψ_i ! This is because Wrenches are duals of Twists and they are pulled-back.

2.4.2 Bi-invariant forms on $se(3)$ and reciprocity of twists

It has been shown in [23] that the space of bi-invariant, and therefore intrinsic, 2-forms defined on $se_i(3)$ is a linear combination of the hyperbolic metric $\mathfrak{H}_{ij}$ and the Killing form $\mathfrak{K}_{ij}$. In the coordinates we used so far, their numerical representations are:

$$\mathfrak{H} = \frac{1}{2} \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix} \text{ and } \mathfrak{K} = -4 \begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix}$$

And a general bi-invariant metric is given by:

$$\mathfrak{B} = h\mathfrak{H} + k\mathfrak{K} \qquad h, k \in \mathbb{R}$$

Among all the possible metrics, there are two which have special properties which allow to directly relate Lie groups to the theory of screws.

If we consider the kernel of $\mathfrak{B}$, this is different than the trivial zero element of $se_i(3)$ iff $h = 0$. A proportional element of the Killing form has therefore a first special property. Furthermore, among all the $\mathfrak{B}$s there is just one metric which is generated by an involutive map, which means that an expression of its inverse is equal to the form itself and this is namely obtained when $k = 0$ and $h = 2$. This involutivity property results fundamental to relate Lie groups to screws and corresponds to the selft-duality of lines.

We indicate $\bar{\mathfrak{H}} = 2\mathfrak{H}$ and $\bar{\mathfrak{K}} = -\frac{1}{4}\mathfrak{K}$. These two special 2-forms are fundamental for what follows.

2.4.3 Identification of twists and wrenches

Using the involutive map $\bar{\mathfrak{H}}$ and the duality product on $se(3)$, one may identify the twists with the wrenches. In the sequel we shall use this identification in order to identify both twists and wrenches with screws.
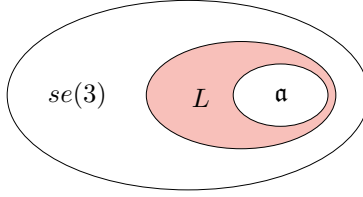
Figure 2: The relation of the sets in $se(3)$.

Therefore one may define the following map, denoted by $\bar{\mathfrak{H}}^{\#}$ from twists $T \in se(3)$ to wrenches $W = \bar{\mathfrak{H}}^{\#}(T)$, as follows:

$$\langle W, T' \rangle = \bar{\mathfrak{H}}(T, T') \quad \forall T' \in se(3) \tag{14}$$

It is remarkable that, using this identification, one may identify the adjoint mapping with its contragradient map, in other words the coordinate change on twists and wrenches.

2.5 From Lie Groups to Screws

The relation between twists (and wrenches) and lines in $\mathcal{P}^3$ is stated by the classical Chasles (and Poinsot) theorem. For expressing these theorems, some subspaces and subsets of $se_i(3)$ need to be defined using the bi-invariant forms [22].

2.5.1 Decomposition of $se_i(3)$

Definition 2 (Self reciprocal sub-set). *We call self reciprocal sub-set the following sub-set of $se_i(3)$:*

$$R := \{t^i \in se_i(3) \ s.t \ \mathfrak{H}_{ij} t^i t^j = 0\}.$$

Definition 3 (Axial sub-space). *We call axial sub-space the following sub-space of $se_i(3)$:*

$$\mathfrak{a} := \{t^i \in se_i(3) \ s.t. \ \mathfrak{K}_{ij} t^i t^j = 0\}.$$

It is the biggest subspace whose numerical representation is invariant for changes of the origin of the coordinates system.

Definition 4 (Lines sub-set). *We call lines sub-space the following sub-set of $se_i(3)$:*

$$L := R - \mathfrak{a}$$

Note that R and L are NOT subspaces, but $\mathfrak{a}$ is. Furthermore, $\mathfrak{a} \subset R$.

Theorem 2 (Geometrical interpretation of L). *Any element $\begin{pmatrix} \omega & r \wedge \omega \end{pmatrix}^T \in L$ is characterised by the fact that it leaves a line of $\mathcal{E}$ invariant in the direction $\alpha\omega$ and passing through r.*

From the previous theorem, we can conclude that a twist $l \in L$ is therefore corresponding to a rotation around this line.

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Theorem 3 (Geometrical interpretation of $\mathfrak{a}$). *Any element $\begin{pmatrix} 0 & v \end{pmatrix}^T \in \mathfrak{a}$ leaves free-vectors invariant.*

From the previous theorem, we can conclude that a twist $a \in \mathfrak{a}$ corresponds to a translation parallel to v.

Based on the special metrics $\bar{\mathfrak{H}}$ and $\bar{\mathfrak{K}}$, we can define an intrinsic surjective but not injective operation which is called *polar map*:

$$\mathrm{p} : se_i(3) \to \mathfrak{a} \quad ; \quad t^i \mapsto \bar{\mathfrak{H}}^{jl} \bar{\mathfrak{K}}_{ij} t^i$$

Note that we have inverted $\bar{\mathfrak{H}}$ in order to get a proper tensor map. It is now possible to express Chasles theorem as follows:

Theorem 4 (Chasles). *For any $t \in se_i(3)$, there exist a $l \in L$ and two constants $\alpha_l, \alpha_{\mathfrak{a}} \in \mathbb{R}$ such that:*

$$t = \alpha_l l + \alpha_{\mathfrak{a}} \mathrm{p}(l) \tag{15}$$

In particular, when $t \in se_i(3) - \mathfrak{a}$, l is unique.

Using the coordinates we have introduced previously, the form of an element of L will be:

$$l = \begin{pmatrix} \omega \\ v \end{pmatrix} \in L \text{ where } \omega \neq 0$$

Since $\omega \neq 0$ due to the fact that $L = R - \mathfrak{a}$, we can consider in these coordinates the orthogonal complement $\omega^\perp$ of ω. The fact that $l \in R$ results in the algebraic condition

$$\omega^T v = 0$$

in these coordinates. This implies that $v \in \omega^\perp$ and therefore there exist a unique $r \in \mathbb{R}^3$ such that $v = r \wedge \omega$. It is therefore possible to express ANY element of L in the following form:

$$\begin{pmatrix} \omega \\ r \wedge \omega \end{pmatrix} \tag{16}$$

where $\omega \neq 0$. Furthermore, as a consequence of Th.4, we can always write a twist as:

$$t = ||\omega|| \begin{pmatrix} \hat{u} \\ r \wedge \hat{u} \end{pmatrix} + ||v|| \begin{pmatrix} 0 \\ \hat{u} \end{pmatrix}$$

where $\hat{u}$ is a unit vector.

The following theorem is also very important because it allows to express a pure translation as the sum of two rotations.

Theorem 5 (Axial decomposition). *For any $a \in \mathfrak{a}$, there exist $l_1, l_2 \in L$ such that $a = l_1 - l_2$.*

As a direct consequence of Th.5, we can express any element $a \in \mathfrak{a}$ as:

$$\begin{pmatrix} 0 \\ (r_1 - r_2) \wedge \omega \end{pmatrix} = \begin{pmatrix} \omega \\ r_1 \wedge \omega \end{pmatrix} - \begin{pmatrix} \omega \\ r_2 \wedge \omega \end{pmatrix}.$$

One of the conclusions we can draw from what said so far, is that:

$$se_i(3) = \mathrm{span}\{L\}$$

and therefore we can get any twist as a linear combinations of elements of L.

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2.5.2 Decomposition of $se_i^*(3)$

The decomposition which has been shown for $se_i(3)$, can be directly mapped to $se_i^*(3)$ defining $R^* = \bar{\mathfrak{H}}^\#(R), \mathfrak{a}^* = \bar{\mathfrak{H}}^\#(\mathfrak{a})$ and $L^* = \bar{\mathfrak{H}}^\#(L)$ where $\bar{\mathfrak{H}}^\#$ corresponds to the hyperbolic metric and it is defined in Eq.14. The Poinsot theorem is then expressed:

Theorem 6 (Poinsot). *For any $w \in se_i^*(3)$, there exist a $l^* \in L^*$ and two constants $\alpha_l, \alpha_\mathfrak{a} \in \mathbb{R}$ such that:*

$$w = \alpha_l l^* + \alpha_\mathfrak{a} \mathrm{p}^*(l^*) \tag{17}$$

In particular, when $w \in se_i^*(3) - \mathfrak{a}^*$, l^* is unique.

A pure moment may be obtained as the sum of two forces.

Theorem 7 (Dual axial decomposition). *For any $a^* \in \mathfrak{a}^*$, there exist $l_1^*, l_2^* \in R^*$ such that $a^* = l_1^* - l_2^*$.*

One of the conclusions we can draw from what said so far, is that:

$$se_i^*(3) = \mathrm{span}\{L^*\}$$

2.5.3 The vector space of screw-vectors

We have seen that any element of $se(3)$ belonging to the lines sub-set L has a line associated to it, namely the line which stays invariant during this rigid rotation. To do this bijectively, we need to associate to a line a direction and a magnitude which would characterise the angular velocity of the corresponding twist belonging to L. In this way, we leave the homegeneous character of lines, and obtain the vectors $\in L$ (or L^*). These vectors are called *bound line vectors* (or *rotors*). Conversely, lines (not at infinity) are members of the projectivization[6] of L.

Considering the whole set of projective lines, including lines at infinity, and leaving their homogeneous character, the set R of self-reciprocal twists is obtained. The vectors of R are called *line vectors* [22]. Those corresponding to lines at infinity, member of $\mathfrak{a}$, are then called *free line vectors* [22] (or also free vectors). Conversely, projective lines are members of the projectivization of R.

Neither L nor R are vector spaces, but L spans a 6-dimensional vector space, that corresponds to $se(3)$. The vectors of this space are usually called *screws* [32], *screw vectors* [22] or also *motors*.

Geometrical screws, defined by a line and a pitch, are members of the projectivization of $se(3)$ [36].

2.5.4 Operations on screw-vectors

Operations on twists, or on wrenches, or between wrenches and twists can now be mapped into operations on screws.

Given two screws s_1 and s_2 associated respectively to two elements $t_1, t_2 \in se(3)$, their *reciprocal product* is defined as:

$$s_1 \bigcirc s_2 := t_1^T \bar{\mathfrak{H}} t_2$$

---

[6]With projectivization is intended the process of considering elements $a, b \in L$ equivalent iff there exist a scalar $\alpha \neq 0$ such that $a = \alpha b$.

The reciprocal product corresponds to the duality product of twists and wrenches. Let s_1 and s_2 associated respectively to $t_1 \in se(3)$ and $w_2 \in se(3)^*$,

$$s_1 \bigcirc s_2 := \langle t_1, w_2 \rangle$$

When $s_1 \bigcirc s_2 = 0$, the two screws are said to be *reciprocal*. Reciprocity of screws is used to express non-working condition. Line-vectors are self-reciprocal, this corresponds to the self-duality of lines.

Let consider the scalar $t_1^T \mathfrak{K} t_2$. The corresponding screw operation is called *Perpendicular product* and it is indicated with:

$$s_1 \Diamond s_2 := t_1^T \mathfrak{K} t_2$$

The vector space $se(3)$ is a Lie algebra and therefore it has an internal product called a commutator and corresponding to the Lie brackets of the associated Lie group $SE(3)$. With the previous association between screws and twists, we can define the *screw-vector product* as:

$$s_1 \wedge s_2 := [t_1, t_2]$$

Using the usual coordinates we have:

$$s_1 \wedge s_2 := \left[\begin{pmatrix} \omega_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} \omega_2 \\ v_2 \end{pmatrix} \right] = \begin{pmatrix} \omega_1 \wedge \omega_2 \\ \omega_1 \wedge v_2 + \omega_2 \wedge v_1 \end{pmatrix}$$

As already mensioned, we can associate to a screw s a screw corresponding to a line at infinity which is called its *polar* and will be indicated with s':

$$s' = \bar{\mathfrak{H}}^{-1} \mathfrak{K} s$$

As screws may be decomposed as sum of bound and free line-vectors, operations on screws may be reduced to operations on line-vectors. Geometric distances and relations between lines may then be used very efficently to perform calculation [22] [6]. In particular the reciprocity of line-vectors is equivalent to the incidence of their lines (axes).

3   Kinestatic and dynamic model

In this section we shall briefly present the dynamical model of spatial mechanisms as the interconnection of elementary subsystems consisting of rigid bodies, spatial springs and kinematic pairs, through a power continuous interconnection.

Indeed formulations of dynamical models of spatial mechanical systems using network representations (linear graphs or bond graphs [33] [15]) have been proposed as an extension of the network representation of electrical circuit models or their analogue in mechanics, thermics or hydraulics [16] [35], [20], [2], [42]. However these representations use as power variables, (i.e. the variables of the interconnection network analoguous to the voltages and currents in Kirchhoff's laws) either scalars (in which case one has to choose a priori some coordinate systems) or real vectors (which means that one decomposes the translations and the rotations). Using these variables may obscure and complicate considerably the modeling and the analysis of the mechanical systems as real vectors or scalar do not reflect the geometric properties of rigid body velocities (and displacements). Therefore other network representations of spatial mechanical systems were proposed using as power variables the intrinsic representation of velocites and forces of rigid bodies as twists and wrenches. T.H.Davies [11] used the oriented graph describing the topological

structure of a mechanisms and showed that a generalisation of Kirchhoff's laws applies to the twists and wrenches of a mechanism. This work was extended to kinestatic models of multi-body systems by C.Bidard [7], who defined a bond graph model using the geometrical definition of twists and wrenches as screw-vectors. Finally an extension of these bond graph models to the complete dynamical model of a mechanical network (including spatial springs and rigid bodies), but using the Lie-algebraic definition of twists and wrenches was proposed in [5, 26, 24, 38].

With such network models one may associate an analytical expression of the dynamics, essentially in terms of port controlled Hamiltonian systems [25]. Indeed it may be shown that, in general, one may map the interconnection structure of network models with the geometrical structure of Hamiltonian systems (Poisson brackets for explicit Hamiltonian systems [30] or Dirac structures for implicit Hamiltonian systems [45]) and the total energy with the Hamiltonian function. For bond graph models of spatial mechanical systems using the Lie-algebraic definition of twists and wrenches, two formulations of the dynamics associated with the network structure were proposed: as constrained Hamiltonian system in [24] and as an implicit Hamiltonian systems in [29].

In the sequel, we shall briefly present the dynamical model of a spatial mechanical system as the interconnection of the elementary subsystems: rigid body, spatial spring and kinematic pair, using as power variables the *twists and wrenches expressed as screw-vectors*.

3.1   Energy storing elements and port controlled Hamiltonian systems

3.1.1   Port controlled Hamiltonian systems

Arizing from the bond graph formalism, more precisely the generalized or thermodynamical bond graph formalism [8], so-called *port controlled Hamiltonian systems* were defined [30]. For the sake of simplicity, hereafter we shall present the definition on $\mathbb{R}^n$ which corresponds actually to local definition of such systems, associated with some local choice of coordinates. On $\mathbb{R}^n$, a port controlled Hamiltonian system is defined by a *skew-symmetric structure matrix*, denoted by $J(x)$, a real function, called *Hamiltonian function*, denoted by $H(x)$, an input matrix, denoted by $G(x)$ and the following equations:

$$\dot{x} = J(x)\frac{\partial H(x)}{\partial x} + G(x)u$$
$$y = G(x)^T \frac{\partial H(x)}{\partial x} \tag{18}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^p$, the output $y \in \mathbb{R}^n$ where $\mathbb{R}^n$ is identified with its dual vector space $\mathbb{R}^{n*}$. Due to the skew-symmetry of the structure matrix : $J^T(x) = -J(x)$, the system is indeed conservative and the change of its internal energy is only due to the power supplied by the interconnecting port characterised by $V \times V^*$:

$$\dot{H} = \frac{\partial H}{\partial x}^T \dot{x} = \underbrace{\frac{\partial H}{\partial x}^T J(x)\frac{\partial H}{\partial x}}_{0 \Leftarrow J^T(x) = -J(x)} + \underbrace{\frac{\partial H}{\partial x}^T G(x)}_{y^T} u = y^T u$$

Hence, if furthermore the Hamiltonian function is bounded from below, then the system is passive and lossless. If furthermore the structure matrix $J(x)$ satisfies the relations

$$\sum_{l=1}^{n} \left[J^{il}\frac{\partial J^{jk}}{\partial x^l} + J^{kl}\frac{\partial J^{ij}}{\partial x^l} + J^{jl}\frac{\partial J^{ki}}{\partial x^l} \right] \equiv 0 \tag{19}$$

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for $i, j, k = 1, .., n$, which is called the *Jacobi identities*, then it corresponds to the local definition of a Poisson tensor [21].

It was shown that the structure of port controlled Hamiltonian systems may be related to the structure of network or bond graph models of physical systems without energy storing elements in excess: the structure matrix $J(x)$ represents the power continuous interconnection structure of the network and the Hamiltonian function corresponds to the total energy of the system [30] [31] [31]. In the case in which $J(x)$ satisfies the Jacobi identity, the drift dynamics of the port controlled Hamiltonian system may be related to standard, i.e. symplectic, Hamiltonian systems. Indeed, if one some neighbourhood the rank of the structure matrix is constant and $m = \text{rank} J(x)$, then there exist so-called *canonical coordinates* (q, p, r) such that the structure matrix gets the following form:

$$J = \begin{pmatrix} 0 & I_m & 0 \\ -I_m & 0 & 0 \\ 0 & 0 & 0_p \end{pmatrix} \tag{20}$$

On this canonical expression of the structure matrix, it appears clearly that it may be split into a first bloc-diagonal term being the standard symplectic matrix and a second one which determines the kernel. The coordinates r, called *redundant* coordinates, corresponds to the kernel of the structure matrix and seen as coordinate functions, they generate the so-called *Casimir functions* [21]. It is clear that *whatever the Hamiltonian function is*, the redundant coordinates are dynamical invariants of the drift dynamics of the port controlled Hamiltonian system in Eq. 18. These dynamical invariants, the Casimir functions, play an essential role in the reduction of Hamiltonian systems with symmetries [21], but they play also an essential role for the passive control [10] [27] as shall be presented heareafter.

Dissipation can be added by defining the structure not only with the skew-symmetric structure matrix $J(x)$ but also with *a positive semi-definite structure matrix* denoted by $R(x)$ [10]. With this extention, one obtains the *port controlled Hamiltonian systems with dissipation*:

$$\dot{x} = (J(x) - R(x))\frac{\partial H(x)}{\partial x} + G(x)u$$
$$y = G(x)^T \frac{\partial H(x)}{\partial x}$$

And the energy balance gives

$$\dot{H} = -\frac{\partial H}{\partial x}^T R(x)\frac{\partial H}{\partial x} + y^T u$$

which shows that part of the internal energy is dissipated and modeled by $R(x)$.

3.1.2 The rigid body element

The first elementary subsystem of a spatial mechanical system is naturally the rigid body. In network (bond graph) models, a rigid body may be represented by a 1-port element [2] with constitutive relations being its dynamical model. This model may be expressed in terms of a port controlled Hamiltonian system which we shall present hereafter. Its bond graph representation was proposed in [24] but note that it differs from this model by using now the representation of twists and wrenches as screws.

The dynamics of a rigid body is defined on the state space composed of the position of the body i with respect to the inertial space $H_i^0 \in SE(3)$ and of its momentum in body frame $P^i \in se^*(3)$

[21]. The rigid body is endowed with a kinetic energy $E_K(P^i)$ which is a quadratic function of the momentum in body frame P^i:

$$E_K(\mathcal{P}^i) = \frac{1}{2}\mathcal{P}^i(\mathcal{I}^i)^{-1}(\mathcal{P}^i)^T$$

where $\mathcal{I}^i$ is a constant tensor called the *inertia tensor*.

It may also be endowed with a potential energy which is a function $E_P(H_i^0)$ of the displacement H_i^0. The potential energy may be, for instance, due to the gravity or may be zero[7]. This element has two (power conjugated) port variables being the *twist in inertial frame* $T_i^{0,0}$ of the body and the conjugated *wrench in inertial frame* W_i^0.

The constitutive relations of this rigid body elements may then be written in the following form of a port controlled Hamiltonian system: (see [29, equation (2.1)]):

$$\frac{d}{dt}\begin{pmatrix} \bar{H}_i^0 \\ \mathcal{P}^i \end{pmatrix} = \begin{pmatrix} 0 & TL_{H_i^0} \\ -TL_{H_i^0}^T & P^i\wedge \end{pmatrix}\begin{pmatrix} dE_P(\bar{H}_i^0) \\ T_i^{i,0} \end{pmatrix} + \begin{pmatrix} 0 \\ Ad_{H_i^0}^T \end{pmatrix}W_i^0$$
$$T_i^{0,0} = \begin{pmatrix} 0 & Ad_{H_i^0} \end{pmatrix}\begin{pmatrix} dE_P(\bar{H}_i^0) \\ T_i^{i,0} \end{pmatrix} \tag{21}$$

where $P^i\wedge := \begin{pmatrix} \tilde{P}_\omega & \tilde{P}_v \\ \tilde{P}_v & 0 \end{pmatrix}$ is a 6×6 matrix composed of 3×3 skew-symmetric matrices representing the vector product of the angular and linear momenta, $\bar{H}_i^0$ is a 6 dimensional vector representation of H_i^0, and $TL_{H_i^0}$ is such that $TL_{H_i^0}T_i^{i,0}$ is equal to the parameterized form of $\dot{H}_i^0 = H_i^0\tilde{T}_i^{i,0}$. Note that the twist $T_i^{i,0}$ is actually seen in these equations as the gradient of the kinetic energy $E_K(\mathcal{P}^i)$ with respect to the momentum $\mathcal{P}^i$.

It may also be noted that the first equation in this system represents simply the definition of the twist of the body i in inertial space and the second one is an extension of the Newton Euler equations to a rigid body which is also endowed with some potential energy [21].

3.1.3 The spring element

The second type of energy storage in a spatial mechanisms consists in spatial springs that means springs with a displacement being a relative position $H_i^j \in SE(3)$ between the Euclidean frames i and j. In general it may be defined by some potential function $E_P(H_i^j)$. In bond graph models it may be represented by a 1-port element with port variables being the twist $T_i^{i,j}$ and elastic force $W_{i,j}^i$ of the spring in [24]. Hence the constitutive relations of the spring element may be expressed as the following port controlled Hamiltonian system:

$$\begin{array}{l} \frac{d}{dt}\bar{H}_i^j = TL_{H_i^j}T_i^{i,j} \\ W_{i,j}^i = TL_{H_i^j}^T dE_P(\bar{H}_i^j) \end{array} \tag{22}$$

where the index notation is the one used in [41]. Note that the actual definition and parameterization of the potential functions for spatial springs is far from being obvious [13] [14].

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[7]Even when the potential energy is zero, the state H_i^0 is necessary to be able to calculate changes of coordinates.

3.2 The kinestatic model, Dirac structures and Hamiltonian dynamics

3.2.1 Dirac structures on vector spaces

Consider a m-dimensional real vector space $\mathcal{V}$ and denote by $\mathcal{V}^*$ its dual. A *Dirac structure* [9] on the vector space $\mathcal{V}$ is an m-dimensional subspace L of $\mathcal{V} \times \mathcal{V}^*$ such that:

$$\forall (f, e) \in L, \ \langle e, f \rangle = 0 \tag{23}$$

where $\langle , \rangle$ denotes the duality product. In order to give some constructive definition of different Dirac structures in the sequel, we shall use a definition of Dirac structures in terms of two linear maps and called *kernel representation of a Dirac structure* [31].

Every Dirac structure $L \subset \mathcal{V} \times \mathcal{V}^*$ is uniquely defined in a basis $\mathcal{B} = \{v_1, .., v_m\}$ by the pair of real matrices (F, E) called *structure matrices* and satisfying the condition:

$$E\,F^T + F\,E^T = 0 \tag{24}$$

by:

$$L = \{(f, e) \in \mathcal{V} \times \mathcal{V}^* \text{ s.t. } F\bar{f} + E\bar{e} = 0\} \tag{25}$$

where $\bar{f}$ is the coordinate vector of f in the basis $\mathcal{B}$ of $\mathcal{V}$ and $\bar{e}$ is the coordinate vector of e in the dual basis to $\mathcal{B}$ of $\mathcal{V}^*$.

A Dirac structure defined on a vector space $\mathcal{V}$ encompasses the two structures of a Poisson vector-space and a pre-symplectic vector-space [9].

Such Dirac structures arise in electrical circuits (or their mechanical analogue) where Kirchhoff's laws actually endows the set of currents and voltages of the circuit or the rate variables and the co-energy variables of an LC-circuit (possibly with elements in excess) with a Dirac structure [28] [1]. In the sequel we shall see how Dirac structures may be associated with the kinestatic model of a mechanism.

3.2.2 The port connection graph

Initially a mechanism is described by the topology of the set of rigid bodies, springs and kinematic pairs that compose it . This topology is described by an oriented graph, called *primary graph*. Its vertices are associated with the bodies (more precisely with the Euclidean space associated with it) and its edges are associated with the generalized springs, kinematic pairs and dampers with arbitrary orientation. However in order to represent explicitly the port variables of the rigid bodies on the topology, one has to augment the primary graph in order to associate an edge also to every body. This augmented graph is called *port connection graph* [18], which is obtained by augmenting the primary graph by a reference vertex and a *Lagrangian tree* [34] connecting the reference vertex with the set of vertices of the primary graph. Examples can be found in [38]. If the inertia of some body is neglected (*i.e.*, is set to zero), this is taken into account by erasing the corresponding edge in the Lagrangian tree.

In this way all the elements of the mechanical network are associated with an edge and the port variables of the elements may be considered as through and across variables of the port connection graph like voltages and currents in an electrical network [34]. Then it may be shown that these variables obey a generalization of Kirchhoff's laws applied to the port interconnection graph [11]:

- the sum of the twists along any cycle (or loop) in the port connection graph vanishes,

- the sum of the wrenches along any co-cycle (or cutset) in the port connection graph vanishes.

The Dirac structure associated with the port connection graph $\mathbf{G}$ may be defined as follows. Denote by $\mathbf{T}$ the maximal tree in $\mathbf{G}$ corresponding to the Lagrangian tree and by $\bar{\mathbf{T}}$ the complementary co-tree in $\mathbf{G}$. Denote by $B \in \{-1, 0, 1\}^{q \times n}$ the *incidence matrix* of the primary graph, where q is the number of rigid bodies corresponding to the order of the Eucledian system and n is the number of edges of the primary graph. Denote by C the *fundamental loop matrix* defined as:

$$C = \begin{pmatrix} B^T & I_{n \times n} \end{pmatrix}$$

and by Q the *fundamental cutset matrix* associated with $\mathbf{T}$ [34]:

$$Q = \begin{pmatrix} I_{q \times q} & -B \end{pmatrix}$$

It is easy to see that

$$C \, Q^T = 0 \tag{26}$$

Then Kirchhoff's loop rules is equivalent to the following constraint relations on the across and through variables of the graph, that is, the twists $T^T := (T_1^T, .., T_n^T) \in se(3)^n$ and the wrenches $W := (W_1, .., W_n) \in se^*(3)^n$:

$$CT = 0 \tag{27}$$

$$QW^T = 0 \tag{28}$$

It is now possible to define the $(n + q) \times (n + q)$ structure matrices of Eq.24 for the corresponding Dirac structure as:

$$F = \begin{pmatrix} C \\ 0_{q \times (q+n)} \end{pmatrix} \qquad \text{and} \qquad E = \begin{pmatrix} 0_{n \times (q+n)} \\ Q \end{pmatrix}$$

Kirchhoff's laws are then equivalent to define the set of admissible twists and wrenches as:

$$\mathcal{D}_{\mathbf{G}, \mathbf{T}} = \{(f, e) \in se(3)^n \times se(3)^{*n} \text{ s.t. } Ff + Ee = 0\} \tag{29}$$

¿From Eq.26 follows that $\ker C$ and $\ker Q$ are dual spaces and it can be seen that $\dim \mathcal{D}_{\mathbf{G}, \mathbf{T}} = 6n$. (This is exactly Tellegen's theorem). Hence the admissible set of twists and wrenches associated with the constraint relation of the port connection graph form a constant Dirac structure on $se(3)^n$.

3.2.3 Dirac structures on differentiable manifolds

The definition of Dirac structures on vector spaces may be extended to differentiable manifolds as follows. Let $\mathcal{M}$ be a differentiable manifold, a *Dirac structure* [9] on $\mathcal{M}$ is given by a smooth vector sub-bundle $L \subset T\mathcal{M} \times T^*\mathcal{M}$ such that the linear space $L(x) \subset T_x\mathcal{M} \times T_x^*\mathcal{M}$ is a Dirac structure on the tangent vector space $T_x\mathcal{M}$ for every $x \in \mathcal{M}$.

More intrinsic definitions in terms of duality and some characteristic distributions and co-distributions may be found in [9] [10]. In the sequel we shall use only a local a kernel representation of Dirac structures.

And the Dirac structure may also be locally characterized by a pair of *structure matrices* (depending now smoothly on the point x in $\mathcal{M}$). Indeed choosing some coordinates $(x_1, .., x_m)$ in some neighborhood of a point $x \in M$, a Dirac structure is defined by a pair of matrices $F(x), E(x)$ depending smoothly on x such that:

$$L(x) = \{(f, e) \in T_xM \times T_x^*M \text{ s.t. } F(x)f + E(x)e = 0\} \tag{30}$$

On this local representation it may be seen that Dirac structures defined on differentiable manifolds generalize as well as (generalized) Poisson manifolds as pre-symplectic manifolds in the same way as Dirac structures do generalize pre-symplectic and Poisson vector spaces [9].

Remark 1. *On Dirac structure there exist also some additional closeness condition which generalize the Jacobi identities of Poisson brackets taking into account also the input port. They play also an essential role in the geometrical structure and representation of Dirac structures; the interested reader is referred to [9] [12] [10].*

3.2.4   The kinematic pair elements

The elementary constraints between two rigid bodies are defined by so-called kinematic pairs. A *kinematic pair* is the kinematic idealization of a set of contacts that occur between two rigid bodies at some configuration of the bodies [22]. The wrench W transmitted by a kinematic pair is constrained to a linear subspace of the space of wrenches $se^*(3)$ called the *space of constraint wrenches* [22] and denoted by $\mathcal{W}^C$. A relative twist between the two bodies is allowed by the kinematic pair when it produces no work with any transmissible wrench. The relative twist is thus constrained to belong to a linear subspace of the space of twists $se(3)$, called the *space of freedom twists* [22] and denoted by $\mathcal{T}^A$. It is the space orthogonal to the space of transmitted wrenches $\mathcal{W}^C$, in the sense of the duality product, because an ideal kinematic pair produces no work:

$$w \bigcirc t = 0 \quad \forall t \in \mathcal{T}^A, \forall w \in \mathcal{W}^C \tag{31}$$

It is important to realise that the Diract structure here used is actually defined on a submanifold of the state manifold $(SE_i^0(3) \times se^*(3))^m$ corresponding to the m rigid bodies from which the mechanism is composed of (see Eq.21). This sub-manifold $\mathcal{K}$, is actually:

$$\mathcal{K} := SE_{i(1)}^{j(1)}(3) \times \ldots \times SE_{i(k)}^{j(k)}(3)$$

where k is the number of kinematic pairs and $i(l), j(l) \in 1, \ldots, n$ are the two rigid bodies which the l-th kinematic pair connects [38]. It is shown in [38], that after references are chosen, it is possible to find a diffeomorphism

$$\pi : SE^k(3) \to \mathcal{K}$$

where $SE^k(3)$ is a Lie group with Lie algebra $se^k(3)$. Using this diffeomorphism, Lie right and left translations, it is therefore possible to map $se(3)^k \times se(3)^{*k}$ to $T_x\mathcal{K} \times T_x^*\mathcal{K}$ with $x \in \mathcal{K}$:

$$\Pi_x : se(3)^k \times se(3)^{*k} \to T_x\mathcal{K} \times T_x^*\mathcal{K} \; ; \; (T, W) \mapsto ((\pi_* \, o \, L_{*x}) \, T, (L_x^* \, o \, \pi^*)^{-1} \, W)$$

We can now define the configuration dependent pair of admissible twists and transmitted wrenches of a kinematic pair at a configuration $x \in \mathcal{K}$ as elements of the following Dirac structure characterizing the kinematic pair:

$$\mathcal{D}_P(x) = \Pi_x(\mathcal{T}^A(x), \mathcal{W}^C(x)) \tag{32}$$

It is also possible, once coordinates are used, to give an expression of the structure matrices expressed for the elements in $se^k(3) \times se^{*k}(3)$:

$$F(x) = \begin{pmatrix} T(x) \\ 0_{p \times n} \end{pmatrix} \qquad \text{and} \qquad E(x) = \begin{pmatrix} 0_{q \times n} \\ W(x) \end{pmatrix}$$

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where $p = \dim \mathcal{T}^A(x)$, $q = \dim \mathcal{W}^C(x)$, $E(x), F(x)$ are $n \times n$, $T(x), W(x)$ are full-rank matrices and such that $\ker W(x) = \operatorname{range} T^T(x)$, $\ker T(x) = \operatorname{range} W^T(x)$ and x is a kinematic state which represents the configurations of the kinematic pairs.

Remark 2. *Note that kinematic pairs which interact with their environment (for instance actuators or other mechanisms along the freedom twists also define a Dirac structure on the space of freedom twists, constraint wrenches and interaction twists and wrenches. The reader is referred to [24, 29, 38] for detailed definitions.*

3.2.5 The kinestatic connection network

By combining the two types of elementary constraints presented here above (*i.e.*, the port connection graph and the kinematic pairs) a complex power conserving interconnection (in the sense of the definition given in the part I of [29]), relating the rigid bodies and the spatial springs of a mechanical system, may be obtained. This interconnection structure defines on its port variables some constraint relations which are called the *kinestatic model* of the mechanical system [22] as they consist of constraint relations on the wrenches and their dual constraint relations on the twists. Therefore we shall call the interconnection structure composed of the port connection graph and the kinematic pairs, the *kinestatic connection network*. The interested reader is referred to [7, 4, 38] for the detailed analysis of this network and for its use in the analysis and design of mechanisms. In this paragraph we shall characterize it in terms of Dirac structures deduced from the Dirac structure associated with the kinematic pairs and with the port connection graph.

In the sequel, we shall give an expression of a Dirac structure associated with a kinestatic model in the case where the inertias of all the bodies are taken into account. Hence a maximal tree of the port connection graph $\mathbf{G}$ is given by the Lagrangian tree $\mathbf{T}_B$ on which the bodies are connected. Denote by $\mathbf{T}_P$ the co-tree composed of the edges of the port connection graph with which the kinematic pairs are associated. Note that the co-tree $\mathbf{T}_P$ is not necessarily maximal if some edges $\mathbf{T}_S$ of the port connection graph correspond to spatial springs or ports: $\mathbf{G} = \mathbf{T}_B \cup \mathbf{T}_P \cup \mathbf{T}_S$. Denote by $\mathcal{D}_{\mathbf{G},\mathbf{T}_B}$ the Dirac structure on $se(3)^n$ defined by the port connection graph (see Sec.3.2.2) and by $\mathcal{D}_P$ the Dirac structure for the kinematic pairs (see Sec.3.2.4).

The "power pairs" belonging to $\mathcal{D}_{\mathbf{G},\mathbf{T}_B}$ must also satisfy the state dependent constraints of the kinematic pairs. The resulting Dirac structure is therefore:

$$\mathcal{D}_K(x) = \left\{ (T, W) \in se(3)^n \times se(3)^{*n} \text{ s.t. } (T, W) \in \mathcal{D}_{\mathbf{G},\mathbf{T}_B} \text{ and } (T, W) \in (\Pi_x)^{-1} \mathcal{D}_P(x) \right\} \quad (33)$$

The space $\mathcal{D}_K(x)$ is therefore defined by assembling the constraints corresponding to the port connection graph described in Eq.28 and the constraints associated with each kinematic pair.

The admissible twists and wrenches satisfy:

$$\sum_{i \in \mathbf{G}} W_i \bigcirc T_i = \sum_{i \in \mathbf{T}_B} W_i \bigcirc T_i + \sum_{i \in \mathbf{T}_P} W_i \bigcirc T_i + \sum_{i \in \mathbf{T}_S} W_i \bigcirc T_i = 0 \quad (34)$$

where the first sum represents the power exchanged to modify the kinetic energy, the second the power exchanged to modify the potential energy and the third one the power exchanged through the kinematic pairs with the environment.

By Proposition 1.1 in [46] $\mathcal{D}_{K_x}$ is a Dirac structure.

Remark 3. *Note that for the sake of simplicity we did not consider that the kinematic pairs are actuated. In the case that some of the kinematic pairs are interacting with the environment, the definition of the Dirac has to be changed by augmenting the wrenches and twists by the wrenches and twists at the ports of the kinematic pairs [29].*

3.2.6   Dirac structures and Implicit Hamiltonian systems

Using Dirac structures the generalized Hamiltonian systems may be extended to *implicit* Hamiltonian systems. Such systems were introduced by Courant [9] and are extensively used in the analysis of symmetries in evolution equations [12]. Contrary to the statement in [9], it seems that such structures play also a fundamental role in finite dimensional systems [44] [10] [28] [43] [29] as we shall see below.

An implicit Hamiltonian system with ports is defined by a state space, a differentiable C^∞ manifold, denoted by $\mathcal{M}$, a vector-bundle over $\mathcal{M}$ called *port interaction space* and denoted by W_x, a Dirac structure $L(x)$ on the vector bundle with fibers: $T_x\mathcal{M} \times W_x \times T_x^*\mathcal{M} \times W_x^*$ and a Hamiltonian function $H(x) \in C^\infty(\mathcal{M})$ by:

$$(\dot{x}, f, \frac{\partial H}{\partial x}(x), -e) \in \mathcal{D}(x), \quad x \in M \tag{35}$$

where $x \in \mathcal{M}$ is the state of the system, the pair (f, e) are the external variables, also called *port variables* and the minus sign corresponds to the sign convention ensuring that the power ingoing the system is counted positively.

¿From the orthogonality condition, it follows that an implicit Hamiltonian system as the one of Eq.35 satisfies the balance equation:

$$\frac{dH}{dt} = \langle e, f \rangle \tag{36}$$

Hence, if the Hamiltonian function is bounded from below, the implicit port controlled Hamiltonian system is passive and lossless. Using a kernel representation of the Dirac structure $L(x)$ with structure matrices:

$F_1(x), F_2(x), E_1(x), E_2(x)$, the Hamiltonian system of Eq.35 becomes:

$$F_1(x)\dot{x} + F_2(x)f + E_1(x)\frac{\partial H}{\partial x}(x) - E_2(x)e = 0 \tag{37}$$

For a detailed exposure and analysis of such implicit port controlled Hamiltonian systems, the reader is referred to [10] [28] [29].

3.2.7   Dynamics of a spatial mechanism

A complete mechanical network is obtained by assembling a set of rigid bodies, spatial springs and kinematic pairs. Using the results of part I of [29] and the network structure described here above, we shall now formulate the dynamics of such a system in terms of an implicit Hamiltonian system [29].

Consider a mechanical network composed of n_B bodies, n_S springs and n_p kinematic pairs interconnected through the port connection graph **G**.

Assume that the inertial properties of the bodies are all taken into account, so that the edges associated with the bodies in **G** form a maximal tree of **G**.

The set of n_B bodies and n_S springs compose a set of port controlled Hamiltonian systems defined in Eq.21 and Eq.22, which define the state space of the complete mechanical network:

$$\mathcal{X} = (SE(3) \times se^*(3))^{n_B} \times SE(3)^{n_S} \tag{38}$$

The port connection graph **G** with the kinematic pairs defines the kinestatic connection network which relates the port variables of the set of rigid bodies and spatial springs. According to Sec.3.2.5,

the constraint relations defined by the kinestatic connection network defines a Dirac structure, denoted by $\mathcal{D}_K(x)$ on its port variables according to Eq.33. Hence the kinestatic connection network defines a power conserving interconnection [29] between the port controlled Hamiltonian systems of bodies and springs.

Hence the dynamics of the mechanical network is an implicit Hamiltonian system defined on the state-space of the energy variables $\mathcal{X}$ with respect to the Dirac structure obtained from the Dirac structures of the energy storing elements and the Dirac structure $\mathcal{D}_K(x)$ according to Theorem 1.2 in [29].

4   Conclusions

In this notes we have recalled in a first part the geometry of rigid body motion in terms of the geometry of lines in $\mathbb{R}^3$ and we have used the Klein form in order to identify the twists and the wrenches. Then the use of reciprocity of twists with respect to the Klein form was used to identify self-reciprocal twists with lines in the projective space $\mathcal{P}^3$ and twists and wrenches with screw-vectors. In the second part we have used this identification of twists and wrenches with screw-vectors in order to define the kinestatic model in terms of a Dirac structure and the dynamics of spatial mechanisms in terms of implicit Hamiltonian systems. This formulation of the dynamical model actually embeds in a *unique* representation different levels of description (in terms of discrete topology, projective geometry, differential geometry) of spatial mechanisms. In this way the symbolic and numerical computation on such models in enhanced and makes at all possible the analysis and design of complex mechanisms. Furthermore the Hamiltonian structure of the dynamical model makes possible the rigorous development of passive nonlinear control laws as was already initiated in [38] [40] [27].

Acknowledgment

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