Robust Control of Multilink Flexible Manipulators

Rex J Theodore and Ashitava Ghosal*

Abstract

This paper deals with some robustness aspects of a model based controller used for trajectory tracking in multi-link flexible manipulators. It is known in literature that the finite element formulation over-estimates the natural frequencies of the original system. We show that over-estimation of natural frequencies may lead to unstable closed-loop response for flexible manipulators using a model based inversion control algorithm. A robust controller design based on the second method of Lyapunov using simple quantitative bounds on the model uncertainties is illustrated for use during the trajectory tracking phase in multi-link flexible manipulator control. In order to actively suppress the link vibrations excited during the trajectory tracking phase, a second controller based on end-point sensing and the rigid Jacobian of the manipulator is used. The performance of the two-stage controller is illustrated with the help of numerical simulations of a flexible elbow manipulator.

Keywords: flexible, manipulators, robustness, control

1 Introduction

A manipulator with flexible links typically respond to motion of the joints with undesirable vibration. To suppress the link vibration, the solutions suggested in the literature range from passive damping methods [1] to active damping methods. In this paper, we deal with the problem of active damping of flexural vibrations of the links by only the joint control inputs. The types of controllers proposed in the literature to actively control the flexural vibrations of the links are typically based on a truncated finite dimensional model [2, 3] of the system, and in this paper, we use a truncated, finite element model of the multi-link flexible robot.

Motion control of a flexible manipulator involves tracking the desired trajectory of the end-effector including active suppression of link vibrations. This can be achieved by model based control techniques (see [4]) for the trajectory tracking phase, followed by a second stage control (see [5, 6]) at the final target position to damp out the link vibrations. Most

*The authors are with Deneb Hitech India(Pvt.) Ltd, Bangalore-560 052(email: rex@india.deneb.com), and Department of Mechanical Engineering, Indian Institute of Science, Bangalore-560 012, India (e-mail: asitava@mecheng.iisc.ernet.in)

of the above mentioned tracking control algorithms require exact knowledge of system parameters, and the stability of the closed-loop system critically depends on how well the actual flexible manipulator dynamics is modeled. In real applications, however, the inertial and elastic characteristics of a flexible manipulator are not known precisely. To this end, solutions suggested in the literature involve use of adaptive methods [7], the robust controllers [8], the variable structure controllers [9] and, more recently, H_∞ optimal control [10].

In this paper, we discuss the robustness and stability issues in a model based trajectory tracking controller which uses a finite element model of the flexible links. It is known in the literature that use of finite element discretization method for approximating the continuum of flexible links, normally over-estimate the natural frequencies of the original system. We analytically show that over-estimation of natural frequencies in a model based inversion control algorithm may lead to unstable closed-loop response in a flexible manipulator system. A robust controller design based on the second method of Lyapunov, using simple quantitative bounds on the model uncertainties, is illustrated for use during the trajectory tracking phase. To damp out the induced flexible vibration at the target point, a second controller based on end point sensing and the rigid Jacobian is used. The results are illustrated by numerical simulation of a flexible spatial elbow manipulator.

2 A Two-Stage Controller for Flexible Manipulators

The control algorithm consist of a model-based joint controller for the trajectory tracking phase and an impedance controller for suppression of unwanted flexural vibrations at the target point. It is similar in concept to the two-stage controllers described in [5, 6]. In this section, we briefly describe the two stages and define the symbols used.

2.1 Model-Based Joint Inversion Control Law

The open-loop equations of motion for the flexible link manipulator system can be written as [11],

$$\mathbf{M}_{rr}\ddot{\mathbf{q}}_r + \mathbf{M}_{rf}^T\ddot{\mathbf{q}}_f + \mathbf{h}_r(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{c}_r(\mathbf{q}) = \mathbf{\Gamma} \tag{1}$$

$$\mathbf{M}_{rf}\ddot{\mathbf{q}}_r + \mathbf{M}_{ff}\ddot{\mathbf{q}}_f + \mathbf{h}_f(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{c}_f(\mathbf{q}) + \mathbf{K}\mathbf{q}_f = \mathbf{0} \tag{2}$$

where $\mathbf{q}_r$ is the n-vector of generalized joint variables, $\mathbf{q}_f$ is the N-vector of flexible deformation variables, $\mathbf{M} = \begin{pmatrix} \mathbf{M}_{rr} & \mathbf{M}_{rf}^T \\ \mathbf{M}_{rf} & \mathbf{M}_{ff} \end{pmatrix}$ is the $(n+N)\times(n+N)$ configuration dependent generalized mass matrix, $\mathbf{h} = (\mathbf{h}_r(\mathbf{q},\dot{\mathbf{q}})^T, \mathbf{h}_f(\mathbf{q},\dot{\mathbf{q}})^T)^T$ is the $(n+N)$-vector of Coriolis, and centrifugal terms and the terms accounting for the interaction of joint variables and their rates with flexible variables and their rates, $\mathbf{c} = (\mathbf{c}_r(\mathbf{q})^T, \mathbf{c}_f(\mathbf{q})^T)^T$ is the $(n+N)$-vector of gravitational terms, $\mathbf{K}$ is the $N \times N$ flexural structure stiffness matrix of the system, $\mathbf{\Gamma}$

is the n-vector of input torques (or forces) applied at the joints, and $\mathbf{0}$ is the zero vector with appropriate dimension.

The model-based control law can be written as

$$\mathbf{\Gamma}_I = (\mathbf{M}_{rr} - \mathbf{M}_{rf}^T\mathbf{M}_{ff}^{-1}\mathbf{M}_{rf})\mathbf{u} \;+\; (\mathbf{h}_r + \mathbf{c}_r - \mathbf{M}_{rf}^T\mathbf{M}_{ff}^{-1}(\mathbf{h}_f + \mathbf{c}_f + \mathbf{K}\mathbf{q}_f)) \qquad (3)$$

where $\mathbf{u}$ in terms of the specified joint trajectory, $\mathbf{q}_r^d(t)$, and its derivatives, can be written as

$$\mathbf{u} = -\mathbf{G}_p\mathbf{q}_r - \mathbf{G}_v\dot{\mathbf{q}}_r + \ddot{\mathbf{q}}_r^d(t) + \mathbf{G}_v\dot{\mathbf{q}}_r^d(t) + \mathbf{G}_p\mathbf{q}_r^d(t) \qquad (4)$$

where $\mathbf{G}_p$ and $\mathbf{G}_v$ are constant diagonal position and velocity gain matrices for the joint variables, respectively. By using the positive definiteness of the matrices $\mathbf{M}_{ff}$ and $\mathbf{K}$, one can show that the closed-loop system is *critically* stable, and if material damping is incorporated in the model, it is *asymptotically* stable[12]. To *actively* damp out the unwanted flexural vibrations, we use a second control law at the target point.

2.2 Impedance Control Using End-Point Sensing

If $\mathbf{z}_n = \mathbf{f}(\mathbf{q}_r, \mathbf{q}_f)$ is used to denote the direct kinematics of a flexible manipulator, the *rigid Jacobian* of the flexible manipulator, $\mathbf{J}_{\mathbf{q}_r}^r$, based on the assumption of small flexible displacement, is given by $\mathbf{J}_{\mathbf{q}_r}^r(\mathbf{q}_r^d) = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{q}_r}\right)_{\mathbf{q}_r=\mathbf{q}_r^d, \mathbf{q}_f=\mathbf{0}}$ It may be noted that $\mathbf{J}_{\mathbf{q}_r}^r$ always exists as it is the conventional Jacobian for rigid link manipulators, and is square for non-redundant robots.

A controller using the above defined rigid Jacobian, for estimating end point displacement, can be written for the multi-link flexible manipulator system with gravity compensation as,

$$\mathbf{\Gamma}_{II} = \mathbf{J}^{rT}_{\mathbf{q}_r}\left(-\widehat{\mathbf{G}}_p\delta\mathbf{z}_n - \widehat{\mathbf{G}}_v\dot{\mathbf{z}}_n\right) + \mathbf{c}_r(\mathbf{q}_r^d, \mathbf{q}_f^d) \qquad (5)$$

where $\delta\mathbf{z}_n \triangleq \mathbf{z}_n - \mathbf{z}_n^d$; $\mathbf{z}_n$, $\dot{\mathbf{z}}_n$ are the measured end-effector generalized displacement and velocity vectors respectively, and matrices $\widehat{\mathbf{G}}_p$, and $\widehat{\mathbf{G}}_v$ are constant diagonal matrices of position and velocity gains respectively. The variable $\mathbf{q}_r^d$ corresponds to the final point of the desired joint trajectory and $\mathbf{q}_f^d$ is $\mathbf{q}_f^d = -\mathbf{K}^{-1}\mathbf{c}_f(\mathbf{q}_r^d, \mathbf{q}_f^d)$, and corresponds to the static deflection of links under gravity at the desired joint configuration $\mathbf{q}_r^d$. The control scheme given in equation (5) can be shown to asymptotically stabilize the equilibrium state $\mathbf{q} = \mathbf{q}^d$, $\dot{\mathbf{q}} = \mathbf{0}$ of the flexible link manipulator system(1-2) with non-zero initial vibrations and for a structural assumption, $\lambda_{min}(\mathbf{K}) > c$ (see [13]), where c is related to the gravity term. It may be noted that $\mathbf{c}_f$ is linear in $\mathbf{q}_f^d$ from the assumption of small flexible displacement.

2.3 The Two-Stage Control Algorithm

The two control laws can be described in a compact manner as

$$\mathbf{\Gamma} = (\mathbf{I} - \mathbf{S})\mathbf{\Gamma}_I + \mathbf{S}\mathbf{\Gamma}_{II} \qquad (6)$$

where,

$$\mathbf{S} = \begin{cases} \mathbf{0} & \text{null matrix during joint trajectory tracking stage} \\ \mathbf{I} & \text{identity matrix to suppress link oscillations at the end point of joint trajectory} \end{cases}$$

and, $\mathbf{\Gamma}_I$, $\mathbf{\Gamma}_{II}$ are given in equations (3), (5) respectively. We discuss some robustness issues of this controller in the next section.

3 Robustness Issues

Practical implementation of the joint inversion control law (equation(3)) requires that the parameters in the dynamic model of the system be known *precisely* and also that the model-based decoupling matrix and the nonlinear feedback terms be computable in real-time. The above requirements are in general not possible to satisfy and so it is much more reasonable instead to suppose that the model-based joint inversion control law is actually of the form

$$\mathbf{\Gamma}_I = \overline{(\mathbf{M}_{rr} - \mathbf{M}_{rf}^T \mathbf{M}_{ff}^{-1} \mathbf{M}_{rf})}\ \mathbf{u} + \overline{(\mathbf{h}_r + \mathbf{c}_r - \mathbf{M}_{rf}^T \mathbf{M}_{ff}^{-1} (\mathbf{h}_f + \mathbf{c}_f + \mathbf{K}\mathbf{q}_f))} \tag{7}$$

where $\overline{(\ \cdot\ )}$ represent the computed versions of respective expressions in equation(3). The parameter uncertainty or modeling error is represented by,

$$\begin{aligned} \Delta M &= \overline{(\mathbf{M}_{rr} - \mathbf{M}_{rf}^T \mathbf{M}_{ff}^{-1} \mathbf{M}_{rf})} - (\mathbf{M}_{rr} - \mathbf{M}_{rf}^T \mathbf{M}_{ff}^{-1} \mathbf{M}_{rf}) \\ \Delta \boldsymbol{\eta} &= \overline{(\mathbf{h}_r + \mathbf{c}_r - \mathbf{M}_{rf}^T \mathbf{M}_{ff}^{-1} (\mathbf{h}_f + \mathbf{c}_f + \mathbf{K}\mathbf{q}_f))} - (\mathbf{h}_r + \mathbf{c}_r - \mathbf{M}_{rf}^T \mathbf{M}_{ff}^{-1} (\mathbf{h}_f + \mathbf{c}_f + \mathbf{K}\mathbf{q}_f)) \end{aligned}$$

With the nonlinear control law(7), and choosing $\mathbf{u}$ as in equation(4), the closed-loop system of equations become,

$$\ddot{\mathbf{e}}(t) + \mathbf{G}_v \dot{\mathbf{e}}(t) + \mathbf{G}_p \mathbf{e}(t) = \Psi(t) \tag{8}$$

$$\mathbf{M}_{ff} \ddot{\mathbf{q}}_f + \mathbf{h}_f(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{c}_f(\mathbf{q}) + \mathbf{K}\mathbf{q}_f = -\mathbf{M}_{rf}(\mathbf{u} + \Psi(t)) \tag{9}$$

where $\Psi(t) = \left(\ \mathbf{M}_{rr} - \mathbf{M}_{rf}^T \mathbf{M}_{ff}^{-1} \mathbf{M}_{rf}\ \right)^{-1} \left(\ \Delta M \mathbf{u} + \Delta \boldsymbol{\eta}\ \right)$, and $\mathbf{e}(t) = \mathbf{q}_r(t) - \mathbf{q}_r^d(t)$.

In a flexible manipulator system, the uncertainty in the model parameters can be due to uncertainty in flexural structure stiffness of the manipulator model (i.e. in $\mathbf{K}$ matrix), and uncertainty in mass parameters of the manipulator model (i.e. in $\mathbf{M}(\mathbf{q})$ matrix). The uncertainty in stiffness and mass matrices can be considered together as uncertainty in the structural natural frequencies of the model.

3.1 Uncertainty in Structural Natural Frequencies

The natural frequencies of the flexible manipulator system at a nominal robot position($\mathbf{q}_r^d$) can be determined from the eigenvalues of $\boldsymbol{\mathcal{W}} \triangleq \mathbf{M}_{ff}^{-1} \mathbf{K}$ as,

$$\omega_i^2 = \lambda_i(\boldsymbol{\mathcal{W}}) = \lambda_i(\mathbf{M}_{ff}^{-1} \mathbf{K}) \qquad i = 1, 2, \ldots, N \tag{10}$$

where $\lambda_i(\cdot)$ denotes the i-th eigenvalue of a matrix. It should be noted that as the matrix $\mathbf{M}_{ff}$ depends on the joint variables of the robot arm, system natural frequencies vary as the robot configuration varies [14].

When an actual partial differential equation model of the flexible link manipulator system is transformed into a system governed by ordinary differential equations by using discretization schemes, we impose constraints that the temporal generalized coordinates of the neglected modes of the infinite dimensional system are zero. It is known that imposing any constraint on a dynamic system tend to render the system stiffer, thus increasing the values of the system natural frequencies [15]. Moreover, in the finite element formulation, low-order polynomial functions which are local in nature are employed to approximate the link mode shapes, and as a result the finite element model tend to always over-estimate the actual natural frequencies of a flexible link manipulator system (see also [11]). In the following, we examine the stability of closed-loop system when the over-estimated natural frequencies are used in the joint inversion control law(3).

Let us rewrite the model based joint inversion control law, with the computed version of matrix $\widehat{\boldsymbol{\mathcal{W}}}$ as,

$$\boldsymbol{\Gamma}_I = (\mathbf{M}_{rr} - \mathbf{M}_{rf}^T \mathbf{M}_{ff}^{-1} \mathbf{M}_{rf})\mathbf{u} \;+\; (\mathbf{h}_r + \mathbf{c}_r - \mathbf{M}_{rf}^T(\mathbf{M}_{ff}^{-1}(\mathbf{h}_f + \mathbf{c}_f) + \widehat{\boldsymbol{\mathcal{W}}}\mathbf{q}_f)) \tag{11}$$

Then the closed-loop equations of motion of the flexible link manipulator system reduce to,

$$\begin{aligned} \ddot{\mathbf{e}}(t) + \mathbf{G}_v \dot{\mathbf{e}}(t) + \mathbf{G}_p \mathbf{e}(t) &= -(\mathbf{M}_{rr} - \mathbf{M}_{rf}^T \mathbf{M}_{ff}^{-1} \mathbf{M}_{rf})^{-1} \mathbf{M}_{rf}^T \Delta \boldsymbol{\mathcal{W}} \mathbf{q}_f \\ \ddot{\mathbf{q}}_f + \mathbf{M}_{ff}^{-1}(\mathbf{h}_f + \mathbf{c}_f) + (\boldsymbol{\mathcal{W}} - \boldsymbol{\mathcal{H}}\Delta\boldsymbol{\mathcal{W}})\mathbf{q}_f &= -\mathbf{M}_{ff}^{-1}\mathbf{M}_{rf}\mathbf{u} \end{aligned}$$

where $\boldsymbol{\mathcal{H}} = \mathbf{M}_{ff}^{-1}\mathbf{M}_{rf}(\ \mathbf{M}_{rr} - \mathbf{M}_{rf}^T\mathbf{M}_{ff}^{-1}\mathbf{M}_{rf}\ )^{-1}\mathbf{M}_{rf}^T$ and $\Delta\boldsymbol{\mathcal{W}} = (\ \widehat{\boldsymbol{\mathcal{W}}} - \boldsymbol{\mathcal{W}}\ )$. We can make the following observations:

1. The $N \times N$ matrix $\boldsymbol{\mathcal{H}}$ is positive definite if $N \leq n$ and is positive semi-definite if $N > n$, where N is the dimension of generalized flexible variables $\mathbf{q}_f$ and n is the dimension of joint variables $\mathbf{q}_r$. The proof of this follows from the symmetry and positive definiteness of the matrix $(\mathbf{M}_{rr} - \mathbf{M}_{rf}^T\mathbf{M}_{ff}^{-1}\mathbf{M}_{rf})$ and from the definition of system mass matrix $\mathbf{M}(\mathbf{q})$.

2. The necessary condition for the above closed-loop system of equations to give bounded, stable response for $\mathbf{q}_f$, is that the closed-loop *frequency matrix* $(\boldsymbol{\mathcal{W}} - \boldsymbol{\mathcal{H}}\Delta\boldsymbol{\mathcal{W}})$ is positive definite.

The above observations clearly indicate that the flexible link manipulator system with $\Delta\boldsymbol{\mathcal{W}} < 0$ is *stable* and if $\Delta\boldsymbol{\mathcal{W}} > 0$ then it can be *unstable.* Hence a flexible manipulator cannot be effectively controlled using a model-based joint inversion control law, by assuming that the model of real system to be *more rigid*, than it actually is[1]. In the next section, we present a robust controller which takes into account uncertainities in stiffness of the flexible links.

[1]The FEM formulation results in*overestimated* natural frequencies and hence $\Delta\boldsymbol{\mathcal{W}} > 0$.

4 Robust Controller Design

There are many approaches in the literature for the design of robust controllers, in this paper, we follow a technique based on the *second method of Lyapunov* as in Leitmann[16].

Let $\mathbf{q}_r^d(t) = (q_{r_1}^d(t), \ldots, q_{r_n}^d(t))^T$ represent a desired trajectory in joint space that we wish the flexible link manipulator system to track. We assume that $\mathbf{q}_r^d$, $\dot{\mathbf{q}}_r^d$, and $\ddot{\mathbf{q}}_r^d$ are *smooth* (i.e. continuously differentiable functions of time) and *bounded*. For the problem of tracking the desired trajectory $\mathbf{q}_r^d(t)$, and its velocity $\dot{\mathbf{q}}_r^d(t)$, the robust model-based joint inversion control law is given by,

$$\mathbf{\Gamma}_{I_r} = \overline{(\mathbf{M}_{rr} - \mathbf{M}_{rf}^T\mathbf{M}_{ff}^{-1}\mathbf{M}_{rf})}\ \ \mathbf{u}_r + \overline{(\mathbf{h}_r + \mathbf{c}_r - \mathbf{M}_{rf}^T\mathbf{M}_{ff}^{-1}(\mathbf{h}_f + \mathbf{c}_f + \mathbf{K}\mathbf{q}_f))} \tag{12}$$

where $\mathbf{u}_r = \ddot{\mathbf{q}}_r^d - \mathbf{G}_p\mathbf{e}(t) - \mathbf{G}_v\dot{\mathbf{e}}(t) + \Delta\mathbf{u}$, and $\Delta\mathbf{u}$ is the additional term that has to be designed to overcome the effects of uncertainty in the system model parameters. The closed-loop system of equations of the system with this control law(12) would then become,

$$\ddot{\mathbf{e}}(t) + \mathbf{G}_v\dot{\mathbf{e}}(t) + \mathbf{G}_p\mathbf{e}(t) = \Delta\mathbf{u} + \Psi(t) \tag{13}$$

$$\mathbf{M}_{ff}\ddot{\mathbf{q}}_f + \mathbf{h}_f + \mathbf{c}_f + \mathbf{K}\mathbf{q}_f = -\mathbf{M}_{rf}(\mathbf{u}_r + \Psi(t)) \tag{14}$$

The control design to follow is based on the fact that although the uncertainty $\Psi(t)$ is unknown, it may be possible to estimate *worst case* bounds on its effects on the tracking performance of the actual flexible link manipulator system. The control law $\mathbf{u}_r$ is then designed to guarantee stability of closed-loop equations(13-14) provided that these quantitative bounds on $\Psi(t)$ are satisfied(see [13] for details). The following steps may be used to compute a stabilizing robust compensator $\Delta\mathbf{u}$.

1. Set the closed-loop system of equations corresponding to joint variables($\mathbf{q}_r$) in state-space form by defining the joint position and velocity error vectors $\mathbf{x}_1 = \mathbf{q}_r(t) - \mathbf{q}_r^d(t)$, and $\mathbf{x}_2 = \dot{\mathbf{q}}_r(t) - \dot{\mathbf{q}}_r^d(t)$, as

$$\dot{\mathbf{x}} = \mathbf{A}_{cl}\mathbf{x} + \mathbf{B}_{cl}(\Delta\mathbf{u} + \Psi(t)) \tag{15}$$

where

$$\mathbf{A}_{cl} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{G}_p & -\mathbf{G}_v \end{pmatrix}, \quad \mathbf{B}_{cl} = \begin{pmatrix} \mathbf{0} \\ \mathbf{I} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \tag{16}$$

2. Given the system(15) with $\mathbf{A}_{cl}$ Hurwitz (i.e. all eigenvalues of the matrix $\mathbf{A}_{cl}$ have negative real parts), suppose we can find a continuous function $\Phi(\mathbf{x}, t)$, which is bounded in time(t), satisfying the inequalities

$$\| \Delta\mathbf{u} \| < \Phi(\mathbf{x}, t) \tag{17}$$

$$\| \Psi(t) \| < \Phi(\mathbf{x}, t) \tag{18}$$

then, we can determine the estimate of this function $\Phi(\mathbf{x},t)$ using equation(17) as

$$\Phi(\mathbf{x},t) \;=\; \frac{1}{1-\alpha_0}\left(\alpha_0 \parallel \mathbf{u} \parallel + \lambda_u \left(\frac{\alpha_0}{\lambda_l}\sum_{i=1}^{n} v_i^2 + \alpha_1 + \alpha_2 \sqrt{\frac{2V_{f,max}}{\lambda_{min}(\mathbf{K})}}\right)\right) \tag{19}$$

where, $\parallel \mathbf{u} \parallel \leq \sqrt{(\sum_{i=1}^{n} a_i^2)} + \parallel \mathbf{G}_v \parallel \cdot \parallel \mathbf{x}_2 \parallel + \parallel \mathbf{G}_p \parallel \cdot \parallel \mathbf{x}_1 \parallel$. In the above equation the quantities α_0, α_1, α_2 are constants related to the bounds on estimates of the mass matrix, the nonlinear Coriolis and centripetal terms and the stiffness matrix respectively, a_i, v_i, $V_{f,max}$ are constants related to the bounds on $\ddot{q}_{r_i}^d$, $\dot{q}_{r_i}^d$ and $\parallel \mathbf{q}_f \parallel$ respectively, and λ_l, λ_u are the minimum and maximum eigenvalues of $\mathbf{M}^{-1}(\mathbf{q})$ respectively(see [13] for more details).

According to the above equation, it is important to have a good estimate of the mass matrix so that α_0 is within the range from 0 to 1. If $\alpha_0 > 1$, the upper bound Φ is negative and hence $\parallel \Psi(t) \parallel$ is bounded by a negative value (see inequality(18)) which is undesirable.

3. Since $\mathbf{A}_{cl}$ is Hurwitz, choose a $2n \times 2n$ symmetric, positive definite matrix $\mathbf{Q}$ and let matrix $\mathbf{P}$ be the unique positive definite symmetric solution to Lyapunov equation

$$\mathbf{A}_{cl}^T\mathbf{P} + \mathbf{P}\mathbf{A}_{cl} + \mathbf{Q} = \mathbf{0} \tag{20}$$

4. Choose the robust compensator $\Delta\mathbf{u}$ then according to

$$\Delta\mathbf{u} = \begin{cases} -\Phi(\mathbf{x},t)\dfrac{\mathbf{B}_{cl}^T\mathbf{P}\mathbf{x}}{\parallel \mathbf{B}_{cl}^T\mathbf{P}\mathbf{x} \parallel} & \text{if } \parallel \mathbf{B}_{cl}^T\mathbf{P}\mathbf{x} \parallel \geq \epsilon \\ -\dfrac{\Phi(\mathbf{x},t)}{\epsilon}\mathbf{B}_{cl}^T\mathbf{P}\mathbf{x} & \text{if } \parallel \mathbf{B}_{cl}^T\mathbf{P}\mathbf{x} \parallel < \epsilon \end{cases} \tag{21}$$

where ϵ is a prescribed non-zero positive constant.

The robust control algorithm with $\mathbf{u}_r$ chosen as $-\Phi(\mathbf{x},t)$ can be shown to be critically stable(see [13]) and the induced flexible vibrations are bounded. Hence, stability of the closed-loop flexible manipulator system is ensured during joint trajectory tracking control(12).

5 Case Study: A Flexible Elbow Manipulator

In this section, we present numerical results of the two-stage control simulation. We use a 3R flexible elbow manipulator (see Figure 1) with forearm and upperarm modeled as slender flexible links. The modeling of the flexibility is by the finite element method as in [11, 13], the dynamic equations of motion are obtained from the kinetic and potential energies and by using the Lagrangian formulation. The physical system parameters of the robot model are given in Table 1.

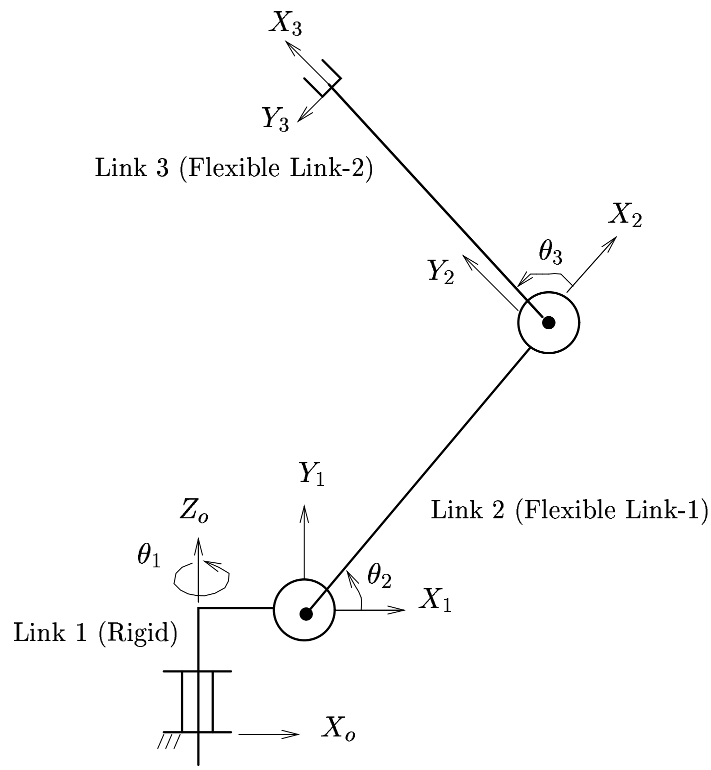

Figure 1: A schematic of a 3R flexible elbow manipulator

Physical system parameters	Value
mass of link 1 (m_1)	3.66 kg
linear mass density of link 2 ($\rho_2 A_2$)	0.331 kgm^{-1}
linear mass density of link 3 ($\rho_3 A_3$)	0.331 kgm^{-1}
mass of payload (m_p)	0.1 kg
length of link 1 (a_1)	0.12 m
length of link 2 (a_2)	1.0 m
length of link 3 (a_3)	1.0 m
rotary inertia of joint 1 (I_{h_1})	0.4 kgm^2
rotary inertia of joint 2 (I_{h_2})	3.275 kgm^2
rotary inertia of joint 3 (I_{h_3})	3.275 kgm^2
flexural rigidity of link 2 ($(EI)_2$)	1165.4916 Nm^2
flexural rigidity of link 3 ($(EI)_3$)	1165.4916 Nm^2

Table 1: Physical system parameters for 3R flexible elbow manipulator

5.1 Numerical Results

The numerical simulation of the two-stage control scheme, for a 3 degree-of-freedom, spatial flexible elbow manipulator (see Figure 1) were performed on a SUN-SPARC-10 Workstation. The first-order differential equations were solved by a variable step, variable order (of interpolation) predictor corrector (PECE) Adams method. The desired trajectory was chosen to be a smooth (sine profile with zero velocity and acceleration at the start and at the end of trajectory) right-circular helix of radius 25cm, pitch 2.5cm and 3π rotations about the helix axis. The time for the entire trajectory was chosen to be relatively small, 1.0 second, to excite oscillations and to evaluate the performance of the two-stage controller in trajectory tracking and suppression of oscillations. The desired trajectory of the end-effector, and the corresponding joint trajectories (assuming the links are rigid) are shown in Figure 2. The parameters of controller gain matrices are as follows: for the I-stage model-based joint inversion controller, the gain matrices are chosen as, $\mathbf{G}_p = \text{diag}\{64.0, 64.0, 64.0\}$ and $\mathbf{G}_v = \text{diag}\{32.0, 32.0, 32.0\}$. For the II-stage impedance controller, the gain matrices are chosen as, $\widehat{\mathbf{G}}_p = \text{diag}\{100.0, 100.0, 400.0\}$ and $\widehat{\mathbf{G}}_v = \text{diag}\{40.0, 40.0, 80.0\}$. The quantitative bounds as required by the robust compensator design for the particular desired joint trajectory are chosen as: $v_1 = 580.23$ deg/sec, $v_2 = 102.89$ deg/sec, $v_3 = 281.92$ deg/sec, $\alpha_0 = 0.25$, $\lambda_l = 0.1034e+01$, $\lambda_u = 0.2708e+05$, $\alpha_2 = 1.0045e+05$, and $\epsilon = 0.001$. We present the simulation results for three cases (for the spatial elbow manipulator):

Case 1: two-stage control algorithm with no uncertainties in model parameters of the model-based joint inversion control law(3).

Case 2: uncertainty in mass parameters of the model-based control law, but without a robust compensator. Mass is underestimated by 25%.

Case 3: uncertainty in both the mass and stiffness parameters of the model-based control law, and with the robust compensator $\Delta\mathbf{u}$ (21). Mass is underestimated by 25% and stiffness is overestimated by 25%.

The results for all three cases are summarized in Table 2 and we present plots of the tip and joint errors (figure 3) and the time history of the flexible variables (figure 4) for Case 3(plots for Case 1 and 2 are available in [13]). For Case 1, the joint errors during the trajectory tracking phase are quite small and the tip errors induced during the trajectory tracking phase (of the order of 5 cm) are suppressed in the second stage by the impedance control law. We can observe from Table 2, that the errors are much larger when there is uncertainty in the model. For the Case 3, we can observe from Figure 3 and Table 2, that the performance of the model-based control law with robust compensator approaches the performance of the control law when there was no uncertainties in model parameters. This clearly illustrates the necessity for a robust compensator in case of uncertainty in model parameters of the model-based joint inversion control algorithms.

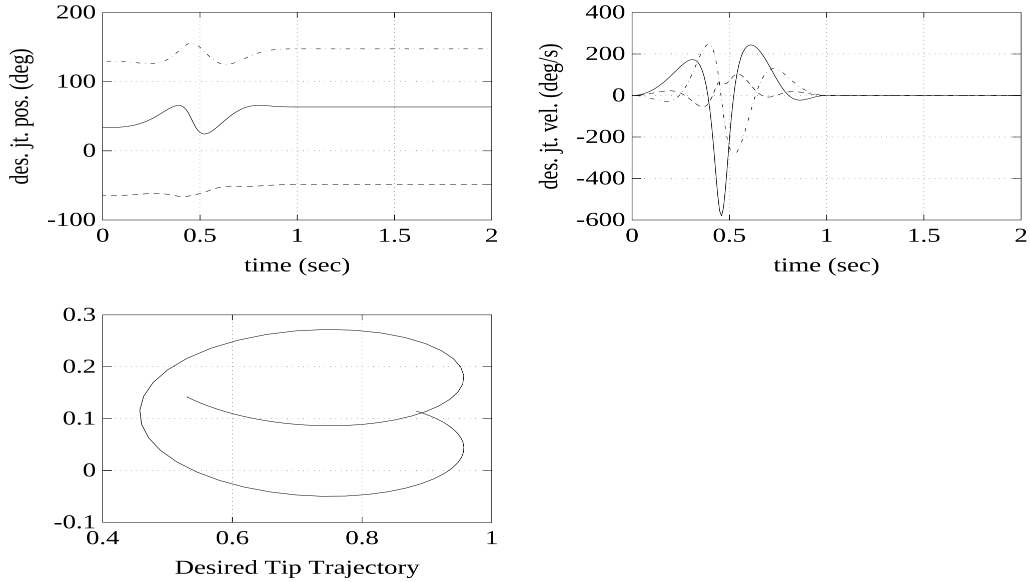

Figure 2: Desired end-effector trajectory: right circular helix, and the corresponding joint position and velocity trajectories of spatial elbow manipulator (— : $q^d_{r_1}(\dot{q}^d_{r_1})$, - - - : $q^d_{r_2}(\dot{q}^d_{r_2})$, -·-·- : $q^d_{r_3}(\dot{q}^d_{r_3})$)

		case 1		case 2		case 3	
		peak value	final value	peak value	final value	peak value	final value
jt. pos. error (deg)	e_1	0.027	0.002	4.515	0.044	0.027	0.002
	e_2	1.186	1.186	8.466	1.551	1.187	1.187
	e_3	0.337	0.039	19.291	1.335	0.329	0.040
jt. vel. error (deg/s)	$\dot{e}_1$	1.209	0.035	81.308	0.066	1.355	0.018
	$\dot{e}_2$	29.342	0.071	61.977	0.882	29.034	0.069
	$\dot{e}_3$	9.638	0.039	106.822	3.101	9.468	0.038
tip pos. error (m)	e_x	0.011	0.002	0.202	0.012	0.011	0.002
	e_y	0.011	0.004	0.299	0.026	0.011	0.004
	e_z	0.046	0.013	0.238	0.014	0.046	0.013
tip vel. error (m/s)	$\dot{e}_x$	0.090	0.0001	0.949	0.025	0.090	0.0006
	$\dot{e}_y$	0.120	0.001	1.814	0.052	0.120	0.0008
	$\dot{e}_z$	0.288	0.0006	1.247	0.003	0.291	0.0005

Table 2: Joint and Tip errors of spatial elbow manipulator for two-stage controller

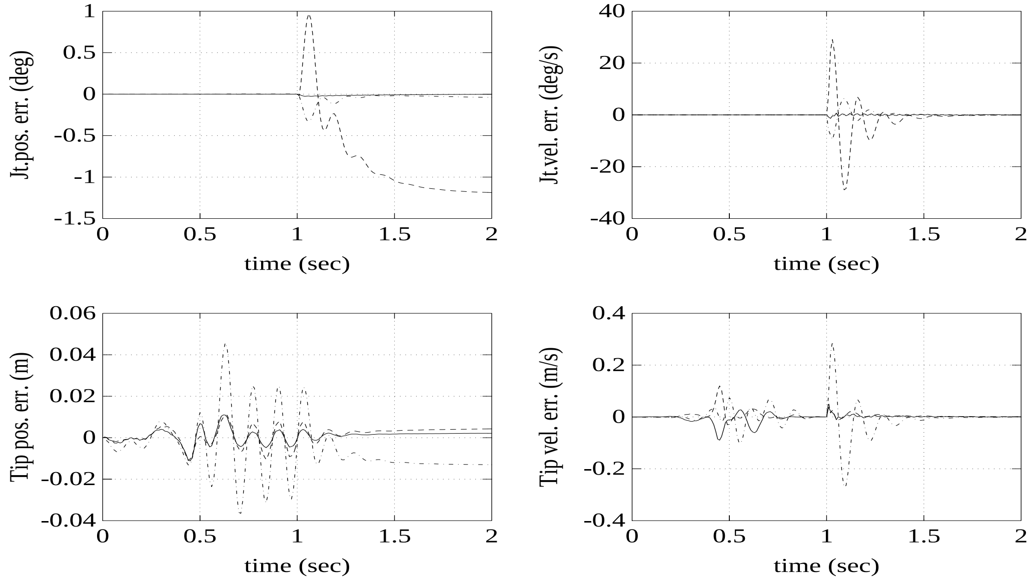

Figure 3: **Case 3 :** time history of the joint position, velocity, and tip position, velocity errors for two-stage controller (Joint error: — : $e_1(\dot{e}_1)$, - - - : $e_2(\dot{e}_2)$, -·-·- : $e_3(\dot{e}_3)$; Tip error: — : $e_x(\dot{e}_x)$, - - - : $e_y(\dot{e}_y)$, -·-·- : $e_z(\dot{e}_z)$)

6 Summary

In this paper, we have addressed the problem of active damping of flexural vibrations of the robot links by only the joint control inputs. We have presented robustness results for a model based control algorithm used for the trajectory tracking in multi-link flexible manipulator systems. We have shown analytically that over-estimation of natural frequencies, a consequence of finite element modeling, may lead to unstable closed-loop response of the actual manipulator system using a model based inversion control algorithm. A robust controller design based on the second method of Lyapunov, using simple quantitative bounds on the model uncertainties, is proposed for use in multi-link flexible manipulator control. Numerical results for a flexible spatial elbow manipulator is used to illustrate the effectiveness of the proposed control schemes.

References

[1] T. E. Alberts, H. Xia, and Y. Chen, "Dynamic analysis to evaluate viscoelastic passive damping augmentation for the space shuttle remote manipulator system", *ASME Journal of Dynamic Systems, Measurement, and Control*, vol. 114, pp. 468–475, 1992.

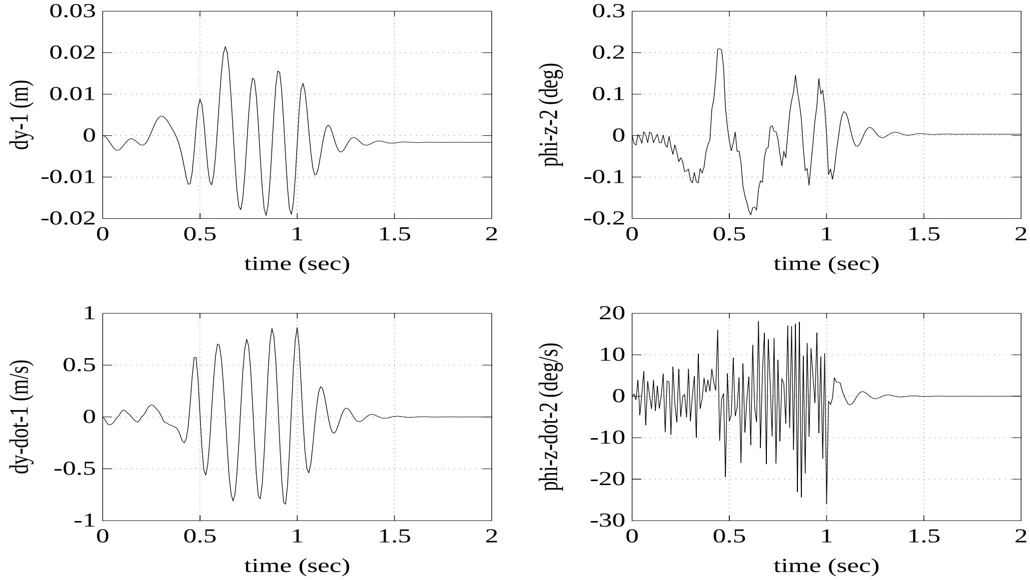

Figure 4: **Case 3 :** time history of the elastic deflection variable along the Y-direction, at the tip of flexible link-1, and its rate; time history of the elastic rotation variable about the Z-direction, at the tip of flexible link-2, and its rate

[2] W. J. Book, O. Maizzo-Neto, and D. E. Whitney, "Feedback control of two-beam, two-joint systems with distributed flexibility", *ASME Journal of Dynamic Systems, Measurement, and Control*, vol. 97, pp. 424–431, 1975.

[3] C. M. Oakley and R. H. Cannon Jr., "Initial experiments on the control of a two-link manipulator with a very flexible forearm", in *Proceedings of American Control Conference*, 1988, pp. 996–1002.

[4] A. De Luca and B. Siciliano, "Trajectory control of a nonlinear one-link flexible arm", *International Journal of Control*, vol. 50, no. 5, pp. 1699–1715, 1989.

[5] S. N. Singh and A. A. Schy, "Elastic robot control: Nonlinear inversion and linear stabilization", *IEEE Transactions on Aerospace and Electronic Systems*, vol. 22, no. 4, pp. 340–347, 1986.

[6] P. Chedmail, Y. Aoustin, and Ch. Chevallereau, "Modelling and control of flexible robots", *International Journal for Numerical Methods in Engineering*, vol. 32, pp. 1595–1619, 1991.

[7] D. M. Rovner and R. H. Cannon Jr., "Experiments toward on-line identification and control of a very flexible one-link manipulator", *The International Journal of Robotics Research*, vol. 6, no. 4, pp. 3–19, 1987.

[8] V. V. Korolov and Y. H. Chen, "Controller design robust to frequency variation in a one-link flexible robot arm", *ASME Journal of Dynamic Systems, Measurement, and Control*, vol. 111, pp. 9–14, 1989.

[9] S. Cetinkunt and W.-L. Yu, "Closed-loop behavior of a feedback-controlled flexible arm: A comparative study", *The International Journal of Robotics Research*, vol. 10, no. 3, pp. 263–275, 1991.

[10] T. Ravichandran, G. K. H. Pang, and D. Wang, "Robust h_∞ optimal control of a single flexible link", *Control-Theory and Advanced Technology*, vol. 9, no. 4, pp. 887–908, 1993.

[11] R. J. Theodore and A. Ghosal, "Comparison of the assumed modes and finite element models for flexible multi-link manipulators", *The International Journal of Robotics Research*, vol. 14, no. 2, pp. 91–111, 1995.

[12] A. De Luca and B. Siciliano, "Inversion-based nonlinear control of robot arms with flexible links", *AIAA Journal of Guidance, Control, and Dynamics*, vol. 16, no. 6, pp. 1169–1176, 1993.

[13] R. J. Theodore, *Dynamic Modeling and Control Analysis of Multilink Flexible Manipulators*, PhD thesis, Department of Mechanical Engineering, Indian Institute of Science, 1995.

[14] A. De Luca and B. Siciliano, "Closed-form dynamic model of planar multilink lightweight robots", *IEEE Transactions on Systems, Man and Cybernetics*, vol. 21, no. 4, pp. 826–839, 1991.

[15] L. Meirovitch, *Computational Methods in Structural Dynamics*, Sijthoff & Noordhoff International Publishers, The Netherlands, 1980.

[16] G. Leitmann, "On the efficacy of nonlinear control in uncertain linear systems", *ASME Journal of Dynamic Systems, Measurement, and Control*, vol. 102, pp. 95–102, 1981.