P4 Kinematics

Dr Alexander M. Korsunsky

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LR1 Thom Building
Wk 5: M. 11, T. 12, Th. 12
Wk 6: M. 11

Scope of the lecture course:

1. **Kinematics of particles**

– Basic definitions and revision
– Rectilinear motion under constant and varying acceleration
– Plane curvilinear motion problems
– Rectangular coordinates (*x*-*y*)
– Tangential and normal coordinates (*t*-*n*)
– Polar coordinates (*r*-*q*)

2. **Plane kinematics of rigid bodies**

– What is rigid body motion?
– Rotation
– Relative velocity
– Instantaneous centre of rotation

3. **Kinematic analysis of mechanisms**

– Graphical and analytical approaches
– Analysis of mechanisms by velocity diagram
– Example applications:
    – Crank-slider mechanisms
    – Four-bar chain
    – Whitworth quick-return mechanism
– Some practical applications

Preface

How this lecture course is organised

This course covers the following key topics:

Plane kinematics of particles: rectilinear and curvilinear motion in rectangular, normal-tangential, and polar coordinates. Relative motion (translating, but not rotating, axes). *Plane kinematics of rigid bodies:* translation, rotation, and general plane motion; relative velocity; rotation about a fixed axis. *Velocity diagrams* for simple planar assemblies of rigid bodies including bars, pin joints, and sliding joints.

Kinematics of particles

Most of this topic will be discussed in the first lecture, and will consist of the development of the equations for the kinematics of a particle moving along a straight or curved path under conditions of constant or varying acceleration. Initially we will use rectangular coordinates, so I expect that this topic will be familiar to most of you from the Physics and Mathematics work that you completed at school. We will then go further, and consider descriptions of particle motion using tangential-normal (t-n) and polar (r-*q*) coordinates. For further information on this topic I recommend Chapter 2 in the second volume (Dynamics) of **Meriam and Kraige** (see reference list below).

Kinematics of rigid bodies

This topic will be covered in the second and third lectures. We will recognise that solid bodies can not only translate, but also rotate, and use the concepts of angular velocity and angular acceleration to describe rotation. We will discuss relative motion and introduce and use the relative velocity equation. We will also develop the idea of the *instantaneous centre of rotation* to describe rigid body motion. Once again, I would like to refer you to the book by **Meriam and Kraige** (Chapter 5) for additional information.

Kinematic Analysis of Mechanisms

This section of the course will be the subject of lectures three and four, and will help you understand how kinematics is used to analyse the behaviour of engineering mechanisms. This will be a new topic for most of you, and will be based mainly on the use of a graphical technique, called the *velocity diagram*, to calculate linear and angular velocities in simple mechanisms involving pins, rigid bars and sliders. This graphical technique may be used to obtain solutions either by scale drawing or on the basis of an analysis of the geometry of the velocity diagram.

Experience shows that undergraduates usually find the fundamental concepts on which the velocity diagram is based to be reasonably straightforward. What is more difficult is learning to apply these concepts to real problems. To help you with this, I will be working through a series of examples, in the hope that you will be able to apply the method yourself when you have seen it demonstrated in this way during the lectures. The various examples I will be describing have been selected to start at a fairly fundamental level, and then to gradually build up in complexity so that you are exposed to new topics at a steady pace.

Unfortunately few textbooks give a good coverage of the topic of velocity diagrams. The books by Drabble and also Grosjean (see reference list below), however, do gie a reasonable (but rather brief) coverage of the topic.

What is the purpose of the lecture notes?

The lecture notes contain a certain amount of theory: I hope that this will be of use to you when you tackle the P4F Kinematics' tutorial sheet and (eventually) paper P4 in the Prelims. Much of the lecture course will consist of working through example problems, however, and in the notes I have provided an outline of each example. The handouts contain space for you to fill in the solutions to the examples during the lectures themselves. I hope you will want to take this opportunity to make your own notes in the spaces provided in the handouts. This should mean that, at the end of the lecture course, you will have obtained a complete set of notes and, in the process, you will have gained some good experience in solving the equations of motion in different coordinates and drawing velocity diagrams.

Recommended reading

Meriam, J.L. and Kraige, L.G. (1999) *Engineering Mechanics vol.2: Dynamics*, MacMillan.
Drabble, G.E. (1990) *Dynamics* Programmes 2 and 4, MacMillan.
Fawcett, J.T. and Burdess, J.S. (1988) *Basic Mechanics with Engineering Applications*, Arnold.
Grosjean, J. (1991) *Kinematics and Dynamics of Mechanisms*, McGraw-Hill.
Norris, C.H., Wilber, J.B. and Utku, S. (1991) *Elementray Structural Analysis,* McGraw-Hill.

Other reading

Mabie, H.H. and Ocvirk, F.W. *Mechanics and Dynamics of Machinery*, John Wiley.
Prentice, J.M. *Dynamics of Mechanical Systems*, Longman.
Hannah, J. And Stephens, R.C. *Mechanics of Machines: Elementary theory and Examples*, Arnold.
Wilson, C.E. and Sadler, J.P. *Kinematics and Dynamics of Machinery*, Addison-Wesley.

Kinematics

Kinematics is the branch of dynamics that consists of the study of motion without the reference to the forces that cause, or are developed by, the motion.

In this lecture course, three distinct areas of kinematic analysis are dealt with. The first section consists of the analysis of the kinematics of linear motion of a particle under constant and varying acceleration; much of this material will be familiar to you from your previous work in Physics.

In the second section of the course the kinematics of curvilinear motion is analysed using tangential-normal and polar coordinates. Relative motion is considered, and expressions for velocities and accelerations with respect to different reference frames are used to solve example problems.

In the third section of the course, a discussion is given of the analysis of the kinematics of simple mechanisms. This analysis requires consideration of the various constraints to the motion of the mechanism that arise as a result of the way in which the mechanism is connected.

1. Kinematics of a Particle

A particle is a body that is assumed to have mass but negligible physical dimensions. Whenever the dimensions of a body are irrelevant to the problem then the use of particle mechanics may be expected to provide accurate results. Typical applications would be the analysis of the motion of a spacecraft orbiting the earth, or the trajectory of a golf ball after it has been struck. Later on in your course you will learn about rigid-body mechanics which is an approach that needs to be adopted when the dimensions of the body cannot be neglected.

1.1 Linear Motion of a Particle under Variable Acceleration

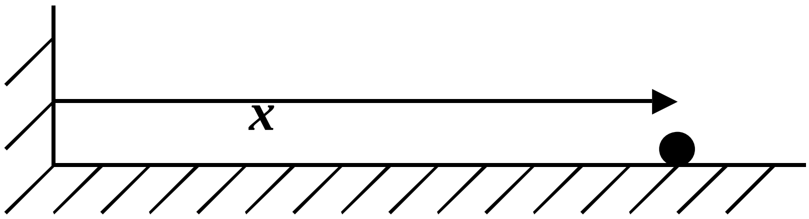

Fig. 1-1

Velocity $v = \frac{dx}{dt} = \dot{x}$ (1.1.1)

Acceleration $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \ddot{x}$ (1.1.2)

The acceleration equation is often the starting point for solving the equations of motion, since Newton's law states that it is equal to F/m, where F is the resultant force acting upon the particle of mass m.

The above equations can be integrated to give:

$$v(t) = v_0 + \int_0^t a(t)\mathrm{d}t \qquad x(t) = x_0 + \int_0^t v(t)\mathrm{d}t \tag{1.1.3}$$

For constant acceleration a this gives

$$v(t) = v_0 + at \qquad x(t) = x_0 + v_0 t + \tfrac{1}{2}at^2 \tag{1.1.4}$$

The expression for particle acceleration may be re-cast in an alternative way:

$$a = \ddot{x} = \frac{d\dot{x}}{dt} \tag{1.1.5}$$

$$\frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx}\frac{dx}{dt} = \dot{x}\frac{d\dot{x}}{dx} = v\frac{dv}{dx} \tag{1.1.6}$$

This new expression for acceleration may be integrated as follows:

$$v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{v^2}{2}\right) = a \tag{1.1.7}$$

$$\frac{v^2}{2} - \frac{{v_0}^2}{2} = \int_{x_0}^{x} a(x)\mathrm{d}x\,. \tag{1.1.8}$$

When the acceleration is constant, this integral may be evaluated to give:

$$\frac{v^2}{2} - \frac{{v_0}^2}{2} = as, \quad \text{where} \quad s = x - x_0\,. \tag{1.1.9}$$

This is the same as the equation familiar from school Physics:

$$v^2 = {v_0}^2 + 2as\,. \tag{1.1.10}$$

Example 1-A. A parachutist jumps out of a plane and free falls. His downward acceleration arises due to combination of gravity and air resistance (drag), and is given by $a = g - kv^2$, where v is the velocity.

(a) Find the expression for the terminal velocity.

(b) Assume that the terminal velocity is 20m/s. How fast will the parachutist be travelling after falling 50m?

(c) Find out how long it takes to fall 100m.

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a. Terminal velocity is reached when no further acceleration or deceleration is taking place, i.e. a=0. Hence $g - kv_t^2 = 0$, and $v_t = vg/k$.

b. Rewrite the expression for acceleration using equation (1.1.7):

$$a = v\frac{dv}{dx} = g - kv^2 .$$

Think of velocity as the variable and position as the function: express the increment dx in terms of v and dv:

$$dx = \frac{vdv}{g - kv^2} .$$

Both sides of this expression can now be integrated. The left hand side yields the total drop x. The right hand side must be integrated from the initial velocity 0 to v:

$$x = \int_0^v \frac{vdv}{g - kv^2} = -\frac{v_t^2}{2g}\ln\left|1 - \left(\frac{v}{v_t}\right)^2\right| .$$

The graph of velocity vs position is shown in Fig.1-2. The speed for x=50m can be determined from the graph, or by expressing v in terms of x from the above formula.

(Ans.: $v = v_t \sqrt{1 - \exp\left(-\frac{2gx}{v_t^2}\right)}$, v=18.3m/s at x=50m)

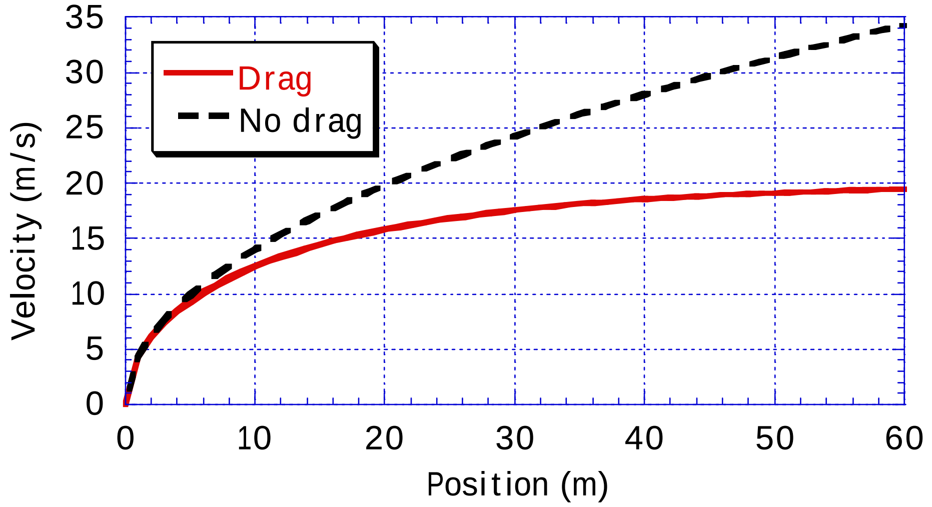


Fig.1-2.

c. A similar trick can be used in order to determine t as a function of v:

$$a = \frac{dv}{dt} = g - kv^2, \qquad dt = \frac{dv}{g - kv^2}, \text{ hence}$$

$$t = \int_0^v \frac{dv}{g - kv^2} = \frac{v_t}{2g} \ln\left|\frac{v_t + v}{v_t - v}\right|.$$

The graph of position vs time can be obtained by combining the above results, and is shown in Fig.1-3.

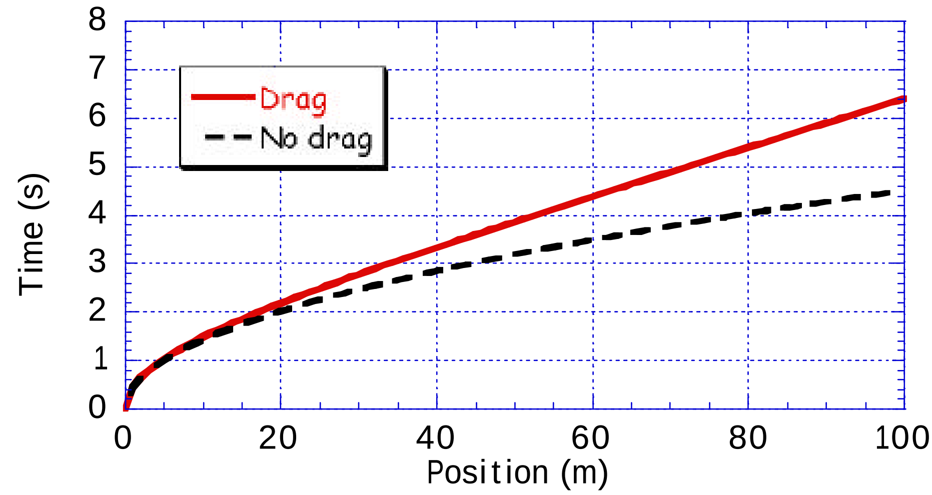


Fig.1-3.

The time to travel 100m can now be read off Fig.1-3.
(Ans.: t=6.4 s)

1.2 Two-Dimensional Motion of a Particle

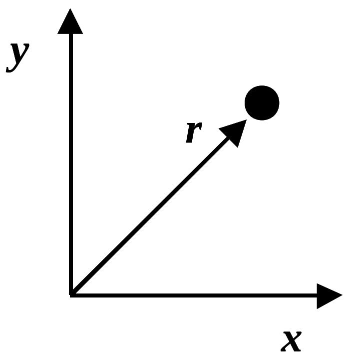


Fig. 1-2

We consider motion of a particle with respect to a *fixed* coordinate system.

Consider a Cartesian rectangular coordinate system, with a pair of orthogonal basis vectors ***i*** and ***j*** defined. The particle position is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} \tag{1.2.1}$$

Since the orientation of the coordinate vectors ***i*** and ***j*** does not change during motion, differentiation of the above equation gives

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} = v_x\mathbf{i} + v_y\mathbf{j} \tag{1.2.2}$$

$$\mathbf{a} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} = a_x\mathbf{i} + a_y\mathbf{j} \tag{1.2.3}$$

When the m-component of acceleration (m=x or m=y) is prescribed as a function of time, the above expressions can be integrated to solve the equations of motion, i.e. determine the position s_m and velocity v_m components for the particle at any moment:

$$v_m(t) = v_{m0} + \int_0^t a_m(t)dt \tag{1.2.4}$$

$$s_m(t) = s_{m0} + \int_0^t v_m(t)dt \tag{1.2.5}$$

If a_m is constant, the integration can be performed explicitly:

$$v_m(t) = v_{m0} + a_m t \tag{1.2.6}$$

$$s_m(t) = s_{m0} + v_{m0}t + \tfrac{1}{2}a_m t^2 \tag{1.2.7}$$

Example 1-B. A ball is thrown from point A to land at B (Fig. 1-5). Find suitable values of v_0 and a.

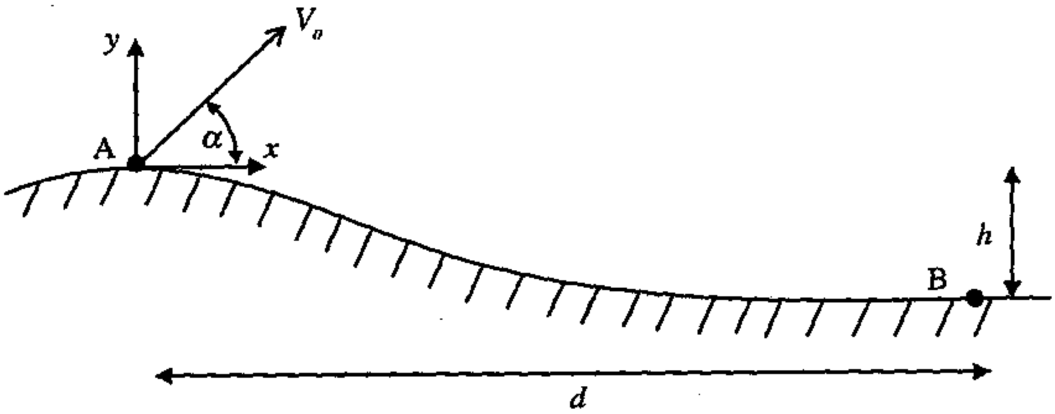


Fig. 1-5.

The acceleration is constant and a_x=0, a_y=-g. The position of the ball as a function of time can be determined from equation (1.2.7):

$$x = v_0 \cos a \; t, \qquad y = v_0 \sin a \; t - gt^2 .$$

Elimination of t from the above pair of equations leads to

$$y = x \tan a - \tfrac{1}{2} g \left(\frac{x}{v_0 \cos a} \right)^2 .$$

We find trajectories passing through point B by putting x=d and y=-h into the above:

$$-h = d \tan a - \frac{gd^2}{2{v_0}^2 \cos^2 a} .$$

Using trigonometry this can be expressed as

$$\frac{h}{d} + \tan a - \frac{1}{2}\left(1 + \tan^2 a\right)\frac{gd}{{v_0}^2} = 0 .$$

The values of tan$a$ (and hence a) and v_0/vgd can be found for any specified value of h/d. The solutions are plotted in Fig.1-6.

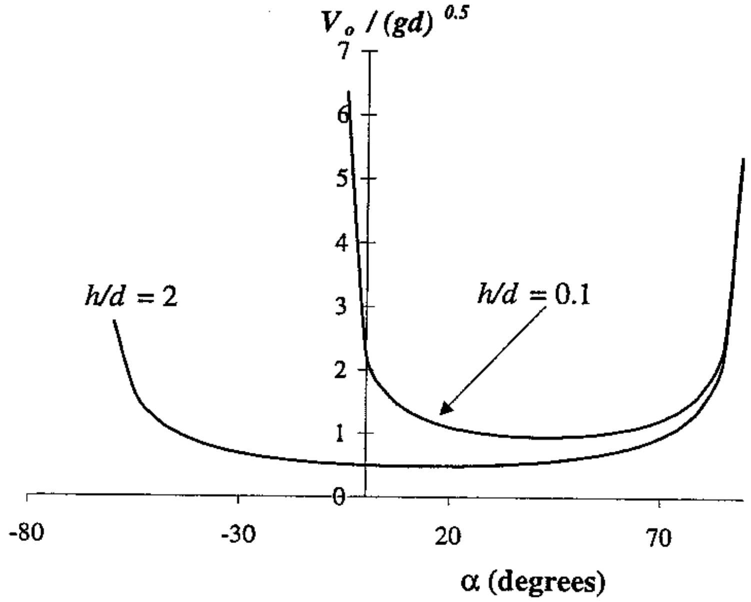


Fig. 1-6.

Note from the graph that, for some values of v_0 and d, there exist two different solutions for a, a high and a low value. In this case for the same initial velocities and target position, two different trajectories are available (Fig.1-7).

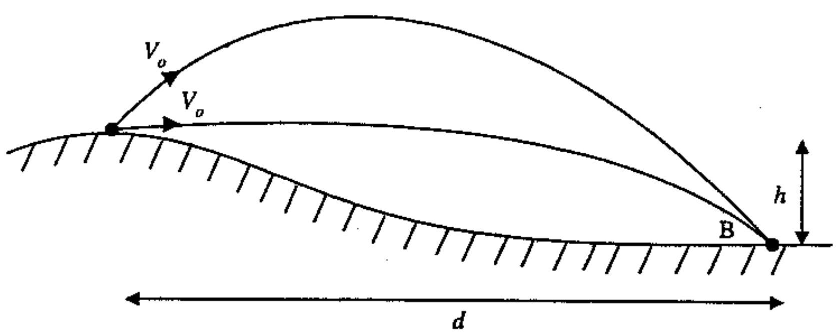


Fig.1-7.

1.3 Normal and Tangential Coordinates

It is often convenient to describe curvilinear motion using path variables, i.e. measurements made along the tangent *t* and normal *n* to the path. The frame can be pictured as a right-angled bracket moving along with the particle. The *t* arm always points in the direction of travel, while the *n* arm points towards the centre of curvature. Unit vectors $\boldsymbol{e}_t$ and $\boldsymbol{e}_n$ are shown in Fig.1-8.

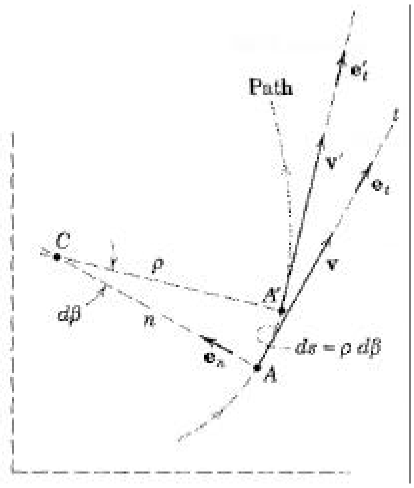


Fig.1-8.

As during time *dt* particle moves from A to A', the increment of path variable *s* is

$$ds = r\ d b, \tag{1.3.1}$$

where r is the path curvature. The speed is then $v = ds/dt = r\ d b/dt = r\ \dot{b}$. The velocity vector is tangential to the path:

$$\mathbf{v} = v\mathbf{e}_t, \tag{1.3.2}$$

$$v = r\ \dot{b}. \tag{1.3.3}$$

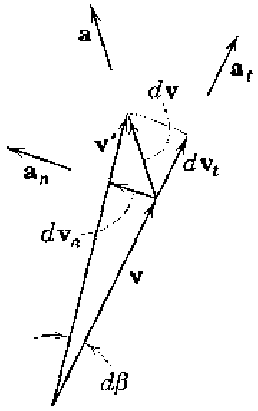


Fig.1-9.

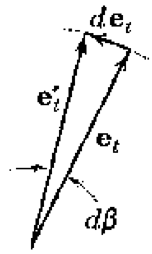


Fig.1-10.

To find the acceleration vector we must differentiate velocity vector, $\mathbf{a} = d\mathbf{v}/dt$. Acceleration vector reflects the change in both magnitude and direction of **v** (Fig.1-9). To perform differentiation, apply standard product rule to equation (1.3.2):

$$\mathbf{a} = \dot{v}\mathbf{e}_t + v\dot{\mathbf{e}}_t . \qquad (1.3.4)$$

Fig.1-10 illustrates how the derivative of vector $\mathbf{e}_t$ is found: its end swings by $d\beta$ in the direction of $\mathbf{e}_n$. Dividing this increment by *dt* we establish

$$\dot{\mathbf{e}}_t = \dot{\beta}\mathbf{e}_n . \qquad (1.3.5)$$

Substituting this into (1.3.4) and using (1.3.3) leads to the following series of expressions for acceleration:

$$\mathbf{a} = \dot{v}\mathbf{e}_t + \frac{v^2}{r}\mathbf{e}_n \qquad (1.3.6)$$

$$a_t = \dot{v} = \ddot{s} \qquad (1.3.7)$$

$$a_n = \frac{v^2}{r} = r\dot{\beta}^2 = v\dot{\beta} \qquad (1.3.8)$$

$$a = \sqrt{a_t^{\,2} + a_n^{\,2}} \qquad (1.3.9)$$

Circular Motion

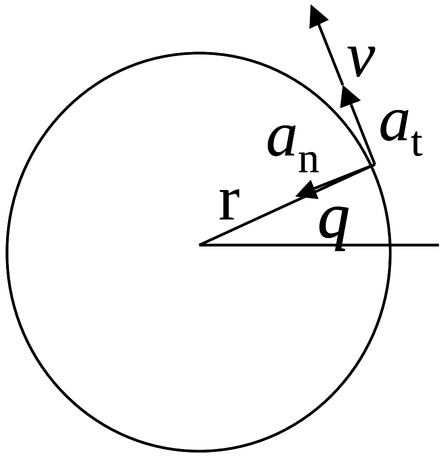


Fig.1-11.

Circular motion is an important special case (Fig.1-11).

We replace *r* with circle radius *r*, angle *b*with *q*, and repeat the formulas:

$$v = r\dot{q} \tag{1.3.3a}$$

$$a_t = \dot{v} = r\ddot{q} \tag{1.3.7a}$$

$$a_n = \frac{v^2}{r} = r\dot{q}^2 = v\dot{q} \tag{1.3.8a}$$

Example 1-C. A water-skier is drawn forward on a line at u =7.1 m/s, and follows a curved path (Fig.1-12), which can be approximated by a circular arc of r=20 m. Find the velocity and acceleration components when angle q=45º.

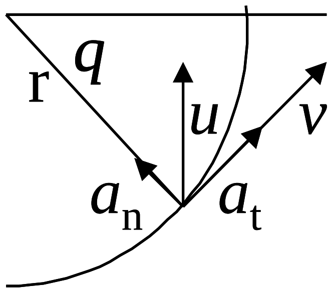


Fig.1-12.

Since forward speed $u = v\cos q = v/\sqrt{2}$ must equal 7.1 m/s, then v=10 m/s.

Since $v = -r\dot{q}$, then $\dot{q} = -\dfrac{u}{r\cos q}$ =-0.5 rad/s (the sign shows that q is decreasing).

Tangential component of acceleration is given by $a_t = \dot{v} = -r\ddot{q}$. By differentiation

$$\ddot{q} = \left(\frac{u}{r\cos^2 q}\right)(-\sin q)\dot{q} = \dot{q}^2 \tan q = 0.25,$$

so that a_t=-5 m/s^2 (the skier is decelerating).

Normal component of acceleration is $a_n = r\dot{q}^2 = 5$ m/s^2.

1.4 Polar Coordinates

The third option is to locate the particle by the radial distance r and angular position q with respect to a chosen fixed direction. We choose unit vectors $\mathbf{e}_r$ and $\mathbf{e}_q$ as shown in Fig.1-13.

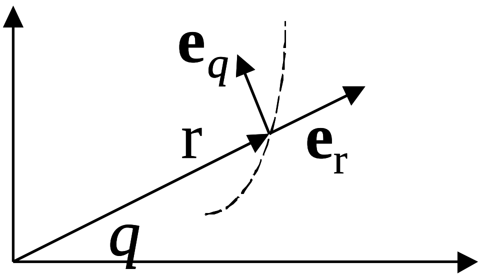


Fig.1-13.

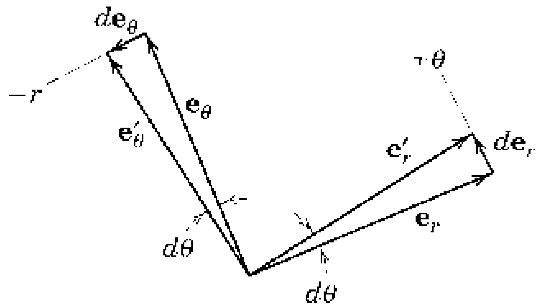


Fig.1-14.

The position vector can be expressed as

$$\mathbf{r} = r\mathbf{e}_r \,. \tag{1.4.1}$$

To find velocity vector **v** we need to differentiate **r** with respect to time using the product rule. Using the same vector construction we used for normal-tangential analysis, we first find (Fig.1-14)

$$d\mathbf{e}_r = \mathbf{e}_q dq, \qquad d\mathbf{e}_q = -\mathbf{e}_r dq \,. \tag{1.4.2}$$

Then diving both sides by dt, we establish

$$\dot{\mathbf{e}}_r = \dot{q}\mathbf{e}_q, \qquad \dot{\mathbf{e}}_q = -\dot{q}\mathbf{e}_r \,. \tag{1.4.3}$$

We now find the velocity

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r = \dot{r}\mathbf{e}_r + r\dot{q}\mathbf{e}_q \,. \tag{1.4.4}$$

In terms of components

$$v_r = \dot{r} \tag{1.4.5}$$

$$v_q = r\dot{q} \tag{1.4.6}$$

$$v = \sqrt{{v_r}^2 + {v_q}^2} \,. \tag{1.4.7}$$

To obtain acceleration, we differentiate the velocity vector **v** according to the same rules. Note that in the term $r\dot{q}\mathbf{e}_q$ each factor must be differentiated in turn.

$$\mathbf{a} = \dot{\mathbf{v}} = (\ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r) + (\dot{r}\dot{q}\mathbf{e}_q + r\ddot{q}\mathbf{e}_q + r\dot{q}\dot{\mathbf{e}}_q).$$

Collecting the terms

$$\mathbf{a} = (\ddot{r} - r\dot{q}^2)\mathbf{e}_r + (r\ddot{q} + 2\dot{r}\dot{q})\mathbf{e}_q. \qquad (1.4.8)$$

In terms of components

$$a_r = \ddot{r} - r\dot{q}^2 \qquad (1.4.9)$$
$$a_q = r\ddot{q} + 2\dot{r}\dot{q} \qquad (1.4.10)$$
$$a = \sqrt{a_r^{\ 2} + a_q^{\ 2}}\ . \qquad (1.4.11)$$

Example 1-D. A rocket is fired vertically and tracked by the radar shown in Fig.1-15. For q=60°, the measurements show r=9 km, $\ddot{r}$=21 m/s^2, and $\dot{q}$=0.02 rad/s. Find the velocity and acceleration of the rocket at this position.

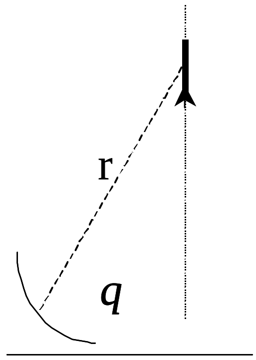


Fig.1-15.

Start with velocity component $v_q = r\dot{q}$ =180 m/s.

Note that $v_q = v\cos q$, hence find v=360 m/s.

Also, $v_r = \dot{r} = v\sin q$ =312 m/s.

Radial acceleration component is $a_r = \ddot{r} - r\dot{q}^2$ =17.4 m/s^2.

Since the rocket moves along a straight vertical line, total acceleration must also be vertical. Hence we find $a = a_r / \sin q$ =20 m/s^2, and $a_q = a_r / \tan q$ =10 m/s^2.

Finally, $\ddot{q} = (a_q - 2\dot{r}\dot{q})/r$ =-0.0003 rad/s^2.

2. Kinematics of a Rigid Body

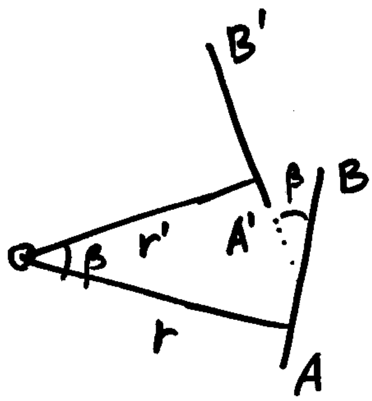
Rigid bodies are different from particles: rigid bodies are extended in space, and the connection between different parts of a rigid body are permanent and unchanged throughout their motion.

When we try to imagine the position of a rigid body, we may first think of an arbitrary point, a marker (let's call it *A*), on that body. The position and motion of this chosen point can be described in exactly the same way we used for particles, e.g. using rectangular or polar coordinates. However, this description is insufficient to describe the motion of the rigid body, since the body as a whole may rotate around point *A*. In order to fix the position of the body we must specify a direction from point *A* to another arbitrarily chosen point on the body, say *B*. Once the position of *A* is fixed, and the direction from *A* to *B* is given as well, position of the rigid body is fully specified. E.g. a rigid rod connecting two points *A* and *B* is a rigid body.

A rigid body is in *plane motion* if all point of the body move parallel to one plane, which is called the *plane of motion*. We can classify the kinds of plane motion into:

(a) translation (rectilinear)
(b) translation (curvilinear)
(c) rotation (around fixed axis)
(d) general plane motion

In *translation* every line in the body remains parallel to its original position, and no rotation is allowed. The trajectory of motion is the line traced by a point in the body. Translation is *rectilinear* if this line is straight, and *curvilinear* otherwise.



Rotation about a fixed axis is the angular motion about this axis, when all points on the body follow concentric circular paths. It is important to picture and understand that all lines on the solid body, even those that do not pass through the centre, rotate through the same angle in the same time.

General plane motion of a rigid body is a combination of translation and rotation. We will discuss the concept of relative motion in order to describe this case.

2.1 Rotation

Consider any two lines 1 and 2 attached to (or drawn on) a rigid body, which have angular orientations described by q_1 and $q_2 = q_1 + b$ Because the body is rigid, b is fixed, so that $\Delta q_2 = \Delta q_1$. Differentiation with respect to time gives

$$\dot{q}_2 = \dot{q}_1, \qquad \ddot{q}_2 = \ddot{q}_1 .$$

> *All lines on a rigid body have the same angular displacement, the same angular velocity and the same angular acceleration.*

The angular velocity w is the time derivative of the angular position of the body,

$$w = \frac{dq}{dt} = \dot{q} . \qquad (2.1.1)$$

The angular acceleration a is the second time derivative of the angular position coordinate of the body,

$$a = \frac{dw}{dt} = \dot{w} = \frac{d^2q}{dt^2} = \ddot{q} . \qquad (2.1.2)$$

When a body rotates about a fixed axis, all points follow concentric circular paths. The linear velocity and acceleration of an arbitrary point can be written using the formulas we introduced in our discussion of particle kinematics:

$$\begin{aligned} v &= w\, r \\ a_n &= w^2 r = v^2 / r \\ a_t &= a\, r \end{aligned} \qquad (2.1.3)$$

It is useful for you to learn how these quantities can be expressed using the cross-product vector notation. A cross-product $w\times\boldsymbol{r}$ of two vectors w and ***r*** is a vector ***v*** of magnitude $v = w\ r \sin f$, where f is the angle between w and ***r***. Vector ***v*** points normal to both w and ***r*** (and therefore normal to the plane of w and ***r***) in the direction defined by the right-hand rule (Fig.2-1). Note that interchanging the order of w and ***r*** in the cross-product changes the sign of ***v***.

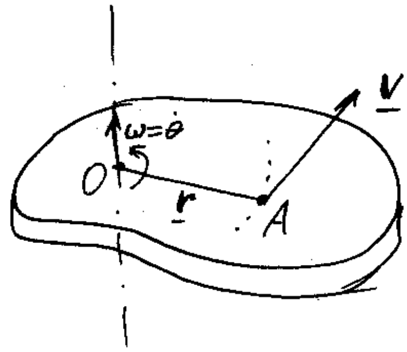


Fig.2-1.

The case is illustrated in Fig.2-1. Angular velocity w is a **vector** of magnitude $\dot{q}$ and pointing in the direction normal to the plane of motion in accordance with the right-hand rule. Using the cross-product notation we can write down the following result:

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \mathbf{?} \times \mathbf{r} \qquad (2.1.4)$$

The significance of this formula is as follows: if a vector ***r*** is constant and attached to a solid body which is rotating around a fixed axis with angular velocity w, then its time derivative is given by the cross-product of w into ***r***.

Note that this applies to **any fixed** vector attached to the solid body. However, it is wrong to use this formula if the vector itself is changing in direction or magnitude.

For the acceleration one obtains:

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{?} \times \mathbf{r}) = \mathbf{?} \times \dot{\mathbf{r}} + \mathbf{?} \times \mathbf{r} = \mathbf{?} \times (\mathbf{?} \times \mathbf{r}) + \mathbf{a} \times \mathbf{r} = \mathbf{a}_n + \mathbf{a}_t . \qquad (2.1.5)$$

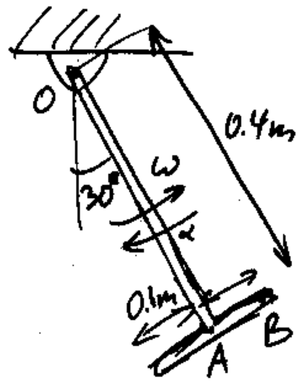


Fig.2-2.

Example 2-A. A T-shaped pendulum rotates about a horizontal axis through point O. At the instant shown in Fig.2-2 its angular velocity is w=3 rad/s and its angular acceleration is a=14 rad/s^2 in the directions indicated. Determine the velocity and acceleration of point A and point B, expressing the results in terms of components along the *n*- and *t*-axes shown.

Velocities: $\mathbf{v} = ? \times \mathbf{r} = ? \times (\mathbf{r}_t + \mathbf{r}_n) = ? \times \mathbf{r}_t + ? \times \mathbf{r}_n = \mathbf{v}_n - \mathbf{v}_t$.

Point A is only moving tangentially (v_n=0) at speed - $w r_n$=1.2m/s.

Point B is moving normally at speed $w r_t$=0.3m/s and tangentially at speed - $w r_n$=1.2m/s.

Total speed of B is 1.24 m/s.

Accelerations: $a_n = w^2 r, \quad a_n = a\,r.$

Point A has normal acceleration a_n= $3^2 \cdot 0.4$ = 3.6 m/s^2 towards point 0, and tangential acceleration a_t= -14 $\cdot$0.4 = -5.6 m/s^2.

Distance OB=0.412m. Point B has acceleration $3^2 \cdot 0.412$ = 3.71 m/s^2 towards point 0, and also acceleration -14 $\cdot$0.412 = -5.77 m/s^2 perpendicular to OB.

Projected onto the t- and n-axes this leads to the following values for the acceleration components:

a_t =(-5.77 $\cdot$0.4/0.412 - 3.71 $\cdot$0.1/0.412) m/s^2 =-6.5 m/s^2
a_n = (3.71 $\cdot$0.4/0.412 + 5.77 $\cdot$0.1/0.412) m/s^2 = 5 m/s^2.

Note that neither component is the same as for point A. Generally, a point on a solid body is likely to experience higher acceleration the further it lies from the centre of rotation.

2.2 Relative motion

Let A and B be two points on the rigid body. We start with the following vector relation:

$$\boldsymbol{r}_A=\boldsymbol{r}_B+\boldsymbol{r}_{A/B} \,. \qquad (2.2.1)$$

Here $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ represent *absolute* position vectors of A and B with respect to some fixed axes, and $\boldsymbol{r}_{A/B}$ stands for the *relative* position vector of A with respect to B.

By differentiating the above equation with respect to time, we obtain the basis of the relative motion analysis, known as the relative velocity equation:

$$\boldsymbol{v}_A=\boldsymbol{v}_B+\boldsymbol{v}_{A/B} \,. \qquad (2.2.2)$$

To determine the velocity of A with respect to the fixed axes (the *absolute* velocity $\boldsymbol{v}_A$) we represent it as the sum of the *absolute* velocity of point B, $\boldsymbol{v}_B$, and the *relative* velocity of point A with respect to point B, $\boldsymbol{v}_{A/B}$ (Fig.2-3).

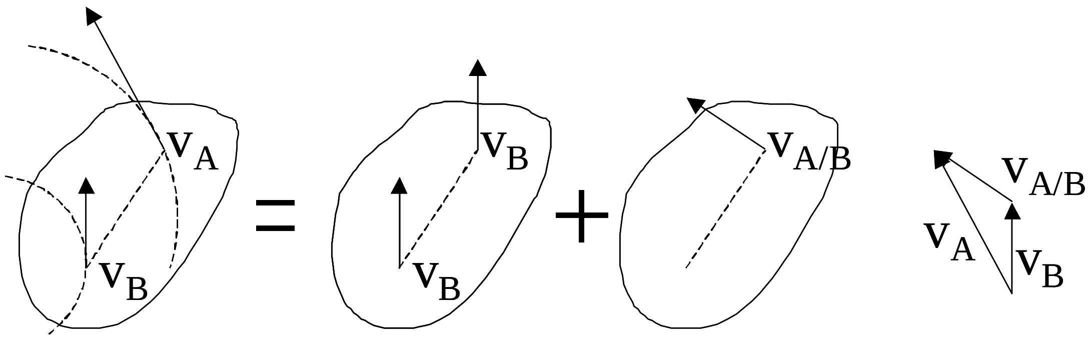


Fig.2-3.

When the solid body is rotating around a fixed axis with angular velocity w, all vectors on that solid body also rotate with the same angular velocity. In fact, we know from equation (2.1.4) that $\boldsymbol{v}_{A/B}$, which is the time derivative of vector $\boldsymbol{r}_{A/B}$, can be represented by the cross-product of w and $\boldsymbol{r}_{A/B}$,

$$\boldsymbol{v}_{A/B} = w \times \boldsymbol{r}_{A/B} \,. \qquad (2.2.3)$$

This result merits some discussion. Some very useful conclusions can be drawn.

Note that vector $\boldsymbol{v}_{A/B}$ is directed normally to both w and $\boldsymbol{r}_{A/B}$. (see e.g. the definition of cross-product). This makes sense: if we observe the motion of A from B, it appears to simply rotate at the end of a fixed link BA.

The component of the relative velocity of A with respect to B along the line AB must be zero. This makes sense again: because the link between B and A is rigid, its length must not change. Coming back to the absolute velocities $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$, this means that points A and B must have the same velocity along the line connecting them. Although most of this is self-evident from the definition of a rigid body, the results are very useful for velocity analysis, as we will see later on in this course.

Example 2-B. The wheel of radius r=300mm rolls to the right without slipping and has a velocity v_O=3m/s of its centre O. Calculate the velocity of point A on the wheel shown in Fig.2-4.

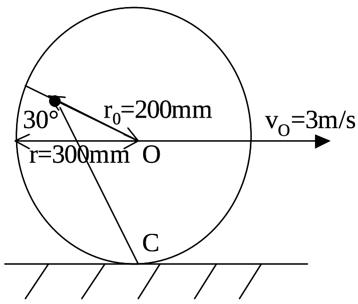


Fig.2-4.

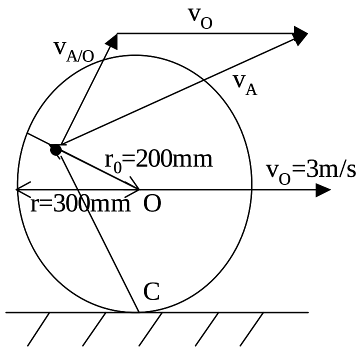


Fig.2-5.

Since the motion of point O is given, choose it as the reference point for the relative velocity equation:

$$\boldsymbol{v}_A=\boldsymbol{v}_O+\boldsymbol{v}_{A/O}\ .$$

Consider also point C in contact with the ground at the instant considered, for which

$$\boldsymbol{v}_C=\boldsymbol{v}_O+\boldsymbol{v}_{C/O} = 0.$$

From equation (2.2.3) we know that the relative velocity $\boldsymbol{v}_{C/O}$ points in the direction of rotation of the wheel, i.e. opposite $\boldsymbol{v}_O$, and has the magnitude wr. We thus establish that v=wr. The angular velocity w of the wheel (as well as of any line fixed on that wheel, e.g. OA) is equal to v_O/r = 10 rad/s.

Returning to the relative velocity equation for points O and A we now determine that the relative velocity $v_{A/O}$ has the magnitude wr_O= (10 ·0.2) m/s = 2 m/s, and is pointing normally to OA in the direction of rotation, i.e. clockwise. It remains to determine $\boldsymbol{v}_A$ from the vector sum of $\boldsymbol{v}_O$ and $\boldsymbol{v}_{A/O}$, as shown on the diagram in Fig.2-5. The magnitude of $\boldsymbol{v}_A$ is found from the cosine rule:

$$\boldsymbol{v}_A^2 = 3^2 + 2^2 + 2 \cdot 3 \cdot 2 \cos 60° = 19\ (\text{m/s})^2, \qquad \boldsymbol{v}_A = 4.36 \text{ m/s}.$$

Note that alternatively the contact point C could be used as the reference point in the relative velocity equation. Since point C is stationary (at the particular moment considered !), the relative velocity of A with respect to C gives the final answer to the problem. The direction of $\boldsymbol{v}_A$ is perpendicular to the line CA.

2.3 Instantaneous centre of rotation

In solving the example problem we discovered that judicious choice of the reference point in the relative velocity equation (2.2.2) may lead to great simplification of the analysis. In particular, if we always choose the point C which is **momentarily stationary** at the instant considered, the relative velocity equation for any point A on the body simplifies to

$$\boldsymbol{v}_A = \boldsymbol{v}_{A/C} = w \times \boldsymbol{r}_{A/C} . \qquad (2.3.1)$$

As far as the velocities are concerned, the body may be thought to be in pure rotation about an axis normal to the plane of motion and passing through point C.

It is very important for you to understand that generally point C is NOT fixed permanently. It moves both with respect to the body and the absolute axes. Point C can be taken as the centre of rotation *only at the given instant*, and is therefore known as the ***instantaneous centre of rotation***.

The location of the instantaneous centre can be easily determined by construction in Fig.2-6. Let us assume first that, for two chosen points A and B the directions of absolute velocities are not parallel, as in Fig.2-6(a). If there is a point with respect to which point A is in a state of pure rotation at that instant, it must lie on the normal to $\boldsymbol{v}_A$ through A. Similar reasoning applies for point B. The intersection of these two normals (which always exists since $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ are not parallel) is the instantaneous centre C. Tha magnitude of w is found from

$$|\boldsymbol{v}_A| = |w||CA|.$$

The magnitudes and directions of all absolute velocities are now determined using equation (2.3.1).

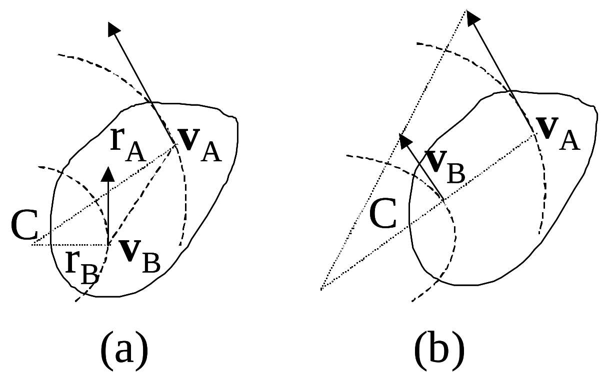


Fig.2-6.

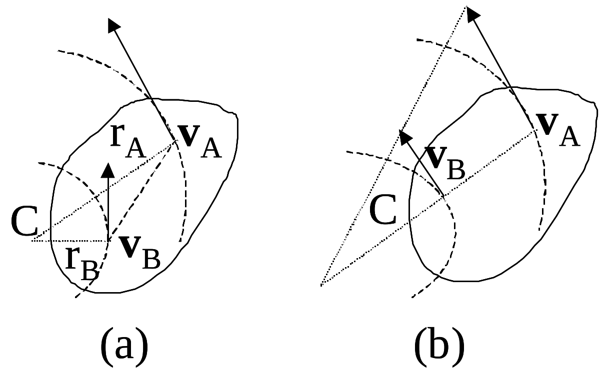


Fig.2-6.

Let's consider what happens if the velocities of the two chosen points A and B are parallel, as in Fig.2-6(b). If $\boldsymbol{v}_A = \boldsymbol{v}_B$, then the body is not rotating, but only translating, and all points have the same velocity. If $\boldsymbol{v}_A$ is not equal to $\boldsymbol{v}_B$, then the line joining them must be normal to both $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ (can you prove that it is so?). The location of the instantaneous centre C is found by direct proportion.

NB: the instantaneous centre does not have to lie within the solid body. However, it is sometimes convenient to imagine extending the solid body to include the instantaneous centre. This thought exercise does not affect the answer in any way.

Example 2-C. Arm OB of the linkage has a clockwise angular velocity of 10 rad/s in the position shown where q=45° (Fig.2-7). Determine the angular velocity of link AB for the instant shown, and the velocities of points A and D on this link.

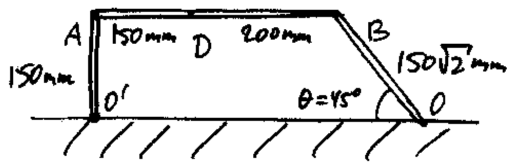


Fig.2-7.

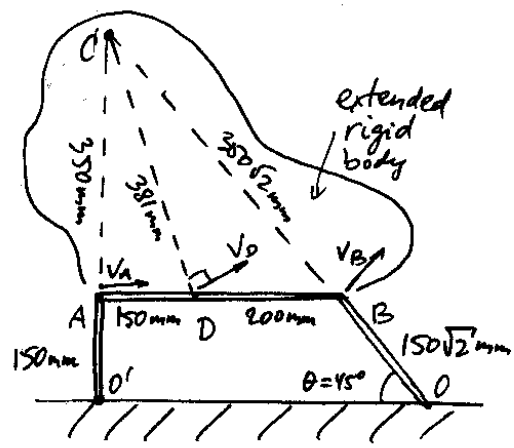


Fig.2-8.

The velocities of A and B are normal to the links AO'and BO, connecting them to the fixed centres O'and O, respectively. The instantaneous centre for the link AB, to which both of these points belong, lies at the intersection of the normals to the velocities, and can be found by extending the links AO'and BO. The distances AC, BC and DC are found from trigonometry or scaled diagram.

We now extend the solid body to include point C (Fig.2-8). The link BC is a line on this body, and rotates with the same angular velocity as the whole. We seek the angular velocity as follows

$$w_{BC} = v_B/BC = w_{OB}\ OB/BC = 4.29 \text{ rad/s CCW} = (w_{AB}).$$

(Since AB and BC belong to the same solid body, they must have the same angular velocity).

The velocities of A and D are now

$$v_A = w_{AB}\ CB = 4.29 \cdot 0.35 = 1.5 \text{ m/s}$$

$$v_D = w_{AB}\ CD = 4.29 \cdot 0.381 = 1.63 \text{ m/s}$$

3. Kinematics of Mechanisms

Ideas discussed in the context of kinematics of particles and rigid bodies will now be used in the analysis of mechanisms. It is often useful to keep this kinematic analysis of a system separate from a consideration of how the mechanism responds to the application of forces.

This section of the lecture course deals, primarily, with the analysis of mechanism *kinematics*. The objective of kinematic analysis of a mechanism is to determine the linear and angular velocities of the various components of the mechanism when some part of it is subjected to a known linear or angular velocity.

Although this is a course on kinematics, examples are given of how a kinematic analysis of a mechanism can lead on naturally to an analysis of the dynamics of the system.

3.1 Methods of Kinematic Analysis of Mechanisms

Various techniques are available to carry out kinematic analysis of mechanisms as described below.

Analytical methods. In this approach a set of equations is written down to describe the geometry of the mechanism and also the various constraints that determine its motion. (Common forms of contraint are (i) the length of a rigid link cannot change , or (ii) the end of a link may be fixed, or (iii) a point on a link may be constrained to move in a particular direction, i.e. in a slider joint). These geometric equations are then differentiated to obtain further equations relating the linear and angular velocities of the various components of the mechanism.

The approach is based on conventional geometry and differential calculus. The approach is not very systematic, however, and careful thought is needed to set up the geometric equations for any particular problem. The main drawback of the method is that it often involves the manipulation and differentiation of rather lengthy expressions. This means that the solution process usually becomes very long-winded and tedious.

Although I will be showing you how to use an analytical approach for a simple mechanism (see example D), I would only recommend this method for general use (at least as far as Paper P4 is concerned) if you feel confident performing the fairly long algebraic manipulations required.

Graphical methods. Graphical methods are based on a geometric representation of the kinematics of a mechanism. Two main types of graphical methods are described in the textbooks on this subject: the method of *instantaneous centres* and the *velocity diagram* method. These lectures will dwell on the velocity diagram method. This method may be developed to include an analysis of accelerations (using an *acceleration diagram*) but this is well beyond the scope of this lecture course.

Computer methods. In engineering practice, of course, any kinematic analysis of a mechanism will use computer analysis in which the various constraints to motion are dealt with in a systematic way. Computer methods of analysis are, however, also beyond the scope of this course.

3.2 Kinematics of a Crank-Slider Mechanism using an Analytical Approach

The crank-slider is used in a variety of mechanisms, most importantly in the piston – connecting rod – crankshaft mechanism in an internal combustion engine.

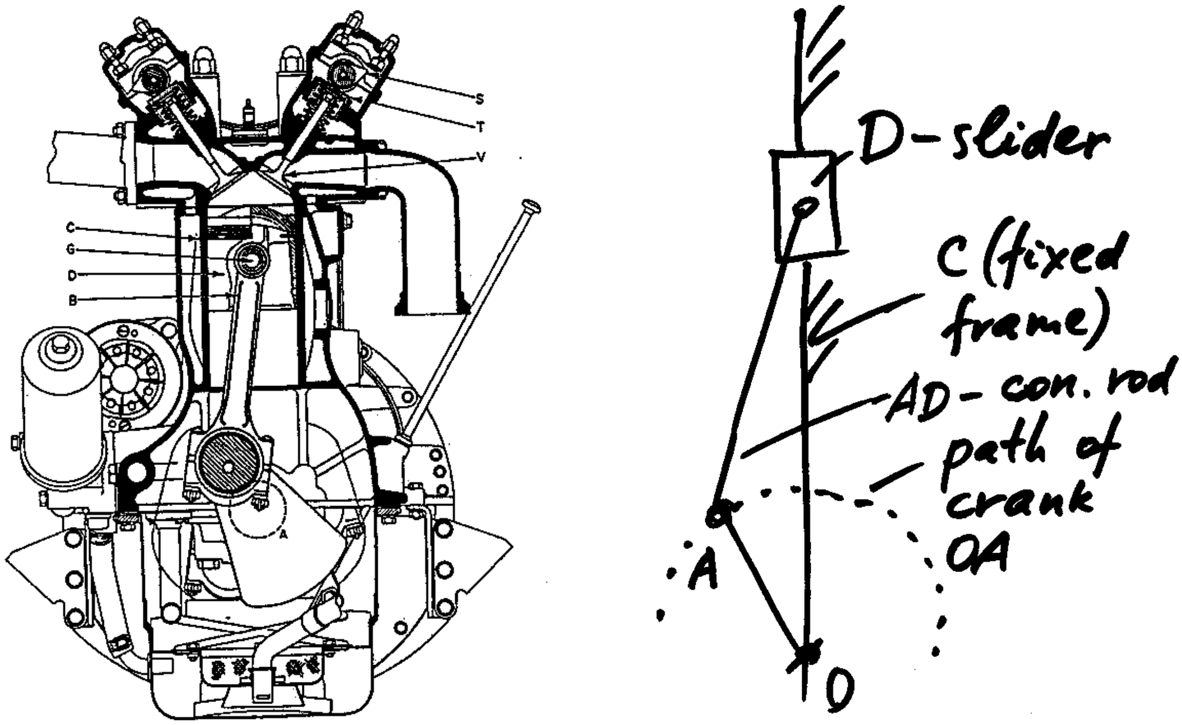


Fig.3-1. Cross-section through a Jaguar 3.8-litre six-cylinder engine (after Morrison and Crossland, 'Mechanics of Machines', Longman), and the schematic mechanism.

Example 3-A (see below) consists of the analysis of a typical crank-slider mechanism of the sort that might be used in the engine of a typical family car. This example is solved below using an analytical procedure to define the geometry of the mechanism; the resulting geometric equations are then differentiated to provide expressions for the linear and angular velocities of the component parts of the mechanism. We will return to this problem to obtain a more rapid solution using the velocity diagram.

Example 3-A. Find the velocity of C and the angular velocity of link BC in the crank-slider mechanism at the instant shown below. The crank AB is rotating anti-clockwise with angular velocity w = *dq/dt* = 500 rad/s (approx. 4800 rpm).

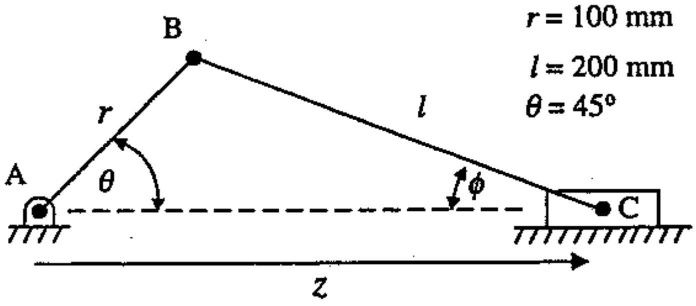


Fig.3-2.

Using the analytical approach, we first write down equations defining the geometry of the mechanism:

$$
\begin{aligned}
& z = r\cos q + l\cos f \\
& 0 = r\sin q - l\sin f \\
& \Rightarrow \quad \sin f = \frac{r}{l}\sin q \\
& \Rightarrow \quad \cos f = \sqrt{1-\left(\frac{r}{l}\sin q\right)^2} \\
& \Rightarrow \quad z = r\cos q + l\sqrt{1-\left(\frac{r}{l}\sin q\right)^2}
\end{aligned}
$$

Next, we need to differentiate the expressions for z and $\cos f$. The derivative of the expression for *z* gives:

$$
\frac{dz}{dt} = -r\sin q\,\frac{dq}{dt} - \frac{r^2\sin q\cos q}{l\sqrt{1-\left(\frac{r}{l}\sin q\right)^2}}\,\frac{dq}{dt}\ .
$$

We are now able to calculate the velocity of point C by inserting the appropriate values of *r*, *q*, *dq/dt* and l into the above expression (noting that *dq/dt* is simply the angular velocity of link AB and therefore equal to 500 rad/s). In this partciular case this leads to the solution

$$
\frac{dz}{dt} = -48.7 \text{ m/s}.
$$

The angular velocity of link BC is given by df/dt. To evaluate this expression we differentiate the expression for $\sin f$ (above) to give:

$$\cos f \frac{df}{dt} = \frac{r}{l}\cos q \frac{dq}{dt}$$

$$\Rightarrow \quad \frac{df}{dt} = \frac{r\cos q}{l\cos f}\frac{dq}{dt}$$

$$= \frac{r\cos q}{l\sqrt{1-\left(\frac{r}{l}\sin q\right)^2}}\frac{dq}{dt}$$

If specific values of r, q, dq/dt and l are substituted into the above expression then this gives a value for the angular velocity of link BC of 189 rad/s.

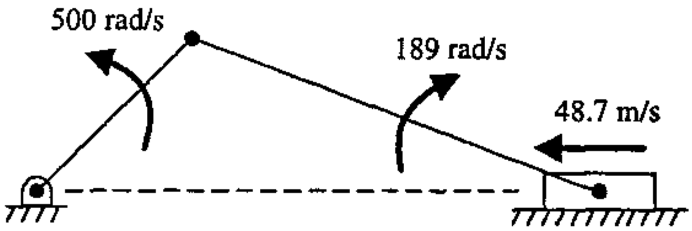
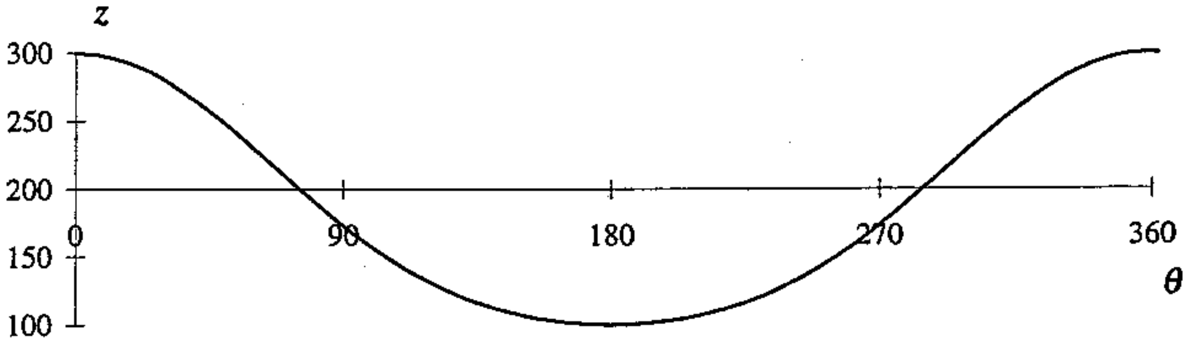
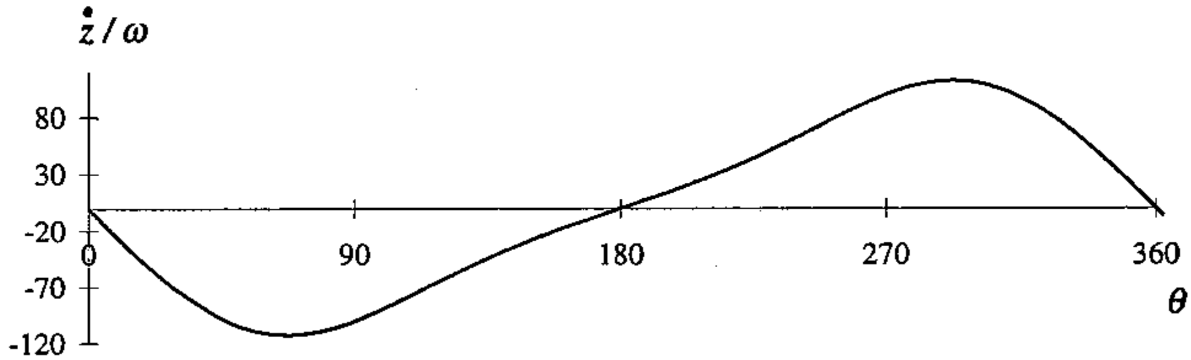


Fig.3-3.

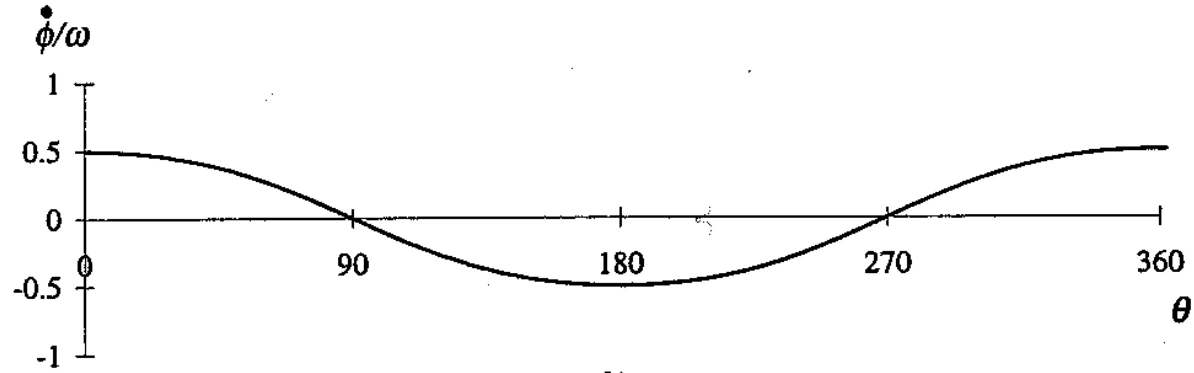
In addition to computing the values of linear and angular velocity for the particular position of the mechanism shown in the example, the results of the analysis can also be used to determine the kinematics of the mechanism for all values of q.



Position (z) as a function of θ



Velocity ($\frac{dz}{dt}/\omega$) as a function of θ



Angular velocity ($\frac{d\phi}{dt}/\omega$) as a function of θ

Fig.3-4.

3.3 Introduction to the use of velocity diagrams

Kinematic constraints. The velocity diagram is a graphical representation of the velocity vectors for different points within the mechanism. When several points are considered, the velocity diagram has a form of a polygon, with vertices representing velocities of different points. The velocity diagram is constructed by considering various *kinematic constraints* that are imposed on the mechanism.

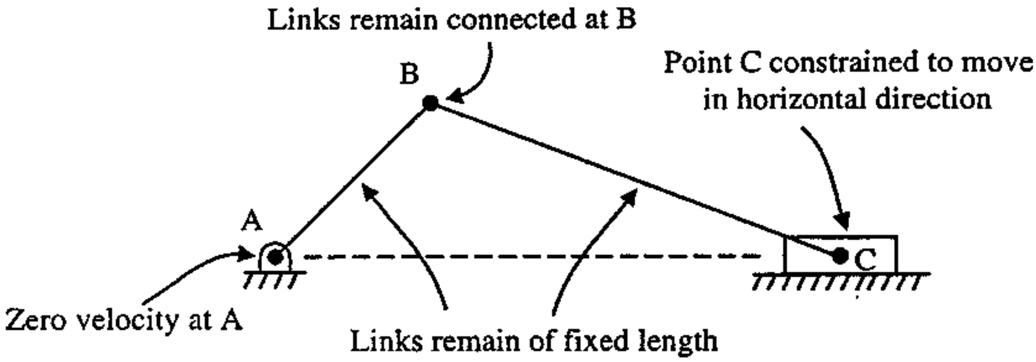


Fig.3-5.

One general constraint is that the bars must remain connected in the same configuration. Several further types of constraint may be identified (Fig.3-5):

(i) Points that are fixed in space (e.g. point A in the crank-slider mechanism)

(ii) Points on different parts of the mechanism which are connected together (e.g. as at point B in the crank-slider mechanism)

(iii) Points which are constrained to move only in certain directions (e.g. point C in the crank-slider mechanism)

(iv) The lengths of all rigid links that must remain constant.

Constraints of type (i) to (iii) are included fairly simply in the velocity diagram.

To deal with constraints of type (iv) we recall our discussion of the *relative velocity*.

We established that two points A and B lying on the same rigid body must have the same velocity along the line connecting them (Fig.3-6).

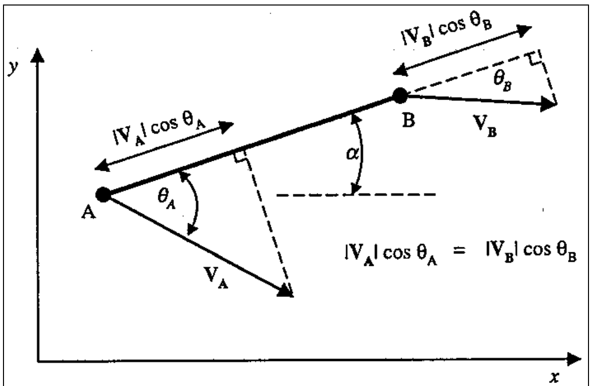
Denote the distance between A and B by *s*, and express the rate of change of this distance with time as

$$ds/dt = |\mathbf{v}_B|\cos q_B - |\mathbf{v}_A|\cos q_A \, .$$

When AB is a rigid link, the distance *s* must remain constant. This constraint may be by the equation

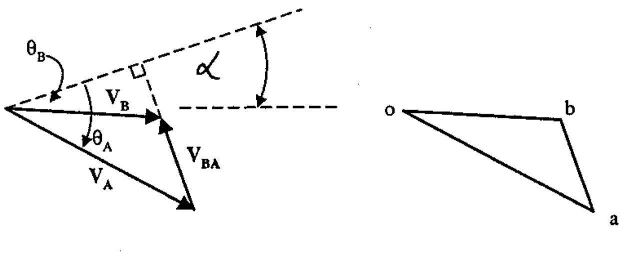
$$|\mathbf{v}_B|\cos q_B = |\mathbf{v}_A|\cos q_A \, . \qquad (3.3.1)$$

Otherwise we can express the same condition by requiring that the *relative velocity* $\mathbf{v}_{A/B}$ is *normal to the orientation* of link BA.



Motion of a rigid link

Fig.3-6.



Vector diagram for velocities of a rigid link

Velocity diagram

Fig.3-7.

A possible velocity diagram for the link AB is shown in Fig.3-7. We use lower case letters o,a,b to denote the ends of velocity vectors for points O,A,B, etc. Point o denotes the (zero) velocity of a stationary point. Let point a be already given or found. Then point b must lie on the line drawn through a perpendicular to link AB. If the direction of velocity of point B is know, this condition is usually sufficient to find point b graphically from intersection.

Example 3-A (revisited). We start by noting the various kinematic constraints on the system, and then express them on the velocity diagram, until velocities of all joints are found.

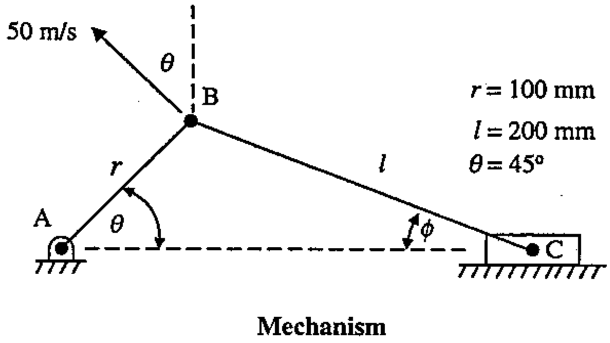


Fig.3-8.

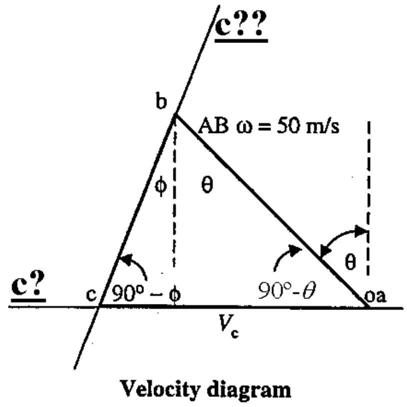


Fig.3-9.

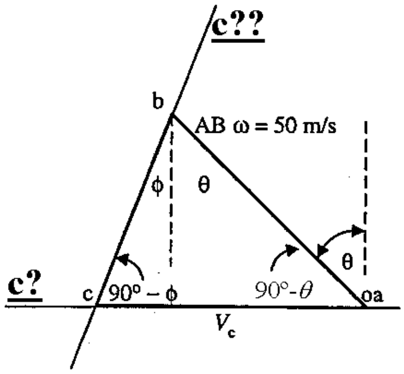
The velocity diagram is constructed by the sequence of operations below:

(i) Find the orientations of all the links (in this case, from simple geometry determine $f = 20.7^{o}$)

(ii) Identify points with zero velocity and plot them on the velocity diagram (in this case, point A is the only one fixed, so a coincides with o, origin)

(iii) Identify points with known velocity and plot them on the velocity diagram (in this case, point B has the velocity of magnitude 50 m/s in the direction shown in Fig.3-8). The line ab on the velocity diagram should have a length corresponding to 50 m/s and be drawn in the direction q to the vertical. At this stage it is necessary to establish a suitable scale for the diagram.

(iv) Draw lines on the velocity diagram corresponding to the remaining kinematic constraints. Since point C is on a slider joint, it is constrained to move horizontally, so point c lies on a horizontal line through o. This can be shown on the diagram by a line indicated c?. Since link BC is rigid, the velocity of C with respect to B must be normal to this link, so point c also lies on a line through point b inclined at f to the vertical. This establishes another c?? line. The intersection of the c? and c?? lines gives the position of point c.



Velocity diagram

Fig.3-9.

Now that the velocity diagram is complete, values of the linear and angular velocities of link BC can be found. The velocity of C is represented by the vector oc on the diagram. The magnitude of $\mathbf{v}_C$ can be found from simple application of the sine rule:

$$v_C = ob\frac{\sin(q+f)}{\sin(90^\circ - f)} = 48.7 \text{ m/s}$$

(given that ob=|$\mathbf{v}_B$|=50 m/s). This result is identical to that obtained from a rather lengthy analytical derivation earlier. The direction of the velocity is given by the relative position of points o and c on the diagram: since c lies to the left of o, $\mathbf{v}_C$ is directed horizontally to the left.

In order to determine the angular velocity of link BC, we note that the relative velocity of C with respect to B, $\mathbf{v}_{C/B}$, is given on the diagram by the vector bc. The sine rule gives

$$bc = ob\frac{\sin(90^\circ - q)}{\sin(90^\circ - f)} = 37.8 \text{ m/s}$$

It remains to find the angular velocity by

$$w = |\mathbf{v}_{C/B}| \;/\; BC = 189 \text{ rad/s.}$$

The direction of rotation is determined simply from the relative positions of points b and c on the velocity diagram. In this case it is clear that BC is rotating in the clockwise direction.

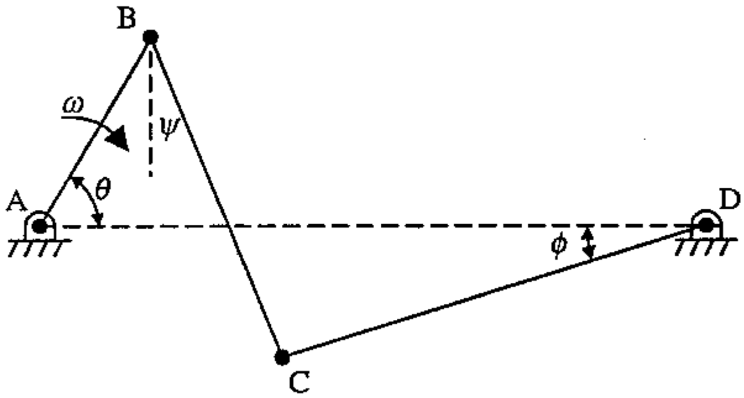
The angular velocity of a link can be determined simply by dividing the magnitude of the relative velocity of two ends by the length of the link.

3.4 Further Use of Velocity Diagrams

If you have followed the previous example then you will have understood the main features of the velocity diagram method of analysis. Further complexities may arise, and we will illustrate them by means of the following three examples. Firstly, we will look at a four-bar chain mechanism which requires more complex procedures to obtain the angular orientations of the links than for the crank-slider. Next, we will consider the useful topic of velocity images in which the velocity diagram is used to determine the velocity of an arbitrary point within a link. Finally, we will study a mechanism containing a more complicated form of a slider joint.

Example 3-B: **four-bar chain**.

The four-bar chain consists of four rigid links connected together. Usually, one of the links is fixed, as in this example. The four-bar chain is capable of a large variety of motions depending on the relative length of the bars.



AB = 0.1m, BC = 0.16m, CD = 0.2m, AD = 0.3m, $\theta = 60°$

Fig.3-10.

The link AB in the four-bar chain shown in Fig.3-10 is rotating clockwise with angular velocity w. Find the velocity of C and the angular velocity of CD.

We use a similar approach to that for the crank-slider mechanism in Example 3-A. In the example considered here there are not one, but two bars with unknown angular velocities. Also, the determination of the angular orientation of the bars in the four-bar chain needs rather more complex geometrical analysis. In order to establish the angular orientation of all bars we use a scale drawing, finding the values f=18° and f=22° approximately.

Alternatively, more accurate values may be obtained by the use of the cosine and sine rules (Fig.3-11).

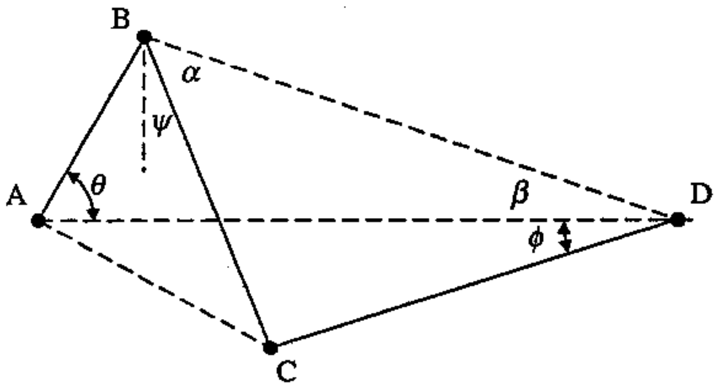


Fig.3-11.

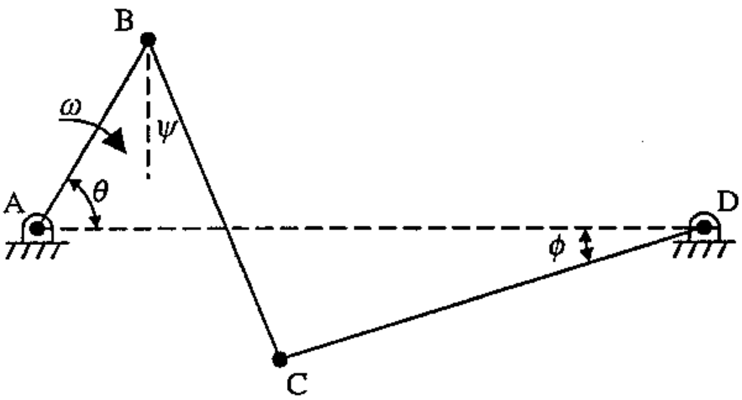
For triangle ABD: $BD^2 = AB^2 + AD^2 - 2\ AB\ AD \cos 60^o$ => $BD = 0.2646$ m

For triangle BCD: $CD^2 = BD^2 + BC^2 - 2\ BD\ BC \cos a$ => $a = 48.943^o$

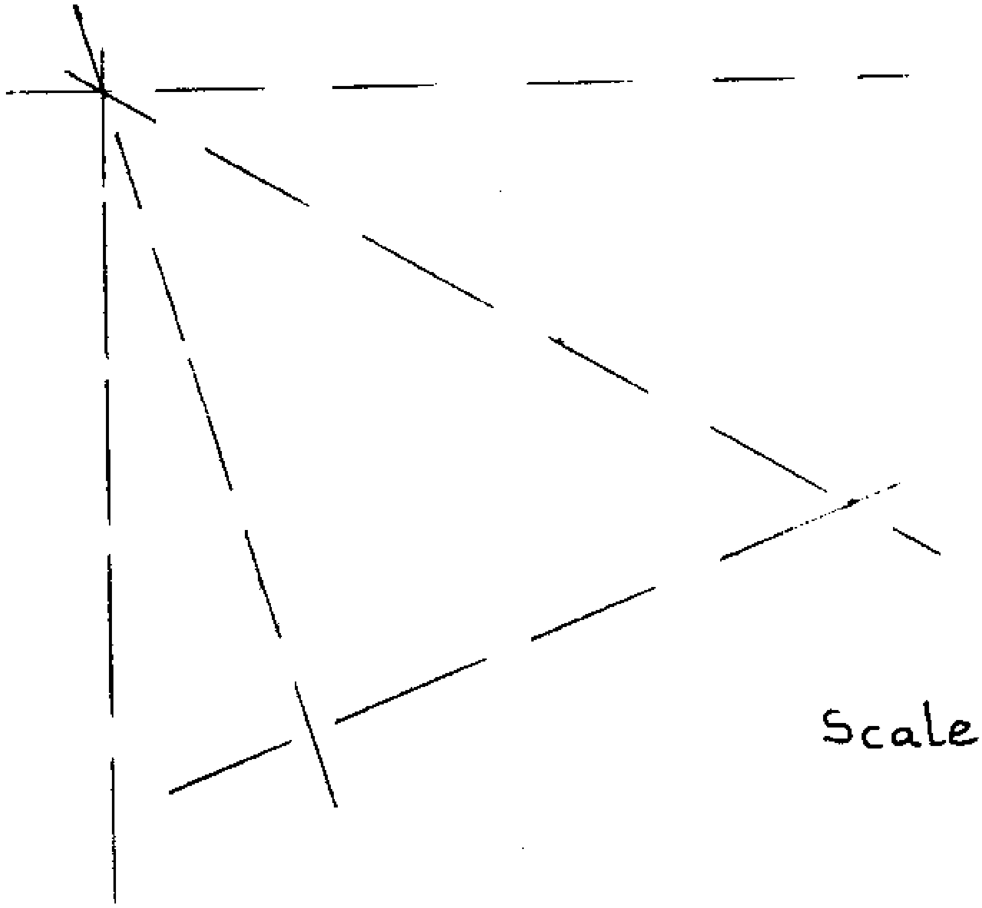
For triangle ABD: $AB^2 = AD^2 + BD^2 - 2\ AD\ BD \cos b$ => $b = 19.08^o$

$y + a + b = 90^o$ => $y = 21.97^o$

For triangle BCD: $CD / \sin a = BC / \sin (b + f)$ => $f = 18.0^o$



AB = 0.1m, BC = 0.16m, CD = 0.2m, AD = 0.3m, $\theta = 60^\circ$



The velocity diagram is now drawn in the following stages:

(i) Choose position of origin o, and draw points a and d to coincide with o.

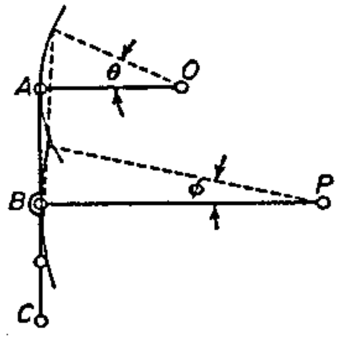
(ii) Draw position of point b (this is straightforward since velocity of B is equal to AB *w* and inclined at an angle *q* to the vertical)

(iii) Draw a line through b in a direction orthogonal to BC (a <u>c?</u> line)

(iv) Draw a line through d in a direction orthogonal to DC (a <u>c??</u> line), and find c as intersection.

The velocity diagram is complete.

Four-bar chains are used in a variety of applications. Some examples are illustrated below:



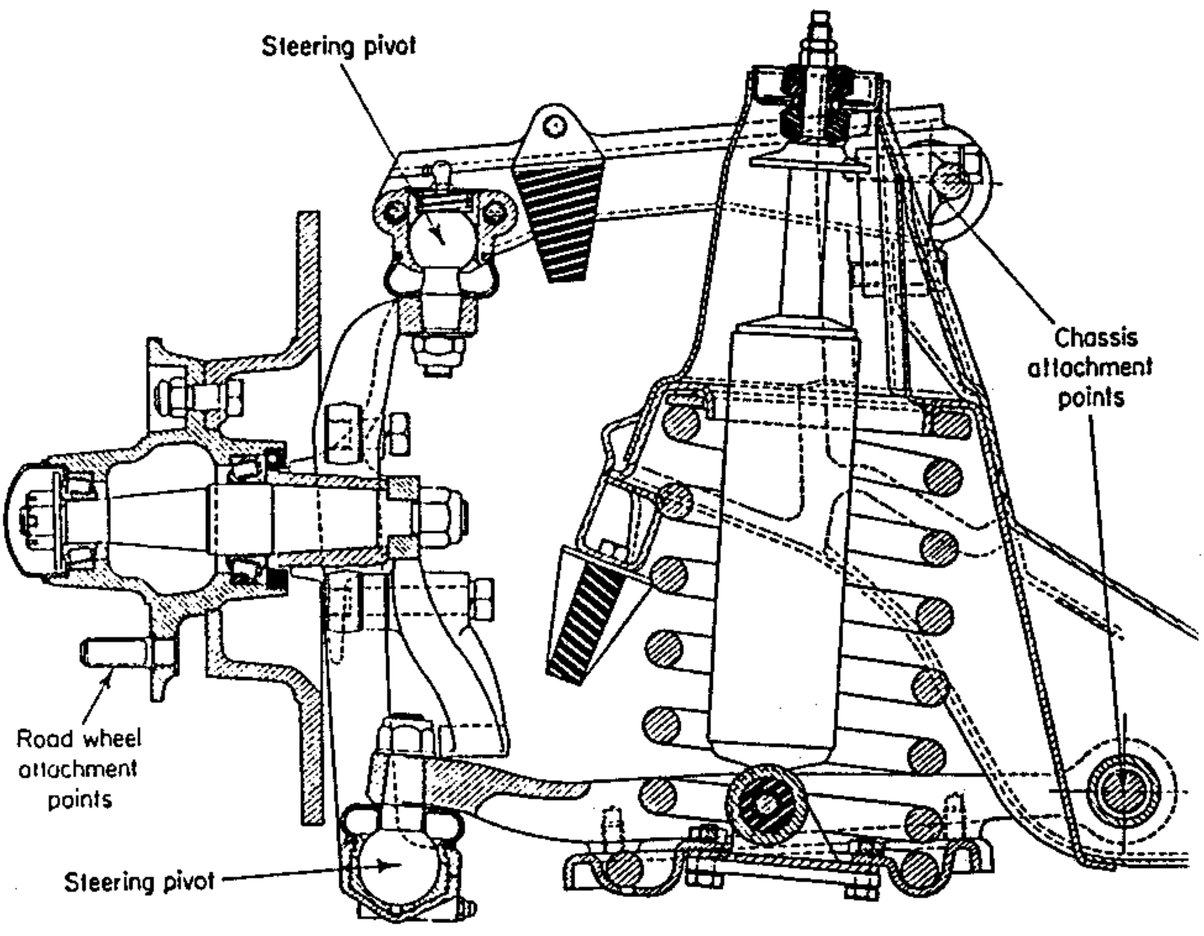


Fig.3-12.

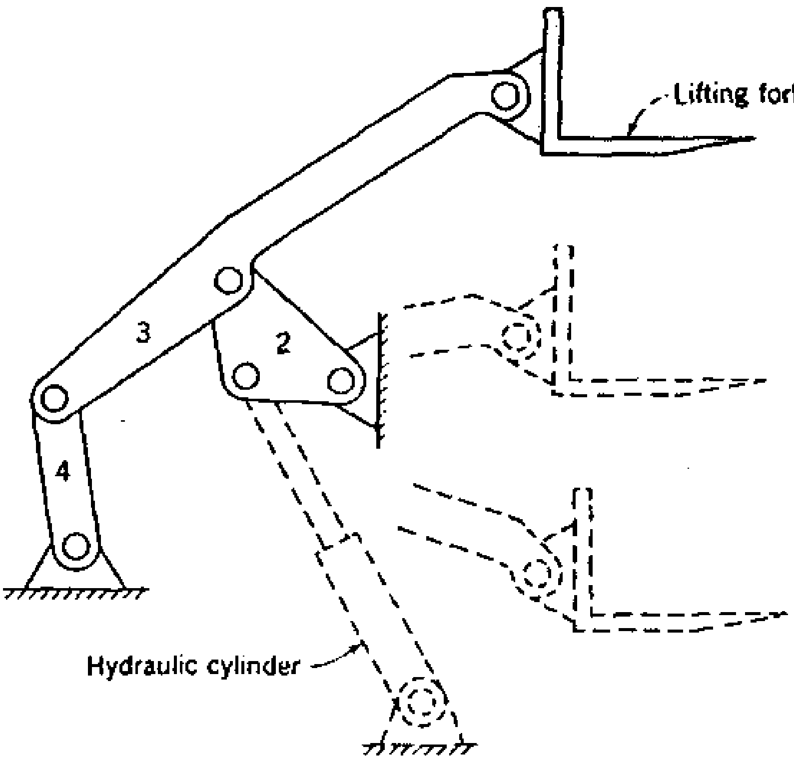


Fig.3-13.

Example 3-C: **velocity images**.

It is often useful to use velocity diagrams to indicate the velocity of points at arbitrary positions within a rigid link, rather than only consider velocities at joints. The idea of velocity images is useful for this, as illustrated below.

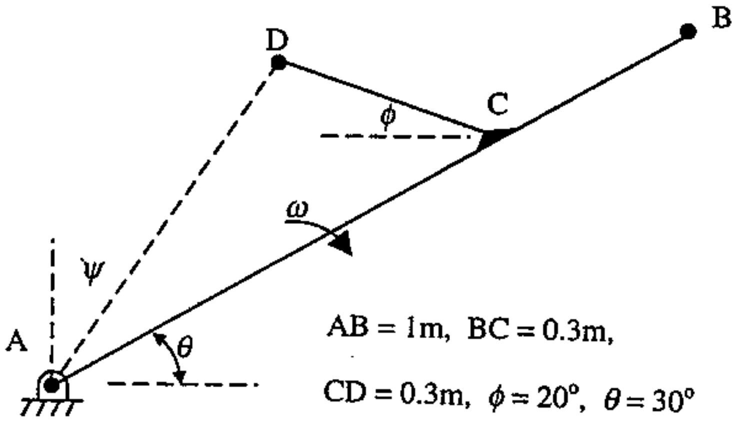
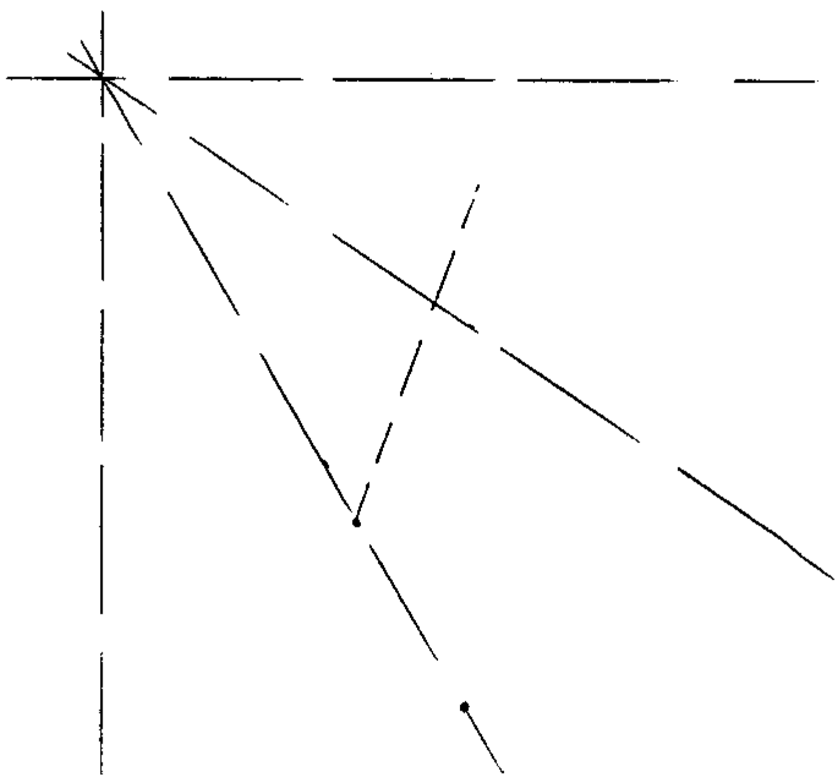
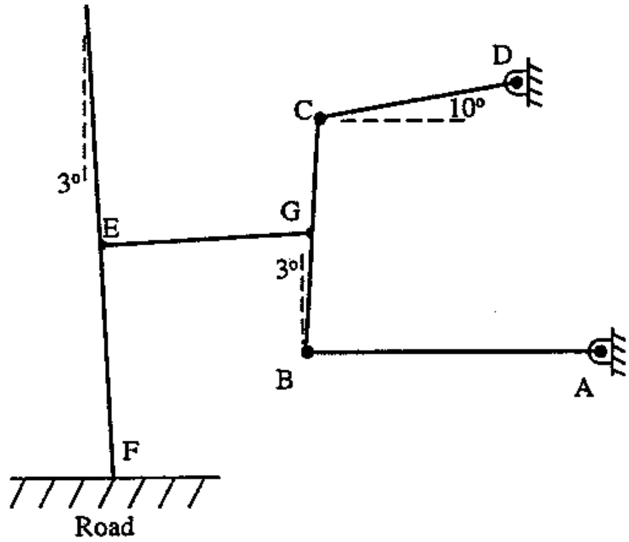


Fig.3-14.

To find the velocities of points B, C and D on the rigid link shown in Fig.3-14, we use the following construction:



We now use this approach to analyse the front wheel suspension system of a car, shown in Fig.3-15.



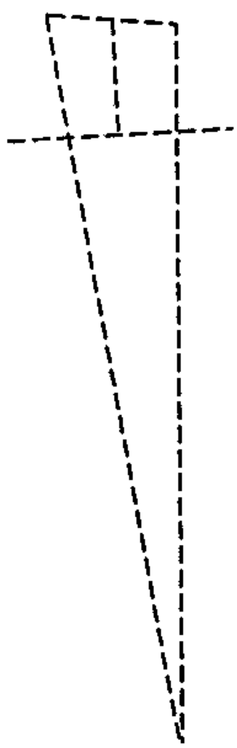
AB = 500 mm, BC = 400 mm, CD = 350 mm, EG = 350 mm, EF = 350 mm

Fig.3-15.

Find the horizontal velocity of the point where the wheel is in contact with the road for the case when the car has a downwards velocity of 1 m/s.

This is a four-bar chain mechanism for which the use of velocity images proves rather useful.

Velocity Diagram



Example 3-D: **mechanisms with sliders**.

Mechanisms can have slider joints rather than pins. This situation is treated by careful consideration of the kinematic constraint set up by the sliding joint, as illustrated below.

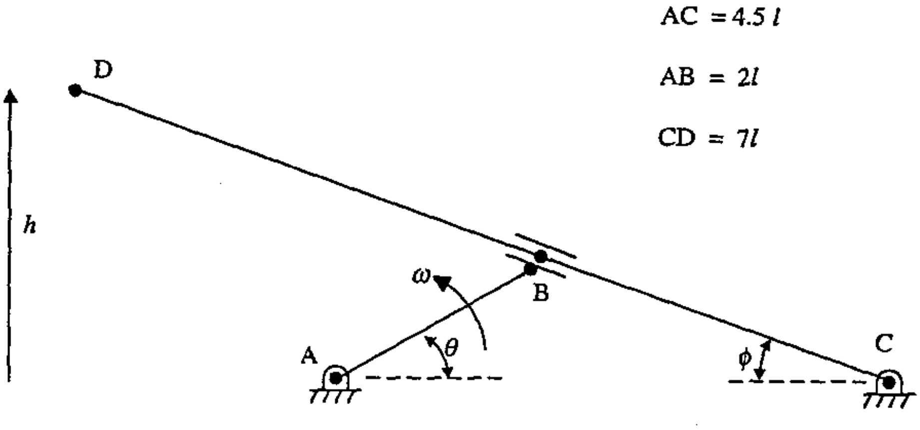


Fig.3-16.

The mechanism shown above is known as the Whitworth quick-return mechanism. Plot the approximate variation of h with *q*. For the case when *q* =30º, find the value of d*h*/d*t* and the angular velocity of DC.

Before proceeding with the velocity diagram it is necessary to distinguish carefully between the following points: point B on the crank AB, and point B'on the rocker CD. The two points, at the moment considered, occupy the same position in space, but they have different velocities.

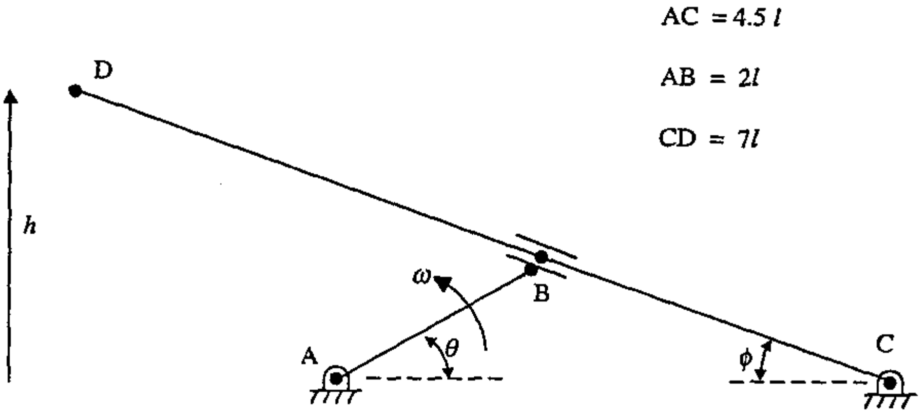
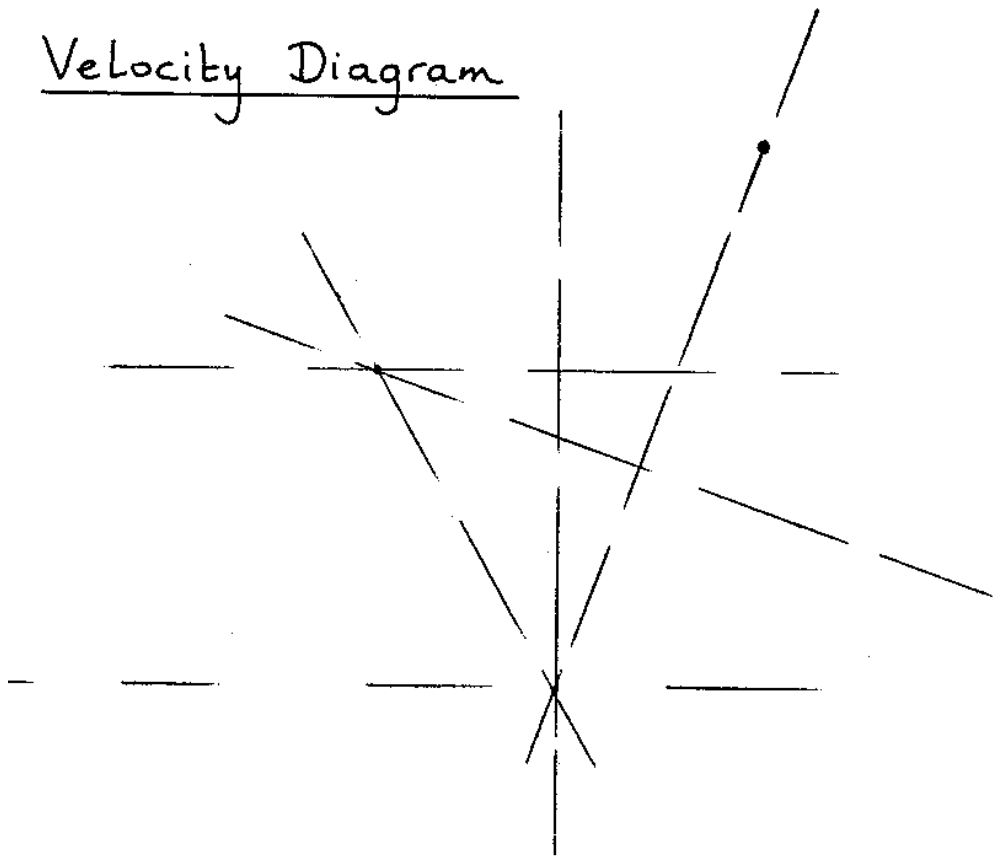


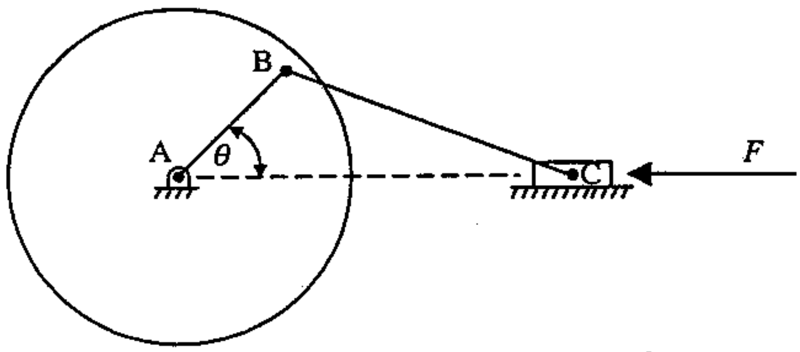
Fig.3-16.



3.5 <u>Application of the Results of Kinematic Analysis</u>

Results of *kinematic* analysis can be used to investigate the *dynamic* behaviour of a mechanism.

Example 3-E.
In the crank-slider mechanism shown in Fig.3-17 the link AB is connected to a flywheel with the moment of inertia 100 kg m^2. A force of 10 kN is applied to the piston. Find the angular acceleration of the flywheel assuming that the masses of the piston and the connecting rod can be neglected.



AB = 100 mm, BC = 200 mm, θ = 45°

Fig.3-17.

In the absence of frictional losses the power produced by the piston, F v_C, is equal to the rate of increase of the flywheel's kinetic energy, Iw^2:

$$Fv_C = \frac{d}{dt}\left(\frac{Iw^2}{2}\right) = Iw\dot{w} \ .$$

The angular acceleration can be found as

$$\dot{w} = \frac{Fv_C}{Iw} = \frac{F\,|AB|}{I}\frac{v_C}{v_B} \ ,$$

where $v_B = |AB|w$ was used. The velocity ratio does not depend on their magnitudes, but only on the dimensions and angles of the mechanism.

The solution from Example 3-A can be used, for which v_C=48.7 m/s and v_B=50 m/s, giving angular acceleration $\dot{w}$ =9.74 rad/s^2.