DUAL QUATERNION SYNTHESIS OF CONSTRAINED ROBOTS

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Abstract This paper presents a synthesis methodology for robots that have less than six degrees of freedom, termed constrained robots. The goal is to determine the physical parameters of the chain that fit its workspace to a given set of spatial positions. Our formulation uses the dual quaternion form of the kinematics equations of the constrained robot. Here we develop the theory and formulate the synthesis equations for the spatial RPR robot. Their solution ensures that the three dimensional workspace of this robot contains a given set of four spatial positions.

1. Introduction

The focus of this paper is on the design of robots that have less than six degrees of freedom that we call *constrained robots.* These robots have workspaces of dimension two through five in the Lie group of spatial displacements. Fitting a constrained robot's workspace to a given set of task positions can be viewed as similar to fitting a surface to a set of points in space. This theory is an extension of the kinematic synthesis of linkages (McCarthy, 2000b).

The geometric design of a robots generally focuses on systems with six or more degrees of freedom. In this case, the workspace is sized to enclose the task space, and the challenge is to ensure that the system has desired differential properties at specified task positions, Kumar and Waldron, 1981. Chedmail, 1998 and Gosselin, 1998 present optimization techniques for the design serial and parallel robotic systems, respectively, that provide desired properties of the workspace.

In constrast, constrained robots have lower dimensional workspaces and the challenge is to locate and shape it such that it passes through the specified task positions.

1.1 Linkage Synthesis

Spatial linkage synthesis uses the geometric properties of a serial chain to formulate algebraic equations that must be satisfied at each of a discrete set of positions in the workspace (Suh and Radcliffe, 1978). This yields algebraic equations that are solved to determine the dimensions of the chain, also see McCarthy, 2000a. See, for example, the synthesis of spatial RR chains (Tsai and Roth, 1973, Perez, 2000), spatial CC chains (Chen, 1969, Huang, 2000) and SS chains (Innocenti, 1994, Liao, 1998).

Recently, new methods have been developed that use the kinematics equations of the robot to form the design equations. In particular, Mavroidis and Lee used the kinematics equations of the spatial RR and RRR robots to formulate its design equations. Larochelle, 2000 uses a similar approach with planar quaternions to define an approximate synthesis for planar robots. This method introduces the joint parameters of the chain at each of the goal positions as additional variables in the design equations (Mavroidis, 2001, Lee, 2002). The advantage is that it can be systematically applied to a broad range of robotic systems.

1.2 Overview

In this paper, we follow Mavrodis' basic ideas, however, we use successive screw displacements (Gupta, 1986, Tsai, 1999) formulated in terms of dual quaternions to represent the kinematics equations of the robot. Dual quaternions were introduced to linkage analysis by Yang and Freudenstein, 1964. They form an eight dimensional Clifford algebra that contains a subset, known as unit dual quaternions, which is isomorphic to the group of spatial displacements (McCarthy, 1990). Also see Angeles, 1998.

There are two advantages in this formulation. The first is that successive screw displacements provide a convenient formulation for the kinematics equations in terms of the joint axes directly. Secondly, it reduces the number of equations needed for each goal position.

2. The Kinematics Equations

The kinematics equations of the robot equate the 4×4 homogeneous transformation $[D]$ between the end-effector and base frame. to the sequence of local coordinate transformations along the chain (Craig 1989),

$$[D] = [G][Z(\theta_1, d_1)][X(\alpha_{12}, a_{12})][Z(\theta_2, d_2)] \ldots \\ [X(\alpha_{n-1,n}, a_{n-1,n})][Z(\theta_n, d_n)][H]. \tag{1}$$

The parameters (θ, d) define the movement at each joint and (α, a) are the length and twist of each link, collectively know as the Denavit-Hartenberg parameters. The transformation $[G]$ defines the position of the base of the chain relative to the fixed frame, and $[H]$ locates the tool relative to the last link frame.

2.1 Successive Screw Displacements

These kinematics equations can be transformed into successive screw displacements choosing a reference position $[D_0]$. We then compute $[D_{0i}] = [D_i][D_0]^{-1}$, that is

$$\begin{aligned}[D_{0i}] &= [D_i][D_0]^{-1} \\ &= ([G][Z(\theta_{1i}, d_{1i})] \ldots [Z(\theta_{ni}, d_{ni})][H])([G][Z(\theta_{10}, d_{10})] \ldots [Z(\theta_{n0}, d_{n0})][H])^{-1}.\end{aligned} \tag{2}$$

This can be viewed as

$$[D_{0i}] = [T(\Delta\theta_1, \mathsf{S}_1)] \ldots [T(\Delta\theta_n, \mathsf{S}_n)], \tag{3}$$

where

$$\begin{aligned}[T(\Delta\theta_1, \mathsf{S}_1)] &= [G][Z(\theta_{1i}, d_{1i})][Z(\theta_{10}, d_{10})]^{-1}[G]^{-1}, \\ [T(\Delta\theta_2, \mathsf{S}_2)] &= ([G][Z(\theta_{10}, d_{10})][X(\alpha_{12}, a_{12})][Z(\theta_{2i}, d_{2i})]) \\ &\qquad ([G][Z(\theta_{10}, d_{10})][X(\alpha_{12}, a_{12})][Z(\theta_{20}, d_{20})])^{-1}, \\ &\vdots \\ [T(\Delta\theta_n, \mathsf{S}_n)] &= ([G][[Z(\theta_{10}, d_{10})] \ldots)[Z(\theta_n, d_n)][Z(\theta_{n0}, d_{n0})]^{-1}([G][[Z(\theta_{10}, d_{10})] \ldots)^{-1}.\end{aligned} \tag{4}$$

The displacements $[T(\Delta\theta_i, \mathsf{S}_i)]$ are the relative transformations along the joint axes of the robot from the reference configuration. Notice that by expressing them in this way, the initial transformation $[G]$ is absorbed in the first joint axis and the final transformation $[H]$ does not appear in the expression.

2.2 Dual Quaternions

The workspace of the robot can also be expressed by using the Clifford algebra of the *dual quaternions*. A spatial displacement can be represented as a dual quaternion,

$$\hat{Q}(\hat{\theta}) = \sin(\frac{\hat{\theta}}{2})\mathsf{S} + \cos(\frac{\hat{\theta}}{2}), \tag{5}$$

where $\mathsf{S} = \mathbf{s} + \epsilon\mathbf{s}^0$, with $\epsilon^2 = 0$, is the screw axis of the transformation. The dual numbers $\cos(\frac{\hat{\theta}}{2}) = \cos\frac{\theta}{2} + \epsilon(-\frac{d}{2}\sin\frac{\theta}{2})$ and $\sin(\frac{\hat{\theta}}{2}) = \sin\frac{\theta}{2} + \epsilon(\frac{d}{2}\cos\frac{\theta}{2})$ contain the information about the rotation and the displacement along the screw axis. The components of the dual quaternions are easily computed from the homogeneous matrix transformation.

The spatial displacements can be represented as the set of points $\mathbf{Z} = (\mathbf{Z}, \mathbf{Z}^0)$ in $\mathbf{R}^8$ which are subject to two constraints: $\mathbf{Z}\cdot\mathbf{Z} = 1$ and $\mathbf{Z}\cdot\mathbf{Z}^0 = 0$. The workspace lies on a six-dimensional submanifold of $\mathbf{R}^8$.

The dual quaternion form for the kinematics equations of the robot are obtained by transforming eq.(3) into

$$\hat{D}^i = \hat{S}_1(\Delta\hat{\theta}_1^i)\dots\hat{S}_n(\Delta\hat{\theta}_n^i), \tag{6}$$

where $\hat{D}^i$ is the dual quaternion for $[D_{0i}]$ and $\hat{S}_i$ is the dual quaternion for $[T(\Delta\theta_i, \mathsf{S}_i)]$.

This approach yields the kinematics equations as successive screw transformations from the reference position. It is a useful formulation from the synthesis point of view because the components of each axis appears explicitly in the base frame coordinates.

3. Synthesis of Constrained Robots

Let $[T(\theta_1, \dots, \theta_k)]$ be the kinematics equations of a serial robot, and let a discrete approximation of the desired workspace be given in the form of n goal transformations $[D_i], i = 0, \dots, n-1$. The synthesis problem consists of solving the n matrix equations

$$[T(\theta_{1,i}, \dots, \theta_{k,i})] = [D_i], \quad \imath = 0, \dots, n-1. \tag{7}$$

We now transform these equations to successive screw displacements in dual quaternion form. The result is $n-1$ goal positions $\hat{D}^i, i = 1, \dots, n-1$ and the kinematics equations $\hat{Q}(\hat{\theta}_1, \dots, \hat{\theta}_k)$ to obtain the $n-1$ equations

$$\hat{Q}_i(\hat{\theta}_1^i, \dots, \hat{\theta}_k^i) = \hat{D}^i, \quad i = 1, \dots, n-1 \tag{8}$$

For each of the $n-1$ positions we have eight component equations. However, due to the structure of the dual quaternions, only six of them are independent.

For a robot chain represented by a series of j revolute joints, each of the joints has an axis defined by six Plucker coordinates, which yields $6j$ unknowns. The j joint variables take different values at each of the $n-1$ positions, which add $j(n-1)$ unknowns. This yields $6j+j(n-1)$ unknowns. For each joint axis, there are two constraints associated with its Plucker coordinates. For each of the $n-1$ goal positions we obtain eight equations, which can be reduced to six. Thus, we have $2j+6(n-1)$ equations. Equating the number of unknowns to the number equations, we obtain

$$6j + j(n-1) = 6(n-1) + 2j. \tag{9}$$

Solving for n we obtain $n = \frac{3j+6}{6-j}$. We have that $2R$, $3R$, $4R$ and $5R$ spatial chains require 3, 5, 9, 21 positions, respectively.This analysis has been extended to include other types of joints and topologies. In general, the number of orientations we can reach is limited by the fact that rotations operate independently in spatial displacements. To compute complete spatial positions, we need to check whether the orientations are limited. Assume that our robot consists of l revolute joints and k prismatic joints. The number of spherical positions we can reach is

$$3l + l(n_R - 1) = 3(n_R - 1) + l, \tag{10}$$

while the accounting for both orientation and translation is

$$6(l+k) + (l+k)(n-1) = 6(n-1) + 2l + k. \tag{11}$$

From the rotation equation, $n_R = \frac{3+l}{3-l}$, and notice that this coincides with the results for spherical robots. If the number of complete positions is restricted by n_R we will reach only that many arbitrary orientations, and the rest will be just translational components of dual quaternions in which rotations will have to be bounded to the given workspace.

Notice also that here we assume that the axes of the rotational and translational joints are not related, but it is easy to adapt the formula to particular cases in which the joints are constrained.This result is used to determine the number of positions we need to define in the synthesis process in order to obtain a finite number of solutions.

4. Solving the Design Equations

The design equations for constrained robots contain joints variables and axis variables. The goal is to eliminate the joint variables and solve for the axis variables, which define the physical dimensions of the robot. The elimination of the joint parameters is called "implicitization" of the parametric equations, which we apply to the kinematics equations for each position.

The methodology consists of solving linearly for two of the revolute parameters in the rotational part of the dual quaternion. The result is substituted in the moment part of the dual quaternion and combined with geometric constraints. This yields equations that can be solved for the remaining translational and rotational parameters. The following derivation for the spatial RPR robot has been applied to spatial RR, CC and RRR robots as well.

5. Synthesis of a Spatial RPR Robot

The RPR robot is a three-degree of freedom robot. The fixed axis G allows rotation of angle θ about it and it is followed by a translation d along an arbitrary direction P and finally a rotation of angle ϕ about an arbitrary axis W, see Figure 1.

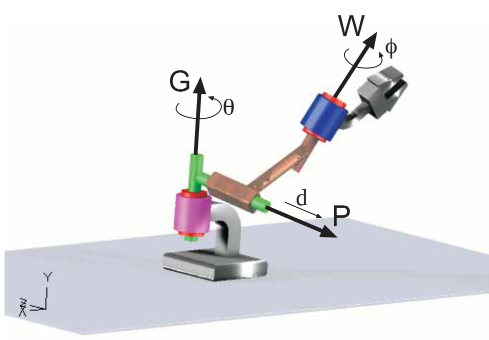

Figure 1. The spatial RPR robot

The dual quaternion representation for the relative displacements of the chain is given by

$$\hat{Q}_{RPR} = \hat{G}(\theta, 0)\hat{P}(0, d)\hat{W}(\phi, 0), \tag{12}$$

When applying the dual quaternion product we obtain the expression $\hat{Q}_{RPR} = Q^0 + \mathsf{Q}$, where the point is

$$Q^0 = \cos\frac{\theta}{2}\cos\frac{\phi}{2} - \epsilon\frac{d}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2}\mathsf{G}\cdot\mathsf{P} - \epsilon\frac{d}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2}\mathsf{P}\cdot\mathsf{W} -$$
$$\sin\frac{\theta}{2}\sin\frac{\phi}{2}\mathsf{G}\cdot\mathsf{W} - \epsilon\frac{d}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2}(\mathsf{G}\times\mathsf{P})\cdot\mathsf{W}, \tag{13}$$

and the vector

$$
\begin{aligned}
\mathsf{Q} = &\sin\frac{\theta}{2}\cos\frac{\phi}{2}\mathsf{G} + \cos\frac{\theta}{2}\sin\frac{\phi}{2}\mathsf{W} + \epsilon\frac{d}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2}\mathsf{P} - \\
&\epsilon\frac{d}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2}(\mathsf{G}\cdot\mathsf{P})\mathsf{W} + \epsilon\frac{d}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2}(\mathsf{G}\times\mathsf{P}) + \epsilon\frac{d}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2}(\mathsf{P}\times\mathsf{W}) + \\
&\sin\frac{\theta}{2}\sin\frac{\phi}{2}(\mathsf{G}\times\mathsf{W}) + \epsilon\frac{d}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2}(\mathsf{G}\times\mathsf{P})\times\mathsf{W}.
\end{aligned} \tag{14}
$$

The expansion of this equations componentwise leads to a set of equations in the components of the fixed rotation axis, $\mathsf{G} = (g_x, g_y, g_z) + \epsilon(g_x^0, g_y^0, g_z^0)$, the moving prismatic axis $\mathbf{P} = (p_x, p_y, p_z)$, and the moving rotation axis $\mathsf{W} = (w_x, w_y, w_z) + \epsilon(w_x^0, w_y^0, w_z^0)$.

The number of positions needed to obtain finite number of solutions is as follows: we have $15 + 3(n-1)$ unknowns (for the prismatic joint P only the direction matters) and $5+6(n-1)$ equations. Therefore we can define up to $n = 4 + \frac{1}{3}$ positions. The fractional value of n means that we can define 4 full positions plus two out of the six parameters that define a fifth position. Another option is to specify some extra relation for the joint axes. For instance, if we specify that the slider P must be perpendicular to the fixed rotation axis G we are adding one constraint and the counting gives finite number of solutions for $n = 4$ positions. This is the case that we use in the example.

To create the design equations we equate the expanded eq.(14) to the goal dual quaternion $\hat{D}$, that is,

$$
\hat{Q}_{RPR}(\theta, d, \phi) - \hat{D} = \vec{0}, \tag{15}
$$

to obtain one of the sets of design equations. To eliminate the joint parameters we consider first the direction equations, which can be solved linearly for θ and ϕ,

$$
\begin{bmatrix} w_x & g_x & g_y w_z - g_z w_y & 0 \\ w_y & g_y & g_z w_x - g_x w_z & 0 \\ w_z & g_z & g_x w_y - g_y w_x & 0 \\ 0 & 0 & -(g_x w_x + g_y w_y + g_z w_z) & 1 \end{bmatrix} \begin{Bmatrix} \cos\frac{\theta}{2}\sin\frac{\phi}{2} \\ \sin\frac{\theta}{2}\cos\frac{\phi}{2} \\ \sin\frac{\theta}{2}\sin\frac{\phi}{2} \\ \cos\frac{\theta}{2}\cos\frac{\phi}{2} \end{Bmatrix} = \begin{Bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{Bmatrix} \tag{16}
$$

The solution exists when the matrix is invertible (the determinant is zero only for the degenerate case when both directions are parallel, hence creating a planar rotation) and when the directions $\mathbf{g}$, $\mathbf{w}$ make the system solvable, which we can assume will be given by the solution of the design equations. The obtained values are substituted in the four moment equations, to obtain four equations which are linear in the parameter d. We eliminate linearly the joint variable d . We obtain three implicit equations per goal position. These 9 equations together with 6 constraints,

$$\begin{aligned}
&g_x^2+g_y^2+g_z^2=1, \quad g_x g_x^0+g_y g_y^0+g_z g_z^0=0\\
&w_x^2+w_y^2+w_z^2=1, \quad w_x w_x^0+w_y w_y^0+w_z w_z^0=0\\
&p_x^2+p_y^2+p_z^2=1, \quad g_x p_x+g_y p_y+g_z p_z=0
\end{aligned} \tag{17}$$

allows us to solve for the 15 unknowns corresponding to the three joint axes.

We present an example in which we want to design the RPR robot to reach the following four positions:

Table 1. The goal positions

Position	*Axis*	*Rotation*	*Translation*
position 1	$(1.0, 0.0, 0.0) + \epsilon(0.0, 0.0, 0.0)$	0^o	0
position 2	$(0.52, 0.59, 0.61) + \epsilon(-0.97, 1.35, -0.47)$	120^o	-0.32
position 3	$(0.54, 0.56, 0.63) + \epsilon(-0.10, 0.84, -0.66)$	44^o	-0.05
position 4	$(-0.33, -0.63, -0.70) + \epsilon(0.00, 1.30, -1.18)$	39^o	0.01

One of the obtained solutions is presented, as the joint axes in the first position,in Table(2). Figure 2 shows the robot while reaching each of the positions.

Table 2. The joint axes

Joint Axis	*Direction*	*Moment*
G	$(0.394, 0.593, 0.701)$	$(-1.379, -0.074, 0.839)$
W	$(0.594, 0.518, 0.615)$	$(0.388, 0.976, -1.199)$
P	$(-0.897, 0.085, 0.432)$	$(-0.629, -0.946, -1.119)$

6. Conclusions

This paper introduces a new formulation for the kinematic synthesis of constrained serial robots. These robots have less than six degrees of freedom. The standard kinematics equations of the chain are transformed into successive screw displacements and then expressed using dual quaternions. The result is an explicit set of axis parameters that define the robot and a set of joint parameters that can be eliminated

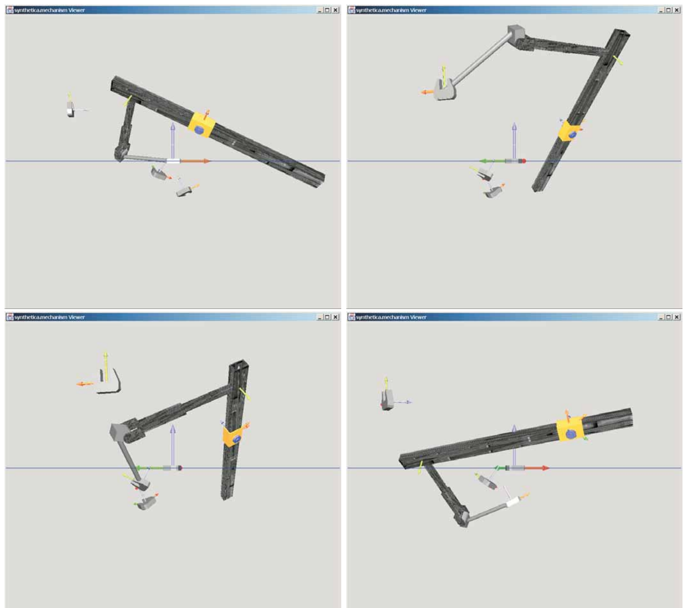

Figure 2. The spatial RPR robot reaching the four goal positions

algebraically. The structure of these equations provides a convenient strategy for this elimination, which we demonstrate with the design of a spatial RPR robot.

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