# A spectral method for stiff differential-delay equations.

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**Abstract**

We present a spectral method for the study of singularly perturbed differential-delay equations and systems of such equations. The method, which can be applied to quite general systems of the form

$$\mathcal{L}[u;\epsilon](t) = f_0(u(t)) + \sum_{i=1}^{k} f_i(u(t-T_i))$$

will be illustrated here for two single-delay problems, the first one linear:

$$\frac{du}{dt} = u(t) - au(t-T)$$

and the second nonlinear:

$$\mathcal{L}_\epsilon[x](t) = g(x(t)) + f(x(t-1);\lambda)\ .$$

Here $T$ is the lag time, $f$ and $g$ are (possibly) nonlinear functions of $u$ and $\mathcal{L}$ is a linear operator of order $k$, possibly with rational function coefficients and depending on a parameter $\epsilon$ singularly, so that the operator's order is reduced at $\epsilon = 0$. Both have periodic solutions for certain ranges of the controlling parameters and forms of the nonlinearity. The method employs a Chebyshev expansion over a delay interval (the smallest delay interval, in the case of multiple delays). In its simplest form, when only a single lag interval or comensurate intervals are involved, the numerical calculation advances the value of the function from one lag interval to the next by taking advantage of the fact that, with a fixed resolution per interval all interpolation points in a given interval are exactly one lag time forward from those on the previous interval. The differential operator is preconditioned by the Chebyshev integration operator $B^k$ and thus converted to banded form [1]. For the cases considered here the operator $\mathcal{L}$ has the simple form

$$\mathcal{L} := \epsilon\frac{d}{dt} + 1\ .$$

For this operator the resulting system of equations for the coefficients in the Chebyshev expansion of $x$ at a given interval $nT \le t \le (n+1)T$ becomes tridiagonal. A highly accurate solution

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is then possible by using methods of 3-term recurrence relations [2]. In the present case, the homogeneous solutions of this recurrence are explicitly known, in terms of Bessel functions, and a rigorous analysis of the error is possible.

The study of the linear problem serves to illustrate the performance of the method against other high order methods for nonstiff problems. We show comparisons with the algorithm *RETARD* of [5] which demonstrate the ability of our algorithm to maintain after a million lag intervals a maximum relative error less by several orders of magnitude than RETARD on the same problem. In this case both algorithms were run with conditions and parameter values to minimize the error for each.

The nonlinear problem serves to illustrate the performance of our scheme on a stiff system. We consider Smith's equation

$$\epsilon \frac{dx}{dt} + x = \lambda x(t-1)\,(1 - x(t-1)) \ .$$

near values of $\lambda$ for which the logistic equation (found if we set $\epsilon = 0$) shows Hopf bifurcations [3], and for values of the parameter $\epsilon$ as small as $10^-7$. The small $\epsilon$ limit leads to relaxation-like oscillations with extremely fast rise times and other extremely sensitive behaviors [4] whose resolution requires the accurate computation of solutions for very long times, possibly in the order of $10^6 T$ with $T$ the delay interval. Comparisons are carried out against the stiff differential-delay solver *RADAR5* which again establish the superior ability of the spectral scheme to maintain high order accuracy even in the computation of problems where there may exist very steep interior layers whose location is not known apriori. The method is ideal for the accurate tracking of oscillatory solutions for very long time intervals, $t > 10^6 * T$ without appreciable phase error.

# References

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