

On computing the pressure for a bounded, 2-d incompressible, viscous flow

Vers. 1: without reference to a-priori estimates and existence theory

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1 Introduction

The mathematical problem of viscous, incompressible hydrodynamics is to determine, in some region \mathcal{D} , the evolution in time of some initial velocity \mathbf{u}_0 subject to the incompressibility condition $\nabla \cdot \mathbf{u}_0 = 0$ and some boundary conditions at $\partial\mathcal{D}$. The momentum equation for incompressible fluid flow is

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \frac{p}{\rho} + \nu \Delta \mathbf{u} + \mathbf{F} \quad (1)$$

with \mathbf{F} an externally applied force, which can be written in the form

$$\mathbf{F} = \nabla \phi + \nabla \times \mathbf{A}.$$

Here, the flow is assumed confined in some region $D \in R^n$, $n = 2, 3$. Since the velocity field is divergence-free, we can take the divergence of eq.(1) to get

$$\Delta \frac{p}{\rho} = -\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) + \Delta \phi \quad (2)$$

while, by considering eq.(1) on the boundary ∂D we find

$$\left. \nabla \frac{p}{\rho} \right|_{\partial D} = (\nu \Delta \mathbf{u} + \mathbf{F})|_{\partial D} \quad (3)$$

(in general \mathbf{u} is given at walls and \mathbf{F} is assumed known, while \mathbf{u}_t could be nonzero but known at walls etc.). Thus we see that determination of the pressure field in terms of the velocity field appears to involve the solution of an overdetermined Poisson problem, since we have both Dirichlet and Neumann data at the boundary.

This would seem to necessitate a proof of compatibility of the two specifications, or, alternatively, an additional restriction on the velocity field so that the problem is solvable. It turns out that no additional restrictions on the velocity field are needed, and that any initial, incompressible velocity field compatible with the boundary conditions does lead to a pressure field satisfying all the required conditions. In this note we discuss the computation of the pressure field in two dimensional viscous, incompressible flow in a bounded region. In Sec. 2 we give the general framework, independent of the boundary conditions. Section 3 addresses the details for noslip conditions and presents an explicit calculation for a vorticity field that, although it corresponds to a velocity field that satisfies the noslip conditions, gives an incompatible pressure [3]. In an appendix we discuss the Dirichlet-to-Neumann operator for a parallel, periodic slab.

2 Two-dimensional flow: preliminaries.

The 2-d Navier-Stokes equations can be written in the form

$$-\omega \nabla \psi = \nabla P + \nabla (\psi_t + \nu \omega) \times \mathbf{e}_z , \quad (4)$$

with $P = p/\rho + u^2/2$ the dynamic pressure, giving the Helmholtz decomposition of the nonlinear advective term as a gradient and a curl, i.e. as an irrotational and a solenoidal field. This decomposition is unique up to the addition of a term ∇g for g harmonic because then there is a harmonic conjugate field h , such that $\nabla g = \nabla h \times \mathbf{e}_z$ and we can write

$$-\omega \nabla \psi = \nabla (P + g) + \nabla (\psi_t + \nu \omega - h) \times \mathbf{e}_z .$$

Including a forcing term would introduce no essential changes in the discussion, and it is omitted here for simplicity.

The relation between vorticity and stream function

$$\Delta \psi = -\omega \quad (5)$$

is also 1-1 up to a harmonic potential. Dirichlet **or** Neumann boundary conditions on $\partial \mathcal{D}$ suffice to completely specify a harmonic potential in \mathcal{D} , and the implication is that the solutions to both problems mentioned above, i.e. the Helmholtz decomposition and Poisson's equation are made unique by imposition of either type of boundary condition or by a **rank 1** combination. However, viscosity (and thus the presence of a non-gradient part of the stress) introduces additional conditions at the boundary, thus making both of the above problems overdetermined, unless certain global solvability conditions are imposed on the vorticity field, ω . There are apparently two such conditions: the first, $\mathcal{C}1$, guaranteeing the solvability of the vorticity-streamfunction problem (5), the second, $\mathcal{C}2$, the proper (i.e. in accord to all boundary conditions) Helmholtz decomposability of the advective term $\omega \nabla \psi$ (4). $\mathcal{C}1$ guarantees that the initial field specified in terms of a vorticity, ω_0 , will satisfy all necessary boundary conditions. This condition is a simple geometric restriction on the vorticity field guaranteeing the compliance of the solution of the Dirichlet Poisson equation (5) with the additional (Neumann or other) condition. It provides also the solvability condition for the Stokes problem. $\mathcal{C}2$, although it can be expressed as a Dirichlet-to-Neumann (or analogous) compatibility is subtler as it involves a non-linear operator. An initial vorticity $\omega_0 \in \mathcal{C}1 \cap \mathcal{C}2$ can be used to compute an initial time derivative for the streamfunction, $\psi_{0,t}$, by solving an overdetermined Poisson problem. This can then serve as the starting point of a numerical computation of the flow as an Initial-Boundary Value Problem (IBVP). Our concern here will be with the question: assuming the initial fields do satisfy the compatibility conditions $\mathcal{C}1 \cap \mathcal{C}2$; is it then enough to require $\mathcal{C}1$ for subsequent evolution (i.e. is $\mathcal{C}2$ implied)? The affirmative answer is the subject of this article. Although the PDE-theoretic aspects of this question have been well known since the work of Leray [4] (see also Heywood, [2]), there seems to be a need for some clear examples for the benefit of the CFD community. Indeed, pressure computations often seem to be performed

without a clear appreciation of compatibility questions. A recent review of this subject by Kress and Montgomery [3] gives references and poses some questions as to the appropriateness of no-slip boundary conditions. Our point here is that one must formulate initial conditions for the NS equations in bounded, no-slip domains with care. Imposition of $\mathcal{C}2$ on the initial vorticity (or velocity) fields is not essential; it will be satisfied for $0 < t$, even if it is initially violated. However, the solution may then tend to its initial conditions (as $t \rightarrow 0^+$) only in a weak sense [2]. Because of the unclear status of the subject in much of the numerical literature we feel that it would be instructive to present a thorough analysis of the initial value problem associated with an elementary example. Indeed this problem is inherent not only under no-slip but also under other conditions that mix the velocity and vorticity fields. The resolution of paradoxes lies with the heat operator, whose smoothing properties are well known.

We will discuss the details of the decomposition (and the solution to the pressure equation) for the important case of no-slip boundary conditions. Although we will keep things general, an explicit calculation will be given for a periodic slab. The condition that the (fixed) boundary is a streamline, also valid in the absence of viscosity, gives:

$$\mathbf{T} \cdot \nabla \psi = 0, \quad \mathbf{r} \in \partial \mathcal{D}$$

with \mathbf{T} an arbitrary tangent at the boundary or, equivalently,

$$\psi = \text{constant}, \quad \mathbf{r} \in \partial \mathcal{D}. \quad (6)$$

This boundary condition uniquely specifies both of the above problems: namely a unique streamfunction is found that corresponds to the given, arbitrary vorticity and to the fixed-boundary condition. Given these vorticity-streamfunction fields, one can perform the Helmholtz decomposition of $\omega \nabla \psi$ for either component uniquely given Dirichlet (or some other rank-1) conditions.

Two routes are available for the decomposition, which lead to equivalent results: we can choose to determine either potential, ϕ or P , from the advection term. By taking the curl and divergence of eq.(1) we find, respectively, equations for the solenoidal potential

$$\phi := \psi_t + \nu \omega,$$

or the irrotational potential

$$P = \frac{p}{\rho} + \frac{u^2}{2}.$$

We have:

$$\Delta \phi = \nabla \omega \times \nabla \psi \cdot \mathbf{e}_z, \quad (7)$$

and

$$\Delta P = -\nabla \cdot (\omega \nabla \psi) = -\nabla \omega \cdot \nabla \psi + \omega^2. \quad (8)$$

Either equation is uniquely solvable given appropriate boundary conditions.

3 No-slip boundary conditions

Adhesion of fluid to the wall demands that the tangential velocity vanishes, leading to the condition

$$\nabla\psi = 0, \quad \mathbf{r} \in \partial\mathcal{D}.$$

This leads to overspecified Poisson equations for the streamfunction ψ as well as for the potentials P and ϕ . We can easily state conditions for the solvability of these equations. First look at the vorticity-stream function problem:

$$\Delta\psi = -\omega, \quad \nabla\psi|_{\partial\mathcal{D}} = 0.$$

The solution of the Dirichlet problem is given in terms of the Dirichlet Green's function G_D :

$$\psi(\mathbf{r}) = \int_{\mathcal{D}} G_D(\mathbf{r}|\xi)\omega(\xi)d\xi.$$

Then, define the Dirichlet-to-Neumann functional:

$$\mathcal{N}[\omega] = \mathbf{n} \cdot \nabla\psi|_{\partial\mathcal{D}} := \left(\mathbf{n} \cdot \nabla \int_{\mathcal{D}} G_D(\mathbf{r}|\xi)\omega(\xi)d\xi \right)_{\partial\mathcal{D}}.$$

Solvability of the streamfunction problem is then guaranteed, provided ω satisfies (C1):

$$\mathcal{N}[\omega] = 0, \tag{9}$$

while solvability of the Poisson problem for ψ_t requires (C2):

$$\mathcal{N}[-\nu \Delta \omega + J(\omega, \psi)] = 0. \tag{10}$$

The subsequent derivations rely on the:

Assumption *The solution to the vorticity equation, with analytic ω_0 satisfying C1 for all time and C2 at $t = 0$ exists together with all its derivatives for some finite interval $0 \leq t < T$ [2].*

Given the above assumption, we now show that ω satisfies C2 while it exists. Indeed, examining the solenoidal potential equation we have:

$$\Delta\phi = \Delta(\psi_t + \nu\omega) = \nabla\omega \times \nabla\psi \cdot \mathbf{e}_z,$$

or

$$(\partial_t - \nu\Delta)\omega = J(\psi, \omega), \quad \mathcal{N}[\omega] = 0.$$

If the latter problem is well posed, then for the time interval of existence, the time derivative of ω will also satisfy C1 and the problem for ψ_t is solvable:

$$\Delta\psi_t = -\nu \Delta \omega + \nabla\omega \times \nabla\psi \cdot \mathbf{e}_z = -\omega_t$$

Thus we see that compatibility of the RHS of the solenoidal potential with C2 follows from compatibility of ω (and thus of ω_t) with C1 for some time interval. Since the evolution operator is the heat operator, we know that we can allow for some initial mismatch with boundary conditions, i.e. solutions may exist in the strong sense for

$\tau < t < T$ but only in the generalized sense as $\tau \rightarrow 0$. Assuming this, we now take up the second question, namely, what if $\mathcal{C}2$ is not satisfied by the initial vorticity ω_0 ? If we only consider initial fields that comply with $\mathcal{C}1$, we can legitimately ask about the mechanism by which the flow adjusts to the boundary conditions and what is the initial pressure.

Since $\omega \in \mathcal{C}1$, $t \geq 0$, and assuming $\lim_{t \rightarrow 0^+} \omega$ exists, we have

$$\lim_{t \rightarrow 0^+} \omega_t = \lim_{t \rightarrow 0^+} \lim_{\delta t \rightarrow 0^+} \frac{\omega(\delta t) - \omega(0)}{\delta t} - \nu \Delta \omega(\delta t) = J(\psi_0, \omega_0) .$$

This Helmholtz equation for $\omega(\delta t)$ can be solved in $\mathcal{C}1$, resulting in a value for $\omega_t(0^+)$ that is accurate to $\mathcal{O}(\delta t)$. In turn, the resulting value for $\psi_t(0^+)$ can be determined to similar accuracy, and having determined the solenoidal part of the Helmholtz decomposition one can now perform the necessary integration to determine the gradient part. Consequently, the error (non-irrotationality) that enters in the numerical computation of the pressure is caused solely by the time-stepping truncation and other numerical errors associated with the numerical approximation of functions and the solutions of the Poisson and Helmholtz partial differential equations.

This leads to a natural question, if ω is not initially in $\mathcal{C}2$: since the field ψ_t computed by the above process is $\mathcal{C}2$, while $\psi_t(0)$, as defined by the vorticity equation clearly is not in $\mathcal{C}2$, how can we reconcile the discrepancy through an initial correction to ω ? The answer is that $\psi_t(0)$ is not defined for the initial conditions, unless they are in $\mathcal{C}2$. Any computations requiring a time derivative (including that of the pressure) for non- $\mathcal{C}2$ fields can only be performed in the limiting sense $t \rightarrow 0^+$.

3.1 An example of Kress and Montgomery

Kress and Montgomery [3] consider the following vorticity:

$$\omega_0 = \cos \lambda x \cos \sigma y \tag{11}$$

which fulfills $\mathcal{C}1$ for $\lambda = \lambda_n = n\pi/L$ for some integer n with L the length of the channel and σ chosen to satisfy

$$\sigma \tan \sigma = -\lambda \tanh \lambda \tag{12}$$

leading to the streamfunction

$$\psi_0 = \frac{1}{\lambda^2 + \sigma^2} \cos \lambda x \left(\cos \sigma y - \frac{\cos \sigma}{\cosh \lambda} \cosh \lambda y \right) . \tag{13}$$

This streamfunction is easily seen to satisfy the no-slip BC. However, $\mathcal{C}2$ is violated. Indeed:

$$\begin{aligned} J(\omega, \psi) &= \begin{vmatrix} \omega_x & \omega_y \\ \psi_x & \psi_y \end{vmatrix} \\ &= \frac{1}{\lambda^2 + \sigma^2} \frac{\cos \sigma}{\cosh \lambda} \begin{vmatrix} -\lambda \sin \lambda x \cos \sigma y & -\sigma \cos \lambda x \sin \sigma y \\ \lambda \sin \lambda x \cosh \lambda y & -\lambda \cos \lambda x \sinh \lambda y \end{vmatrix} \\ &= \frac{1}{2} \frac{\lambda}{\lambda^2 + \sigma^2} \frac{\cos \sigma}{\cosh \lambda} \sin 2\lambda x (\lambda \cos \sigma y \sinh \lambda y + \sigma \sin \sigma y \cosh \lambda y) \\ &=: \sin 2\lambda x J_{2\lambda} \end{aligned}$$

Solvability of the equation for ψ_t requires that

$$I_2 := \int_0^1 \sinh 2\lambda y J_{2\lambda} dy = 0 . \quad (14)$$

Defining Q_2, Q_3 by:

$$Q_2 := \int_0^1 \sinh 2\lambda y \cos \sigma y \sinh \lambda y dy , \quad (15)$$

$$Q_3 := \int_0^1 \sinh 2\lambda y \sin \sigma y \cosh \lambda y dy \quad (16)$$

we have:

$$\frac{\lambda^2 + \sigma^2}{\lambda} \frac{\cosh \lambda}{\cos \sigma} I_2 = \frac{1}{2} (\lambda Q_2 + \sigma Q_3) = -2 \frac{\lambda^2 + \sigma^2}{9\lambda^2 + \sigma^2} \cos \sigma \cosh \lambda \sinh 2\lambda \quad (17)$$

where we have used the relation between σ and λ ($\sigma \tan \sigma = -\lambda \tanh \lambda$). But the latter expression is nonzero, for any choice of aspect ratio for the channel (the only available free parameter) and the Kress-Montgomery fields cannot be made to satisfy $\mathcal{C}2$ except trivially. Thus, no compatible pressure field can be computed for this initial velocity, even though the latter satisfies the no-slip boundary conditions. The initial pressure exists only in a weak sense, requiring, as we shall see, appropriately placed boundary layers.

3.2 Time evolution

The vorticity equation, whose solution we have assumed to exist for $0 < t < T$, is an IBVP needing initial as well as boundary conditions. Without resorting to hard estimates, we can only give a formal discussion of a procedure to generate approximations to the time evolution of the KM field since $\omega_0 \notin \mathcal{C}2$ and we will pursue it only in order to understand how a simple numerical scheme will advance in time. However the procedure can be also considered as a method for generating a vorticity field which is compatible with $\mathcal{C}2$ with an error that can be made arbitrarily small. We have (with $\omega_1 := \omega(\delta t)$):

$$(1 - \nu \delta t \Delta) \omega_1 = \omega_0 + \delta t J(\psi_0, \omega_0) + \mathcal{O}(\delta t^2) .$$

This Helmholtz equation is uniquely solvable for $\omega_1 \in \mathcal{C}1$. Clearly, solving the above problem begins with a $\mathcal{C}1$ field at $t = 0$ and produces one at $t = \delta t$. If the limit

$$\lim_{t \rightarrow 0^+} \omega_t = \lim_{t \rightarrow 0^+} \lim_{\delta t \rightarrow 0^+} \frac{\omega(t + \delta t) - \omega(t)}{\delta t}$$

exists in some sense, then we can use it as a source for computing ψ_t . The sense in which we will compute the corrected initial field is the following:

$$\hat{\omega}_0 = \lim_{\delta t \rightarrow 0^+} \omega_1(\delta t)$$

and for the initial t-derivative:

$$\hat{\omega}_{0,t} = \lim_{\delta t \rightarrow 0^+} \frac{\omega_1(\delta t) - \hat{\omega}_0}{\delta t}$$

We now give the necessary computations. Let $\epsilon^2 := \nu \delta t$; then the problem becomes

$$\mathcal{H}\omega_1 := (1 - \epsilon^2 \Delta) \omega_1 = \omega_0 + \delta t J(\psi_0, \omega_0) .$$

We seek ω_1 in the form

$$\begin{aligned} \omega_1 &= a_1 \Omega_1 \cos \lambda x + (a_2 \Omega_2 + a_3 \Omega_3) \sin 2\lambda x + a_4 \omega_h \\ &:= a_1 \cos \lambda x \cos \sigma y + \sin 2\lambda x \left(a_2 \cos \sigma y \sinh \lambda y + a_3 \sin \sigma y \cosh \lambda y + a_4 \sinh \left(\frac{y}{z} \right) \right) , \end{aligned} \quad (18)$$

where we defined

$$z^{-1} = \sqrt{\frac{1}{\epsilon^2} + 4\lambda^2} = \frac{1}{\epsilon} \sqrt{1 + 4\lambda^2 \epsilon^2} = \frac{1}{\epsilon} + 2\lambda^2 \epsilon + \mathcal{O}(\epsilon^3) . \quad (19)$$

Since the homogeneous solution ω_h is included so ω_1 can be made to comply with $\mathcal{C}1$, only a mode $\sin 2\lambda x$ part is needed.

Substitution of ω_1 into the Helmholtz equation gives:

$$\begin{aligned} \mathcal{H}\omega_1 &= a_1 [1 + \epsilon^2(\sigma^2 + \lambda^2)] \Omega_1 \sin \lambda x \\ &+ (a_2 [1 + \epsilon^2(3\lambda^2 + \sigma^2)] - a_3 2\epsilon^2 \sigma \lambda) \Omega_2 \sin 2\lambda x \\ &+ (a_3 [1 + \epsilon^2(3\lambda^2 + \sigma^2)] + a_2 2\epsilon^2 \sigma \lambda) \Omega_3 \sin 2\lambda x \end{aligned}$$

which, when set equal to the expression for $\omega_0 + \delta t J(\psi_0, \omega_0)$ gives for the coefficients:

$$a_1 = \frac{1}{1 + \epsilon^2(\sigma^2 + \lambda^2)} = 1 + \mathcal{O}(\epsilon^2) , \quad (20)$$

$$\begin{aligned} a_2 &= -\frac{\epsilon^2 \lambda}{2\nu(\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} \frac{\lambda \{1 + \epsilon^2(3\lambda^2 + \sigma^2)\} + \sigma \{2\epsilon^2 \sigma \lambda\}}{[1 + \epsilon^2(\sigma^2 + 3\lambda^2)]^2 + [2\epsilon^2 \sigma \lambda]^2} \\ &= -\epsilon^2 \frac{\lambda^2}{2\nu(\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} + \mathcal{O}(\epsilon^4) , \end{aligned} \quad (21)$$

and

$$\begin{aligned} a_3 &= -\frac{\epsilon^2 \lambda}{2\nu(\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} \frac{\sigma \{1 + \epsilon^2(3\lambda^2 + \sigma^2)\} - \lambda \{2\epsilon^2 \sigma \lambda\}}{[1 + \epsilon^2(\sigma^2 + 3\lambda^2)]^2 + [2\epsilon^2 \sigma \lambda]^2} \\ &= -\epsilon^2 \frac{\lambda \sigma}{2\nu(\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} + \mathcal{O}(\epsilon^4) . \end{aligned} \quad (22)$$

Compatibility for ω_1 gives:

$$\int_0^1 \sinh 2\lambda y \omega_1(y) dy = a_2 Q_2 + a_3 Q_3 + a_4 Q_4 = 0$$

from which a_4 is computed:

$$a_4 = -\frac{a_2 Q_2 + a_3 Q_3}{Q_4}$$

with Q_2, Q_3 defined previously and Q_4 given by:

$$\begin{aligned} Q_4 &:= \int_0^1 \sinh 2\lambda y \sinh\left(\frac{y}{z}\right) dy \\ &= \frac{2z}{2z\lambda + 1} \sinh\left(2\lambda + \frac{1}{z}\right) - \frac{2z}{2z\lambda - 1} \sinh\left(2\lambda - \frac{1}{z}\right). \end{aligned}$$

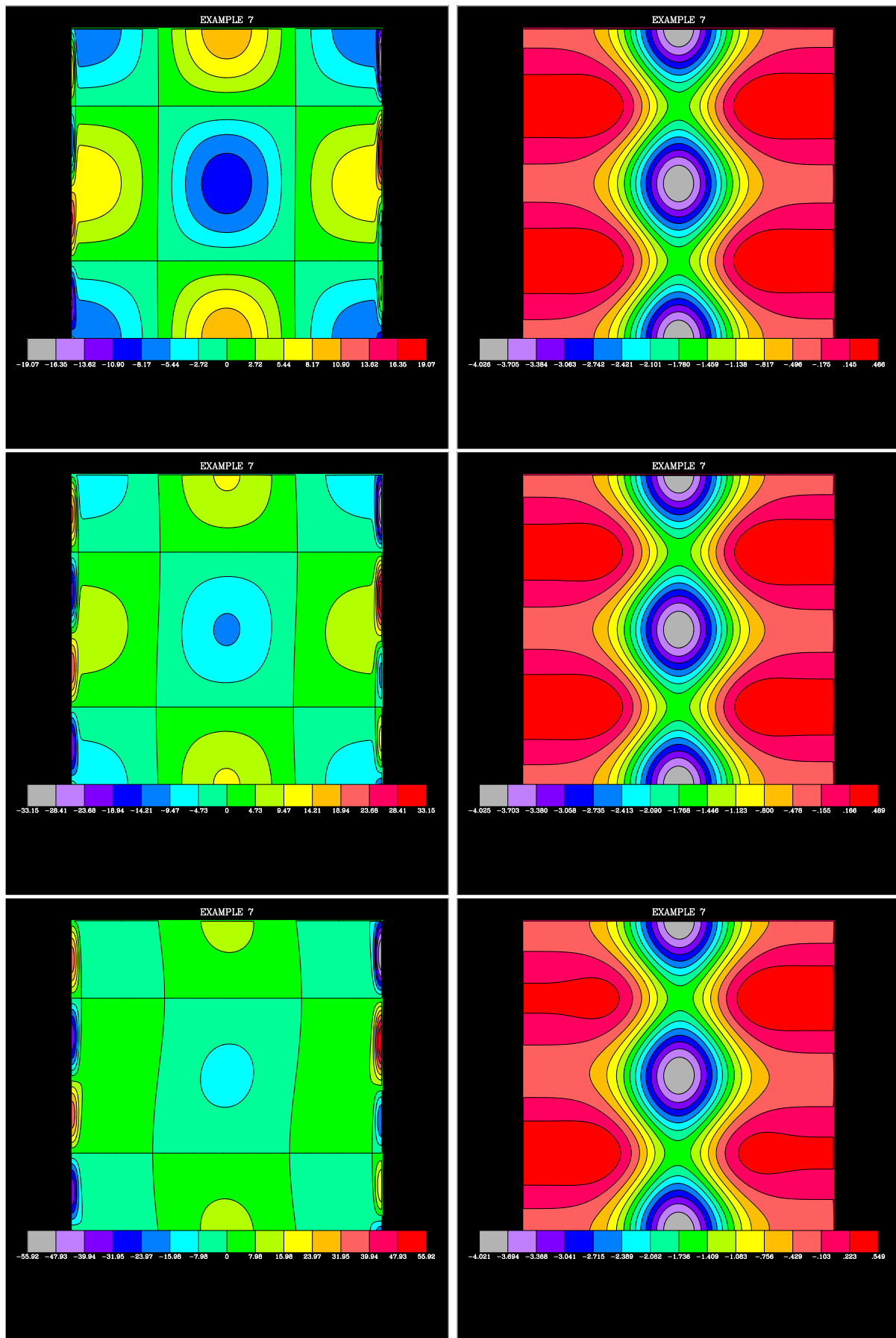


Figure 1. Vorticity and pressure for the $\mathcal{K}M$ field: $10 \cos 2\pi x/L_x \cos 2.4490372y$ at $t=.001, .004, .012$

3.2.1 The limit $t \rightarrow 0^+$

Introducing in (18) the leading expressions for the coefficients a_1, a_2, a_3 given in (20, 21, 22) we find:

$$\begin{aligned}\omega_1(\epsilon) &= (1 + \mathcal{O}(\epsilon^2)) \cos \lambda x \cos \sigma y \\ &- \left(\epsilon^2 \frac{\lambda^2}{2\nu(\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} + \mathcal{O}(\epsilon^4) \right) \sin 2\lambda x \cos \sigma y \sinh \lambda y \\ &- \left(\epsilon^2 \frac{\lambda \sigma}{2\nu(\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} + \mathcal{O}(\epsilon^4) \right) \sin 2\lambda x \sin \sigma y \cosh \lambda y \\ &+ a_4 \omega_h\end{aligned}$$

where the homogeneous term ω_h is the most complicated and will be treated separately; the particular solution clearly approaches ω_0 uniformly as $\epsilon \rightarrow 0$.

To treat the homogeneous term we consider leading order behavior for its factors as $\epsilon \rightarrow 0$:

$$a_2 Q_2 + a_3 Q_3 = \frac{2\epsilon^2 \lambda}{\nu(9\lambda^2 + \sigma^2)} \cos^2 \sigma \sinh 2\lambda + \mathcal{O}(\epsilon^4) \quad (23)$$

and

$$\frac{Q_4}{2z} = e^{z^{-1}} (\sinh 2\lambda + 2z\lambda \cosh 2\lambda + \mathcal{O}(z^2)) + \mathcal{O}(e^{-z^{-1}}). \quad (24)$$

We are interested in the behavior of ω_h for z small. Defining the constant C as:

$$C := \frac{\lambda}{(9\lambda^2 + \sigma^2)} \cos^2 \sigma$$

we arrive at the following:

$$\omega_h(\epsilon) = -\epsilon \frac{C}{\nu} \sin 2\lambda x e^{-z^{-1}} \sinh\left(\frac{y}{z}\right) + \mathcal{O}(\epsilon^2)$$

Collecting terms, we can give the leading part of the vorticity field:

$$\omega_1(\epsilon) = \cos \lambda x \cos \sigma y - \epsilon \frac{C}{2\nu} (e^{(y-1)/z} - e^{-(y+1)/z}) \sin 2\lambda x + \mathcal{O}(\epsilon^2)$$

The two $\mathcal{O}(\epsilon)$ boundary layers at $y = \pm 1$ clarify the nature of the nonuniform approach to the limit $t \rightarrow 0^+$: although $\omega(\delta t) \rightarrow \omega_0$, we see that its y -derivatives approach their limiting value in a singular fashion, with $\mathcal{O}(1)$ boundary layers. We will need these derivatives, so we give them here:

$$\partial_x \omega_1 = -\lambda \sin \lambda x \cos \sigma y - 2\lambda \epsilon \frac{C}{2\nu} (e^{(y-1)/z} - e^{-(y+1)/z}) \cos 2\lambda x + \mathcal{O}(\epsilon^2)$$

and

$$\partial_y \omega_1 = -\sigma \cos \lambda x \sin \sigma y - \frac{C}{2\nu} (e^{(y-1)/z} + e^{-(y+1)/z}) \sin 2\lambda x + \mathcal{O}(\epsilon^2)$$

The role of these boundary layers on the pressure computation is to adjust the boundary conditions that the pressure gradient must satisfy. Otherwise, they only contribute to higher order for either potential. As we will now see, the streamfunction is only altered to higher order as well.

3.2.2 The streamfunction ψ_1

We give now the computation of the streamfunction $\psi_1 := \psi(\delta t)$. We have

$$\begin{aligned}\Delta\psi_1 &= -\omega_1 \\ &= \cos \lambda x \cos \sigma y + \epsilon \frac{C}{\nu} e^{-z^{-1}} \sinh\left(\frac{y}{z}\right) \sin 2\lambda x + \mathcal{O}(\epsilon^2) .\end{aligned}$$

The mode $\cos \lambda x$ term has been treated before, in the computation of ψ_0 , so we concentrate on the mode $\sin 2\lambda x$ term. We let

$$\psi_1 = \frac{1}{\lambda^2 + \sigma^2} \cos \lambda x \left(\cos \sigma y - \frac{\cos \sigma}{\cosh \lambda} \cosh \lambda y \right) + \left(b_1 \sinh\left(\frac{y}{z}\right) + b_2 \sinh 2\lambda y \right) \sin 2\lambda x$$

Then

$$\Delta \left(b_1 \sinh\left(\frac{y}{z}\right) \sin 2\lambda x \right) = b_1 \left(\frac{1}{z^2} - 4\lambda^2 \right) \sinh\left(\frac{y}{z}\right) \sin 2\lambda x = \epsilon \frac{C}{\nu} e^{-z^{-1}} \sinh\left(\frac{y}{z}\right) \sin 2\lambda x$$

gives

$$b_1 = \epsilon^3 \frac{C}{\nu} e^{-z^{-1}}$$

and it is clear that, to leading order, ψ_1 can be replaced by ψ_0 in all subsequent computations.

3.3 Computation of the solenoidal potential ϕ

For purposes of numerical computation, it turns out to be advisable to integrate the components of ∇P after calculating the solenoidal potential ϕ (see Coutsias et al. [1]). It satisfies the equation

$$\Delta\phi := \Delta(\psi_t + \nu\omega) = J(\omega, \psi) , \quad \nabla\phi|_{\partial\mathcal{D}} = \nu\nabla\omega|_{\partial\mathcal{D}} .$$

This computes the lumped quantity $\psi_t + \nu\omega$ which, after all, is what appears on the right hand side of the pressure gradient equation. However, it is easier to solve for ψ_t , and we give that calculation here.

3.3.1 Compatibility for ψ_t

Before computing ϕ and P , we verify that the compatibility condition is satisfied to $\mathcal{O}(\epsilon^2)$. For this, it is simplest to work with ψ_t , so that we will verify that the right hand side for the problem

$$\Delta\psi_t = -\nu \Delta \omega_1 + J(\omega_1, \psi_1)$$

is in $\mathcal{C}2$, at least to $\mathcal{O}(\epsilon^2)$. Thus we must compute $\mathcal{N}[-\nu \Delta \omega_1 + J(\omega, \psi)]$. Based on the computations in the previous subsection, we can easily show that

$$\mathcal{N}[J(\omega_1, \psi_1)] = \mathcal{N}[J(\omega_0, \psi_0)] + \mathcal{O}(\epsilon^2)$$

while

$$\mathcal{N}[-\nu \Delta \omega_1] = \mathcal{N}[-\nu \Delta \omega_h] + \mathcal{O}(\epsilon^2)$$

since $\mathcal{N}[-\nu \Delta \omega_0] = 0$. Note that cross-terms like $\mathcal{N}[J(\omega_h, \psi_0)]$ etc which involve y -derivatives of ω_h and are therefore of $\mathcal{O}(1)$, contribute only to higher order once integrated, and the same is true in the computation for ψ_t , and thus we omit such terms. Now

$$-\nu \Delta \omega_h = -\frac{\nu a_4}{\epsilon^2} \omega_h = \left(-\frac{\nu a_4}{\epsilon^2}\right) \sin 2\lambda x \sinh\left(\frac{y}{z}\right)$$

Compatibility for the mode $\sin 2\lambda x$ term now reads

$$\begin{aligned} \int_0^1 \sinh 2\lambda y \left(-\frac{\nu a_4}{\epsilon^2} \sinh\left(\frac{y}{z}\right) + J_{2\lambda}\right) dy &= \\ \frac{\nu(a_2 Q_2 + a_3 Q_3)}{\epsilon^2} + \frac{\lambda}{\lambda^2 + \sigma^2} \frac{\cos \sigma}{\cosh \lambda} \frac{\lambda Q_2 + \sigma Q_3}{2} &= \mathcal{O}(\epsilon^2) \end{aligned}$$

as anticipated, using (23). Throughout we have neglected computations that contribute only to $\mathcal{O}(\epsilon^2)$.

3.3.2 Computation of ψ_t

We now carry out the computation of ψ_t . We begin by evaluating

$$\Delta(A_2 \Omega_2 + A_3 \Omega_3) \sin 2\lambda x = \left(\partial_y^2 - 4\lambda^2\right) (A_2 \cos \sigma y \sinh \lambda y + A_3 \sin \sigma y \cosh \lambda y) \sin 2\lambda x$$

giving

$$\left[A_2 (3\lambda^2 + \sigma^2) - A_3 2\sigma\lambda\right] \Omega_2 + \left[A_3 (3\lambda^2 + \sigma^2) + A_2 \sigma\lambda\right] \Omega_3 = \frac{\lambda}{2(\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} (\lambda \Omega_2 + \sigma \Omega_3)$$

leading to the system

$$\begin{aligned} A_2 (3\lambda^2 + \sigma^2) - A_3 2\sigma\lambda &= \frac{\lambda^2}{2(\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} =: A_1 \lambda \\ A_3 (3\lambda^2 + \sigma^2) + A_2 \sigma\lambda &= \frac{\lambda\sigma}{2(\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} =: A_1 \sigma \end{aligned}$$

We find:

$$A_2 = \frac{3\lambda^2}{2(\lambda^2 + \sigma^2)(9\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda}, \quad (25)$$

$$A_3 = -\frac{\lambda\sigma}{2(\lambda^2 + \sigma^2)(9\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda}. \quad (26)$$

This gives for ψ_t :

$$\begin{aligned} \psi_t &= -\nu \omega_1 + A_1 \cos \lambda x \cosh \lambda y - \nu a_4 \omega_h + (A_2 \Omega_2 + A_3 \Omega_3 + A_4 \sinh 2\lambda y) \sin 2\lambda x + \mathcal{O}(\epsilon^2) \\ &= -\nu \cos \lambda x \left(\cos \sigma y - \frac{\cos \sigma}{\cosh \lambda} \cosh \lambda y\right) \\ &\quad + \left(-\nu a_4 \sinh\left(\frac{y}{z}\right) + A_2 \Omega_2 + A_3 \Omega_3 + A_4 \sinh 2\lambda y\right) \sin 2\lambda x + \mathcal{O}(\epsilon^2) \end{aligned}$$

We now set A_4 so as the Dirichlet condition $\psi_t = 0$ at $y = \pm 1$ is satisfied; the Neumann condition should be automatically satisfied to $\mathcal{O}(\epsilon)$ (the expression for A_1 is the same as for ψ_0).

For the mode $\sin 2\lambda x$ the Dirichlet boundary conditions give the equation (keeping only terms of $\mathcal{O}(1)$):

$$A_2 \cos \sigma \sinh \lambda + A_3 \sin \sigma \cosh \lambda + A_4 \sinh 2\lambda + \mathcal{O}(\epsilon) = 0$$

so that

$$A_4 = -\frac{\lambda^2}{(\lambda^2 + \sigma^2)(9\lambda^2 + \sigma^2)} \left(\frac{\cos \sigma}{\cosh \lambda} \right)^2$$

A straightforward computation shows that the Neumann condition is satisfied to $\mathcal{O}(\epsilon)$.

Finally,

$$\begin{aligned} \phi &= \psi_t + \nu \omega_1 \\ &= \nu \frac{\cos \sigma}{\cosh \lambda} \cos \lambda x \cosh \lambda y \\ &+ \frac{\lambda^2}{2(\lambda^2 + \sigma^2)(9\lambda^2 + \sigma^2)} \left(\frac{\cos \sigma}{\cosh \lambda} \right) \left(3\Omega_2 - (\sigma/\lambda)\Omega_3 - 2 \left(\frac{\cos \sigma}{\cosh \lambda} \right) \sinh 2\lambda y \right) \sin 2\lambda x \end{aligned}$$

3.4 Computation of the initial pressure

We now give an approximation to $P(0) := \lim_{\delta t \rightarrow 0^+} P(\delta t)$. We must solve the overdetermined Poisson problem:

$$\Delta P = -\nabla \cdot (\omega \nabla \psi) = -\nabla \omega \cdot \nabla \psi + \omega^2, \quad \nabla P|_{\partial \mathcal{D}} = -\nu \nabla \omega \times \mathbf{e}_z.$$

It is clear, in light of the computation of ψ_1 given earlier, that no contribution is made by the layer terms to leading order in P or ∇P . The only leading contribution comes from the y -component of $\nabla \omega_1$ at the boundary. Thus we solve exactly the same problem as we would using just ω_0 and ψ_0 but with modified boundary values!

We have:

$$\nabla \omega_0 = -\lambda \sin \lambda x \cos \sigma y \hat{\mathbf{e}}_x - \sigma \cos \lambda x \sin \sigma y \hat{\mathbf{e}}_y$$

$$\nabla \psi_0 = \frac{-1}{\lambda^2 + \sigma^2} \left(\lambda \sin \lambda x \left(\cos \sigma y - \frac{\cos \sigma}{\cosh \lambda} \cosh \lambda y \right) \hat{\mathbf{e}}_x + \sigma \cos \lambda x \left(\sin \sigma y - \frac{\sin \sigma}{\sinh \lambda} \sinh \lambda y \right) \hat{\mathbf{e}}_y \right)$$

so that

$$\begin{aligned} \omega_0^2 - \nabla \omega_0 \cdot \nabla \psi_0 &= \frac{1}{2} \cos 2\lambda x \cos 2\sigma y \\ &+ \frac{\lambda^2}{2(\lambda^2 + \sigma^2)} \cos 2\lambda x + \frac{\sigma^2}{2(\lambda^2 + \sigma^2)} \cos 2\sigma y \\ &+ \frac{\lambda^2}{2(\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} \cos \sigma y \cosh \lambda y + \frac{\sigma^2}{2(\lambda^2 + \sigma^2)} \frac{\sin \sigma}{\sinh \lambda} \sin \sigma y \sinh \lambda y \\ &- \left(\frac{\lambda^2}{2(\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} \cos \sigma y \cosh \lambda y - \frac{\sigma^2}{2(\lambda^2 + \sigma^2)} \frac{\sin \sigma}{\sinh \lambda} \sin \sigma y \sinh \lambda y \right) \cos 2\lambda x \end{aligned}$$

So P has the form

$$P = C_{20} \cos 2\lambda x + C_{02} \cos 2\sigma y + C_{0c} \cos \sigma y \cosh \lambda y + C_{0s} \sin \sigma y \sinh \lambda y \\ + (C_{22} \cos 2\sigma y + C_{2c} \cos \sigma y \cosh \lambda y + C_{2s} \sin \sigma y \sinh \lambda y) \cos 2\lambda x$$

The coefficients are determined now in turn:

$$C_{20} = C_{02} = C_{22} = \frac{-1}{8(\lambda^2 + \sigma^2)}$$

$$C_{0c}/\lambda = C_{0s}/\sigma = \frac{\lambda}{2(\lambda^2 + \sigma^2)^2} \frac{\cos \sigma}{\cosh \lambda}$$

and

$$C_{2c}/(3\lambda) = C_{2s}/\sigma = \frac{\lambda}{2(\lambda^2 + \sigma^2)(9\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda}$$

so that we finally find

$$P = \frac{-1}{8(\lambda^2 + \sigma^2)} (\cos 2\lambda x + \cos 2\sigma y + \cos 2\sigma y \cos 2\lambda x) \\ + \frac{\lambda}{2(\lambda^2 + \sigma^2)^2} \frac{\cos \sigma}{\cosh \lambda} (\lambda \cos \sigma y \cosh \lambda y + \sigma \sin \sigma y \sinh \lambda y) \\ + \frac{\lambda}{2(\lambda^2 + \sigma^2)(9\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} (3\lambda \cos \sigma y \cosh \lambda y + \sigma \sin \sigma y \sinh \lambda y) \cos 2\lambda x \\ + C_1 \sin \lambda x \sinh \lambda y + C_2 \cos 2\lambda x \cosh 2\lambda y$$

The homogeneous solution that needs to be added to satisfy the boundary conditions is of the given form since only the 0 and $\cos 2\lambda x$ modes are present in the forcing, while the $\sin \lambda x$ mode is driven by the boundary conditions.

Now compute ∇P :

$$\partial_x P = \frac{\lambda}{4(\lambda^2 + \sigma^2)} \sin 2\lambda x (1 + \cos 2\sigma y) \\ - \frac{\lambda^2}{(\lambda^2 + \sigma^2)(9\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} (3\lambda \cos \sigma y \cosh \lambda y + \sigma \sin \sigma y \sinh \lambda y) \sin 2\lambda x \\ + \lambda C_1 \cos \lambda x \sinh \lambda y - 2\lambda C_2 \sin 2\lambda x \cosh 2\lambda y$$

$$\partial_y P = \frac{\sigma}{4(\lambda^2 + \sigma^2)} \sin 2\sigma y (1 + \cos 2\lambda x) \\ + \frac{\lambda}{2(\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} \cos \sigma y \sinh \lambda y \\ + \frac{\lambda}{2(\lambda^2 + \sigma^2)(9\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} ((3\lambda^2 + \sigma^2) \cos \sigma y \sinh \lambda y - 2\lambda \sigma \sin \sigma y \cosh \lambda y) \cos 2\lambda x \\ + \lambda C_1 \sin \lambda x \cosh \lambda y + 2\lambda C_2 \cos 2\lambda x \sinh 2\lambda y$$

3.4.1 P satisfies all boundary conditions!

Now compute ∇P at the boundary and compare to $-\nu \nabla \omega_1 \times \hat{\mathbf{e}}_z$:

$$\begin{aligned}\partial_x P &= \frac{\lambda}{4(\lambda^2 + \sigma^2)} \sin 2\lambda x (1 + \cos 2\sigma) \\ &- \frac{2\lambda^2}{(\lambda^2 + \sigma^2)(9\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} (3\lambda \cos \sigma \cosh \lambda + \sigma \sin \sigma \sinh \lambda) \sin 2\lambda x \\ &+ \lambda C_1 \cos \lambda x \sinh \lambda - 2\lambda C_2 \sin 2\lambda x \cosh 2\lambda\end{aligned}$$

and

$$-\nu \partial_y \omega_1 = \nu \sigma \cos \lambda x \sin \sigma + \frac{1}{2} C \sin 2\lambda x + \mathcal{O}(\epsilon^2)$$

$$\begin{aligned}\partial_y P &= \frac{\sigma}{4(\lambda^2 + \sigma^2)} \sin 2\sigma (1 + \cos 2\lambda x) \\ &+ \frac{\lambda}{\lambda^2 + \sigma^2} \frac{\cos \sigma}{\cosh \lambda} \cos \sigma \sinh \lambda \\ &+ \frac{\lambda}{(\lambda^2 + \sigma^2)(9\lambda^2 + \sigma^2)} \frac{\cos \sigma}{\cosh \lambda} ((3\lambda^2 + \sigma^2) \cos \sigma \sinh \lambda - 2\lambda \sigma \sin \sigma \cosh \lambda) \cos 2\lambda x \\ &+ \lambda C_1 \sin \lambda x \cosh \lambda + 2\lambda C_2 \cos 2\lambda x \sinh 2\lambda\end{aligned}$$

$$\nu \partial_x \omega_1 = -\nu \lambda \sin \lambda x \cos \sigma - \lambda \epsilon C \cos 2\lambda x + \mathcal{O}(\epsilon^2)$$

We find

$$C_1 = \nu \frac{\sigma \sin \sigma}{\lambda \sinh \lambda} = -\nu \frac{\cos \sigma}{\cosh \lambda}$$

for the $\sin \lambda x$ mode, by the definition of σ , while for C_2 the two components give the same value:

$$C_2 = \frac{\lambda^2}{2(9\lambda^2 + \sigma^2)(\lambda^2 + \sigma^2)} \frac{\cos^2 \sigma}{\cosh^2 \lambda}$$

4 Appendix A

4.1 Dirichlet-to-Neumann Solvability for the periodic channel.

After performing a Fourier expansion (with Fourier index n) in the x -direction of a periodic channel of length L_x with no-slip walls located at $y = \pm 1$, the Poisson equation reads

$$\frac{d^2 \psi_n}{dy^2} - \left(\frac{2\pi n}{L_x}\right)^2 \psi_n = -\omega_n(y), \quad (27)$$

with the Dirichlet boundary conditions

$$\psi_n(-1) = \psi_n(1) = 0, \quad (28)$$

and the Neumann boundary conditions

$$\psi'_n(-1) = \psi'_n(1) = 0. \quad (29)$$

We also define

$$\lambda_n = \frac{2\pi n}{L_x}. \quad (30)$$

From now on, we will assume a fixed value of $n \neq 0$ and drop the subindex n .

Introducing the Dirichlet Green's function $g(y|s)$, defined by

$$g_{yy} - \lambda^2 g = \delta(y - s), \quad (31)$$

with the boundary conditions

$$g(-1) = g(1) = 0, \quad (32)$$

we can formally write the solution to the Dirichlet Poisson equation as

$$\psi_{\text{Dir}}(y) = - \int_{-1}^1 \omega(s) g(y|s) ds. \quad (33)$$

The solvability constraint then arises by requiring that the function ψ not only satisfies (33) but also the Neumann boundary conditions, which means that we must impose the integral constraint

$$\int_{-1}^1 \omega(s) \left(\frac{\partial g(y|s)}{\partial y} \right)_{y=\pm 1} ds = 0. \quad (34)$$

In the coming sections we will describe how eq. (34) can be computed and expressed in a closed form by various methods.

4.1.1 Calculation of $g(y|s)$

We can immediately split the Green's function into the two parts

$$g(y|s) = \begin{cases} g^-(y) = A^- e^{\lambda y} + B^- e^{-\lambda y} & \text{for } y < s \\ g^+(y) = A^+ e^{\lambda y} + B^+ e^{-\lambda y} & \text{for } y > s. \end{cases} \quad (35)$$

Employing the outer boundary conditions

$$\begin{aligned} g^-(-1) &= 0 \\ g^+(1) &= 0, \end{aligned} \quad (36)$$

we find

$$\begin{aligned} g^-(y) &= A \sinh(\lambda(y + 1)) \\ g^+(y) &= B \sinh(\lambda(y - 1)). \end{aligned} \quad (37)$$

At $y = s$ we must require that $g(y)$ is continuous and that the jump in $g_y(y)$ is consistent with eq. (31), so

$$\begin{aligned} g^-(s) &= g^+(s) \\ g_y^+(s) - g_y^-(s) &= 1, \end{aligned} \quad (38)$$

Imposing these two inner boundary conditions, we can determine the constants A and B in eq. (37), which gives

$$g(y|s) = \begin{cases} \frac{\sinh[\lambda(s-1)] \sinh[\lambda(y+1)]}{\sinh[\lambda(s+1)] \sinh[\lambda(y-1)]} / [\lambda \sinh(2\lambda)] & \text{for } -1 \leq y \leq s \\ \frac{\sinh[\lambda(s+1)] \sinh[\lambda(y-1)]}{\sinh[\lambda(s-1)] \sinh[\lambda(y+1)]} / [\lambda \sinh(2\lambda)] & \text{for } s \leq y \leq 1. \end{cases} \quad (39)$$

From this expression, we can easily find

$$\frac{\partial g(y|s)}{\partial y} = \begin{cases} \frac{\sinh[\lambda(s-1)] \cosh[\lambda(y+1)]}{\sinh[\lambda(s+1)] \cosh[\lambda(y-1)]} / \sinh(2\lambda) & \text{for } -1 \leq y \leq s \\ \frac{\sinh[\lambda(s+1)] \cosh[\lambda(y-1)]}{\sinh[\lambda(s-1)] \cosh[\lambda(y+1)]} / \sinh(2\lambda) & \text{for } s \leq y \leq 1. \end{cases} \quad (40)$$

Inserting this expression into (34) and dropping the constant factors we find the Green's function solvability constraints

$$\int_{-1}^1 \omega(s) \sinh(\lambda(s \pm 1)) ds = 0. \quad (41)$$

Since

$$\sinh(\lambda(s \pm 1)) = \sinh(\lambda s) \cosh(\lambda) \pm \cosh(\lambda s) \sinh(\lambda), \quad (42)$$

we can rewrite eq. (41) as

$$\int_{-1}^1 \omega(s) \sinh(\lambda s) ds \pm \tanh(\lambda) \int_{-1}^1 \omega(s) \cosh(\lambda s) ds = 0. \quad (43)$$

The function $\omega(s)$ will eventually be expanded in terms of Chebyshev polynomials

$$\omega(s) \approx \sum_{m=0}^M \omega_m T_m(s), \quad (44)$$

so we will formulate the solvability constraints (43) in terms of weighted sums of the Chebyshev coefficients

$$\sum_{m=0}^M \omega_m b_m^{\pm} = 0. \quad (45)$$

These conditions are equivalent to requiring that the vector with elements ω_m is orthogonal to both of the vectors b_m^+ and b_m^- .

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