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Introduction

In this dissertation we study the Hamiltonian character of certain singular solutions of the Euler equations. In particular, we approximate the well-known Hamiltonian dynamical system of N point vortices with a family of smooth flows indexed by ϵ, whose energies converge as $\epsilon \to 0$ to within a constant from the sum of the Hamiltonian of the system of N point vortices and a function independent of the locations of the vortices. Thus we solve the problem of the derivation of the energy of the point vortices from the kinetic energy of shrinking vorticity patches using a combination of asymptotic and geometrical methods.

In the first chapter, using no geometry, we establish the equivalence of the Euler equations in their velocity form:

$$[EU] \begin{cases} \vec{\nabla} \cdot \vec{v} = 0, \\ \vec{v}_t + (\vec{v} \cdot \vec{\nabla})\vec{v} = \vec{\nabla} P, \\ \vec{v} \parallel \partial\Omega, \end{cases}$$

or their vorticity form:

$$[VORT] \begin{cases} \vec{\omega} = \vec{\nabla} \times \vec{v}, \\ \vec{\omega}_t = \vec{\nabla} \times (\vec{v} \times \vec{\omega}), \\ \vec{v} \parallel \partial\Omega. \end{cases}$$

with the Hamiltonian system

$$\omega_t = [\omega, H],$$

with a properly defined bracket operation. The approach is classical, the smoothness of the solutions we consider is not addressed and the Hamiltonian setting we are in is not clear.

In the second chapter, we introduce a more powerful and precise description of [EU]. The space of solutions we are dealing with is defined and it turns out that its geometrical structure provides an enviroment for expressing the canonical Hamiltonian character of [EU]. We show that [EU] are equivalent to a Hamiltonian evolution on the Poisson manifold $\mathit{\Delta}^*_{vol}$, the dual of the Lie algebra of $\mathcal{D}_{vol}$, the group of diffeomorphisms of the fluid region onto itself. And we compute the Kirillov symplectic form on an orbit of the coadjoint action of $\mathcal{D}_{vol}$ on $\mathit{\Delta}^*_{vol}$.

In the third chapter, we present the problem we solve. We study the Hamiltonian system of N point vortices of Kirchoff:

$$[KIR] \quad \begin{cases} \dfrac{dX_m}{dt} = -\dfrac{1}{2\pi} \displaystyle\sum_{\substack{j=1 \\ j\neq m}}^{N} \Gamma_j \frac{Y_m - Y_j}{(X_m - X_j)^2 + (Y_m - Y_j)^2}, \\ \dfrac{dY_m}{dt} = \dfrac{1}{2\pi} \displaystyle\sum_{\substack{j=1 \\ j\neq m}}^{N} \Gamma_j \frac{X_m - X_j}{(X_m - X_j)^2 + (Y_m - Y_j)^2}, \end{cases}$$

where (X_m, Y_m) is the position of the mth vortex and Γ_m is its intensity, and $1 \leq m \leq N$. We describe several integrals of the motion and compute the values of the bracket associated with the symplectic structure in $\mathbf{R}^{2N}$ on all pairs of them, demonstrating the integrability of the motion of 1,2,3 point vortices and 4 point vortices with zero total circulation.

We observe that the energy of a vorticity distribution that approximates the singular one corresponding to point vortices blows up failing to converge to the Hamiltonian function of [KIR]. And we analyze a method of deriving [KIR] from [EU], although a basic assumption in this method is unjustified till the next chapter. This derivation works also for the other constants of the motion we describe.

In the fourth chapter, we give a solution to the problem of the energy. We prove that a family of solutions of [EU] indexed by ϵ, whose vorticities converge initially as

$\epsilon \to 0$ to a linear combination of δ functions, will keep that property for an interval of time whose length we estimate. And we show the convergence of the symplectic structure of the coadjoint orbits where the vortex patch approximation lives to a symplectic structure that is essentially the same one in [KIR]. In this way we present a complete derivation of [KIR] from [EU].

Chapter 1

The Euler equations

In this chapter we take a classical approach to the Euler equations and show a way of understanding their Hamiltonian character. In this approach velocity and vorticity are both vector fields but we allow very different types of singularities in them.

1.1 The classical approach.

We begin with Euler equations for the velocity field $\vec{v}$ of an ideal incompressible fluid in a region $\Omega \subseteq \mathbf{R}^3$:

$$[EU] \begin{cases} \vec{\nabla} \cdot \vec{v} = 0, \\ \vec{v}_t + (\vec{v} \cdot \vec{\nabla})\vec{v} = \vec{\nabla} P, \\ \vec{v} \parallel \partial\Omega. \end{cases} \tag{1.1}$$

The incompressibility condition

$$\vec{\nabla} \cdot \vec{v} = 0 \tag{1.2}$$

implies that there exists a vector field $\vec{\psi}$ satisfying

$$\vec{v} = \vec{\nabla} \times \vec{\psi} \tag{1.3}$$

and such that

$$\vec{\nabla} \cdot \vec{\psi} = 0. \tag{1.4}$$

We call $\vec{\psi}$ the stream vector. It follows that the vorticity

$$\vec{\omega} \stackrel{\text{def}}{=} \vec{\nabla} \times \vec{v} \tag{1.5}$$

is such that

$$\vec{\omega} = -\nabla^2 \vec{\psi} \tag{1.6}$$

since

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\psi}) = -\nabla^2 \vec{\psi} + \vec{\nabla}(\vec{\nabla} \cdot \vec{\psi}). \tag{1.7}$$

Using the vector identity

$$\frac{1}{2}\vec{\nabla} u^2 = \vec{u} \times (\vec{\nabla} \times \vec{u}) + (\vec{u} \cdot \vec{\nabla})\vec{u} \tag{1.8}$$

we can rewrite the second equation in $[EU]$ as

$$\vec{v}_t + \vec{\omega} \times \vec{v} = \vec{\nabla}(P - \frac{1}{2}v^2). \tag{1.9}$$

And curling we obtain

$$\vec{\omega}_t = \vec{\nabla} \times (\vec{v} \times \vec{\omega}). \tag{1.10}$$

Therefore the vorticity fields of solutions of $[EU]$ satisfy the following equations on Ω:

$$[VORT] \begin{cases} \vec{\omega} = \vec{\nabla} \times \vec{v}, \\ \vec{\omega}_t = \vec{\nabla} \times (\vec{v} \times \vec{\omega}), \\ \vec{v} \parallel \partial\Omega. \end{cases} \tag{1.11}$$

On the other hand, solving a Poisson equation, we can obtain the stream vector $\vec{\psi}$ from the vorticity $\vec{\omega}$, namely

$$\vec{\psi}(\vec{x}) = \frac{1}{4\pi} \int_\Omega \|\vec{x} - \vec{y}\|^{-1} \vec{\omega}(\vec{y})\, dV(\vec{y}), \tag{1.12}$$

assuming that the vorticity has no normal component at $\partial\Omega$. And curling we get

$$\vec{v}(\vec{x}) = -\frac{1}{4\pi} \int_\Omega \|\vec{x} - \vec{y}\|^{-3} (\vec{x} - \vec{y}) \times \vec{\omega}(\vec{y})\, dV(\vec{y}). \tag{1.13}$$

In order to explain in what sense we say that the Euler equations are Hamiltonian we define the point-value functionals $A^i_{\vec{x}}$, for $1 \leq i \leq 3$, by the formula

$$A^i_{\vec{x}}[\vec{u}\,] = \int_\Omega u_i(\vec{y})\,\delta(\vec{y} - \vec{x})\,dV(\vec{y}). \tag{1.14}$$

So $A^i_{\vec{x}}$ acting on a field $\vec{u}$ produces the value of the ith component of the field at the point $\vec{x}$, i.e.

$$A^i_{\vec{x}}[\vec{u}\,] = u_i(\vec{x}). \tag{1.15}$$

Since the functionals $A^i_{\vec{x}}$, $1 \leq i \leq 3$, are linear in $\vec{u}$ we have that

$$\frac{\delta A^i_{\vec{x}}}{\delta\,\vec{u}}(\vec{y}) = \delta(\vec{y} - \vec{x})\delta_{ij}\,\vec{e}_j. \tag{1.16}$$

Given two functionals F, G we define another functional $\{F, G\}$ by

$$\{F, G\}[\vec{u}\,] = \int_\Omega \vec{u}(\vec{x}) \cdot [(\vec{\nabla} \times \frac{\delta F}{\delta \vec{u}}) \times (\vec{\nabla} \times \frac{\delta G}{\delta \vec{u}})](\vec{x})\,dV(\vec{x}). \tag{1.17}$$

This definition of the bracket was implicit in Arnold [**?**] and explicitly given in Kuznetsov and Mikhailov [**?**]. The kinetic energy of the fluid is a functional of the velocity that can be expressed as a functional of the vorticity as follows. We have that

$$\frac{1}{2}\int_\Omega v^2(\vec{x})\,dV(\vec{x}) = \frac{1}{2}\int_\Omega \vec{\omega}(\vec{x}) \cdot \vec{\psi}(\vec{x})\,dV(\vec{x}), \tag{1.18}$$

because

$$v^2 = \vec{v} \cdot (\vec{\nabla} \times \vec{\psi}) = -\vec{\nabla} \cdot (\vec{v} \times \vec{\psi}) + \vec{\psi} \cdot (\vec{\nabla} \times \vec{v}) \tag{1.19}$$

and the integral of the first summand gives a surface integral over $\partial\Omega$ that is assumed to be zero. Thus expressing the stream vector $\vec{\psi}$ in terms of the vorticity $\vec{\omega}$ we obtain

$$H[\vec{\omega}\,] = \frac{1}{8\pi}\int_\Omega \int_\Omega \vec{\omega}(\vec{x}) \cdot \vec{\omega}(\vec{y})\,\|\vec{x} - \vec{y}\|^{-1}\,dV(\vec{x})\,dV(\vec{y}). \tag{1.20}$$

Next we compute the variational derivative of H.

Proposition 1.1.1

$$\frac{\delta H}{\delta \vec{\omega}} = \vec{\psi} \tag{1.21}$$

Proof:

$$\langle \frac{\delta H}{\delta \vec{\omega}}, \vec{\gamma} \rangle = \frac{d}{d\epsilon} H[\vec{\omega} + \epsilon \vec{\gamma}]\Big|_{\epsilon=0} \quad (1.22)$$

$$= \frac{1}{4\pi} \int_\Omega \vec{\gamma}(\vec{x}) \cdot (\int_\Omega \vec{\omega}(\vec{y}) \, \|\vec{x} - \vec{y}\|^{-1} \, dV(\vec{y})) dV(\vec{x}) \quad (1.23)$$

$$= \langle \vec{\psi}, \vec{\gamma} \rangle. \quad (1.24)$$

Q.E.D.

Now we are able to demonstrate the Hamiltonian character of the Euler equations.

Theorem 1.1.1

The equation

$$\vec{\omega}_t = \vec{\nabla} \times (\vec{v} \times \vec{\omega}) \quad (1.25)$$

(i.e.

$$\frac{\partial \vec{\omega}}{\partial t}(\vec{x}, t_0) = \vec{\nabla} \times (\vec{v}(\vec{x}, t_0) \times \vec{\omega}(\vec{x}, t_0)) \quad (1.26)$$

for all $\vec{x}$ and all t_0) is equivalent to the Hamiltonian equations

$$\frac{d}{dt} A^i_{\vec{x}} \, [\vec{\omega}^t]\Big|_{t=t_0} = \{A^i_{\vec{x}} \, , H\}[\vec{\omega}^{t_0}], \quad (1.27)$$

for $1 \leq i \leq 3$, and for all $\vec{x}$ and all t_0 , where $\vec{\omega}^t(\vec{x}) \stackrel{\text{def}}{=} \vec{\omega}(\vec{x}, t)$.

Proof:

$$\frac{d}{dt} A^i_{\vec{x}} \, [\vec{\omega}^t]\Big|_{t=t_0} = \lim_{h \to 0} \frac{1}{h} \int_\Omega (\omega_i(\vec{y}, t_0 + h) - \omega_i(\vec{y}, t_0)) \, \delta(\vec{y} - \vec{x}) \, dV(\vec{y}) = \frac{\partial \omega_i}{\partial t}(\vec{x}, t_0). \quad (1.28)$$

And we have that

$$(\vec{\nabla} \times \frac{\delta A^i_{\vec{x}}}{\delta \vec{\omega}})(\vec{y}) = \vec{\nabla} \times \delta(\vec{y} - \vec{x}) \vec{e}_i, \quad (1.29)$$

$$(\vec{\nabla} \times \frac{\delta H}{\delta \vec{\omega}})(\vec{y}) = (\vec{\nabla} \times \vec{\psi})(\vec{y}) = \vec{v}(\vec{y}). \quad (1.30)$$

Therefore

$$
\begin{aligned}
\{A^i_{\vec{x}}, H\}[\vec{\omega}] &= \int_\Omega \vec{\omega}(\vec{y}) \cdot [(\vec{\nabla} \times \frac{\delta A^i_{\vec{x}}}{\delta \vec{\omega}}) \times (\vec{\nabla} \times \frac{\delta H}{\delta \vec{\omega}})](\vec{y})\, dV(\vec{y}) \qquad (1.31)\\
&= \int_\Omega \vec{\omega}(\vec{y}) \cdot [(\vec{\nabla} \times \delta(\vec{y} - \vec{x})\vec{e}_i) \times \vec{v}(\vec{y})]\, dV(\vec{y}) \qquad (1.32)\\
&= \int_\Omega \vec{\nabla} \times \delta(\vec{y} - \vec{x})\vec{e}_i \cdot [\vec{v}(\vec{y}) \times \vec{\omega}(\vec{y})]\, dV(\vec{y}) \qquad (1.33)\\
&= \int_\Omega \vec{\nabla} \cdot [\delta(\vec{y} - \vec{x})\vec{e}_i \times (\vec{v}(\vec{y}) \times \vec{\omega}(\vec{y}))]\, dV(\vec{y})\\
&\quad + \int_\Omega \delta(\vec{y} - \vec{x})\vec{e}_i \cdot \vec{\nabla} \times (\vec{v}(\vec{y}) \times \vec{\omega}(\vec{y}))\, dV(\vec{y}) \qquad (1.34)\\
&= \vec{e}_i \cdot \vec{\nabla} \times (\vec{v}(\vec{x}) \times \vec{\omega}(\vec{x})). \qquad (1.35)
\end{aligned}
$$

Q.E.D.

The ideas for this proof are in Morrison [**?**].

In this classic approach to the Euler equations it is not clear what type of singularities can be allowed in the vorticity field . And the Hamiltonian formalism presented above does not make much sense when the vorticity is singular. However, we intend to study how the symplectic structure of certain singular solutions relates with the Hamiltonian character of the Euler equations. In order to get a more geometrical language to deal with the Euler equations, in the following sections of this chapter we address the question of the spaces on which the solutions live, the concept of variational derivative and the exact definition of what we mean by the expression "Hamiltonian structure".

1.2 The spaces.

We begin by describing the appropiate configuration space for ideal incompressible fluid flow in a region Ω. Such configuration space is $\mathcal{D}_{vol}$, the group of volume preserving diffeomorphisms of Ω onto itself. A curve in $\mathcal{D}_{vol}$ through the identity

$$\{\eta(t) : t \in [0, T], \eta(0) = I_\Omega\} \tag{1.36}$$

describes a fluid motion, namely $\eta(t)(p)$ is the position at time t of that particle which at time zero was at $p \in \Omega$.

The phase space is the cotangent bundle $T^*\mathcal{D}_{vol}$ of $\mathcal{D}_{vol}$. Since the Euler flow is invariant under the action of $\mathcal{D}_{vol}$ on itself by right (or left) multiplication we can reduce the phase space to the cotangent space of $\mathcal{D}_{vol}$ at the identity, i.e. to the dual of the "Lie" algebra of $\mathcal{D}_{vol}$. There are two possible ways in which $\mathcal{D}_{vol}$ can be made into a Lie group. The first is the obvious one and it does not work: we consider the elements of $\mathcal{D}_{vol}$ to be C^∞ diffeomorphisms and make $\mathcal{D}_{vol}$ a Fréchet manifold, i.e. an infinite dimensional manifold modelled in locally convex, Hausdorff, complete vector spaces. But the differential calculus available in Fréchet spaces does not provide tools like the Implicit Function Theorem necessary to perform the geometrical constructions outlined by Arnold [**?**]. The second way to proceed is to consider the elements of $\mathcal{D}_{vol}$ to be diffeomorphisms of Sobolev or Hölder class. It turns out that if the Sobolev class $W^{s,p}$, or Hölder class $C^{k+\alpha}$, is high enough so that such diffeomorphisms are at least C^1, then they form a C^∞ Banach manifold and we have the usual existence and uniqueness theorems for solutions of differential equations. But even so only right translation is smooth, whereas left translation and taking inverses are continuous maps. Thus $W^{s,p}\mathcal{D}_{vol}$ (or $H^{k+\alpha}\mathcal{D}_{vol}$) is a topological group with a structure of a Banach manifold with respect to which right translation is smooth. We need to go a step further and consider the inverse limit as the differentiability class goes to ∞, obtaining an ILB-Lie group (inverse limit of Banach) structure on $\mathcal{D}_{vol}$.

The task of getting the right definition for $\mathcal{D}_{vol}$ was accomplished by Ebin and Marsden [**?**] in 1970 but the study of the groups of diffeomorphisms and their differentiable structures has continued to be an active area of research to the present

[?].

The Lie algebra $\mathit{\Delta}_{vol}$ of the Lie group $\mathcal{D}_{vol}$ consists of the divergence-free vector fields on Ω and its bracket turns out to be minus the usual bracket for vector fields, i.e.

$$[\vec{u}, \vec{v}] = \sum_{i,j=1}^{3} \{ \frac{\partial u_i}{\partial x_j} v_j - \frac{\partial v_i}{\partial x_j} u_j \}. \tag{1.37}$$

And the elements of its L^2-dual $\mathit{\Delta}^*_{vol}$ are the linear functionals

$$f \colon \mathit{\Delta}_{vol} \to \mathbf{R} \tag{1.38}$$

for which there exists $\vec{F} \in L^2(\Omega)$ such that

$$f(\vec{v}) = \int_\Omega \vec{v} \cdot \vec{F} \, dV. \tag{1.39}$$

The elements of the dual space corresponding to smooth elements of $L^2(\Omega)$ are in one-to-one correspondence with the classes of one-forms modulo exact one-forms on Ω. Note that by this passage to the dual generalized functions enter into the picture naturally.

By the Hodge Decomposition Theorem the quotient module

$$\mathit{\Lambda}^1(\Omega)/E^1(\Omega) \tag{1.40}$$

is isomorphic with the divergence-free vector fields on Ω. We describe this isomorphism with the following notation:

$$\begin{array}{ccc} \mathit{\Lambda}^1(\Omega)/E^1(\Omega) & \stackrel{\approx}{\longrightarrow} & \mathit{\Delta}_{vol} \\ [\alpha] & \longmapsto & \vec{u_\alpha} \end{array} \tag{1.41}$$

where $[\alpha]$ stands for the class of the element $\alpha \in \mathit{\Lambda}^1(\Omega)$. Note that if we define $\Lambda^1_{\vec{v}}$ as the one-form such that

$$\Lambda^1_{\vec{v}}(\vec{w}) \stackrel{\text{def}}{=} \vec{v} \cdot \vec{w}, \tag{1.42}$$

we have a map

$$\begin{array}{ccc} \mathit{\Delta}_{vol} & \xrightarrow{\Lambda^1} & \mathit{\Lambda}^1(\Omega) \\ \vec{v} & \longmapsto & \Lambda^1_{\vec{v}} \end{array} \tag{1.43}$$

such that

$$[\Lambda^1_{\vec{u_\alpha}}] = [\alpha]. \tag{1.44}$$

If Ω is simply-connected, to each element of the smooth part of $\mathit{\Delta}^*_{vol}$ there corresponds a unique closed two-form on Ω and conversely:

$$\begin{array}{ccc} \mathit{\Lambda}^1(\Omega)/E^1(\Omega) & \longrightarrow & E^2(\Omega) \\ [\alpha] & \longmapsto & d\alpha \end{array} \tag{1.45}$$

In this language a careful computational distinction is made between the velocity and the vorticity: the distinction between one-forms modulo exact differentials and exact two forms.

1.3 The variational derivative.

Given a smooth function

$$F \colon \mathit{\Delta}^*_{vol} \to \mathbf{R} \tag{1.46}$$

we define

$$\frac{\delta F}{\delta[\alpha]} \in \mathit{\Delta}_{vol} \tag{1.47}$$

by the equality

$$DF([\alpha]) \cdot [\sigma] = \int_\Omega (\frac{\delta F}{\delta[\alpha]}, [\sigma]) dV \tag{1.48}$$

where (,) is the pairing between Δ_{vol} and its dual. We denote the divergence-free velocity field $\dfrac{\delta F}{\delta[\alpha]}$ by $\vec{v_F}$ and the corresponding one-form by $\Lambda^1_{\vec{v_F}}$ where $[\alpha]$ is clear from the context.

So the variational derivative of a functional with respect to a given velocity field is the Riesz representative of the linear form $DF([\alpha])$.

Any smooth function

$$F\colon \mathit{\Lambda}^1(\Omega)/E^1(\Omega) \longrightarrow \mathbf{R} \tag{1.49}$$

induces a smooth function

$$\tilde{F}\colon E^2(\Omega) \longrightarrow \mathbf{R} \tag{1.50}$$

defined by

$$\tilde{F}(\omega) = F([\delta\Delta^{-1}\omega]) = F([\alpha]), \tag{1.51}$$

where $\omega = d\alpha$ and Δ stands for the Laplace-Beltrami operator. The derivative of $\tilde{F}$ at ω corresponds to an element

$$\frac{\delta\tilde{F}}{\delta\omega} \in E^2(\Omega) \tag{1.52}$$

defined by

$$\langle\frac{\delta\tilde{F}}{\delta\omega}, \eta\rangle_2 = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\{\tilde{F}(\omega+\epsilon\eta) - \tilde{F}(\omega)\}. \tag{1.53}$$

And since we have that

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{1}{\epsilon}\{\tilde{F}(\omega+\epsilon\eta) - \tilde{F}(\omega)\} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\{DF([\delta\Delta^{-1}\omega])\cdot(\epsilon[\delta\Delta^{-1}\eta]) + o(\epsilon)\} && (1.54)\\
&= DF([\delta\Delta^{-1}\omega])\cdot[\delta\Delta^{-1}\eta] && (1.55)\\
&= \int_\Omega (\frac{\delta F}{\delta[\alpha]}, [\delta\Delta^{-1}\eta])dV && (1.56)\\
&= \langle\Lambda^1_{\vec{v_F}}, \delta\Delta^{-1}\eta\rangle_1 && (1.57)\\
&= \langle\Delta^{-1}d\Lambda^1_{\vec{v_F}}, \eta\rangle_2 && (1.58)
\end{aligned}$$

we see that $\dfrac{\delta\tilde{F}}{\delta\omega}$ is the stream two-form corresponding to the velocity $\vec{v_F}$ and we denote it by $\Lambda^2_{\vec{\psi_F}}$ when ω is clear from the context.

So we see that there are two different variational derivatives associated with a functional on $\mathit{\Delta}^*_{vol}$: one of them is in $\mathit{\Delta}_{vol}$ and the other one is a closed 2-form, i.e. one is a velocity and the other one is a stream function.

1.4 The Lie-Poisson bracket.

In Δ^*_{vol}, as in the dual of any Lie algebra, we have a natural Poisson structure, i.e. we have a bracket

$$\{\ ,\ \}: C^\infty(\Delta^*_{vol}) \times C^\infty(\Delta^*_{vol}) \to C^\infty(\Delta^*_{vol}) \tag{1.59}$$

such that:

(i) $\{\ ,\ \}$ is bilinear and skew-symmetric,

(ii) $\{F,\{G,H\}\} + \{H,\{F,G\}\} + \{G,\{H,F\}\} = 0$ (Jacobi's identity),

(iii) $\{FG,H\} = F\{G,H\} + G\{F,H\}$ (Leibniz' identity).

To define $\{\ ,\ \}$ in $C^\infty(\Delta^*_{vol})$ we use as in the classical approach the variational derivative. We define the bracket by

$$\{F,G\}([\alpha]) = \int_\Omega ([\frac{\delta F}{\delta[\alpha]}, \frac{\delta G}{\delta[\alpha]}], [\alpha]) dV \tag{1.60}$$

where the bracket between the variational derivatives is the one of the Lie algebra Δ_{vol}.

The bracket $\{\ ,\ \}$ coincides with the classic one defined above:

$$\begin{aligned}
\{F,G\}([\alpha]) &= = \int_\Omega ([\frac{\delta F}{\delta[\alpha]}, \frac{\delta G}{\delta[\alpha]}], [\alpha]) dV && (1.61)\\
&= \int_\Omega (L_{\frac{\delta G}{\delta[\alpha]}} \frac{\delta F}{\delta[\alpha]}, [\alpha]) dV && (1.62)\\
&= -\int_\Omega (\frac{\delta F}{\delta[\alpha]}, L_{\frac{\delta G}{\delta[\alpha]}} [\alpha]) dV && (1.63)\\
&= -\langle \Lambda^1_{\vec{v_F}}, L_{\vec{v_G}} \Lambda^1_{\vec{u_\alpha}} \rangle_1 && (1.64)\\
&= \langle L_{\vec{v_G}} \Lambda^1_{\vec{v_F}}, \Lambda^1_{\vec{u_\alpha}} \rangle_1 && (1.65)\\
&= \langle L_{\vec{v_G}} \delta \Lambda^2_{\vec{\psi_F}}, \Lambda^1_{\vec{u_\alpha}} \rangle_1 && (1.66)\\
&= \langle \delta L_{\vec{v_G}} \Lambda^2_{\vec{\psi_F}}, \Lambda^1_{\vec{u_\alpha}} \rangle_1 && (1.67)
\end{aligned}$$

$$= \langle L_{\vec{v_G}} \Lambda^2_{\vec{\psi_F}}, d\Lambda^1_{\vec{u_\alpha}} \rangle_2 \tag{1.68}$$

$$= \int_\Omega d\Lambda^1_{\vec{u_\alpha}} \wedge \star L_{\vec{v_G}} \Lambda^2_{\vec{\psi_F}} \tag{1.69}$$

$$= \int_\Omega d\Lambda^1_{\vec{u_\alpha}} \wedge L_{\vec{v_G}} \star \Lambda^2_{\vec{\psi_F}} \tag{1.70}$$

$$= \int_\Omega d\Lambda^1_{\vec{u_\alpha}} \wedge L_{\vec{v_G}} \Lambda^1_{\vec{\psi_F}} \tag{1.71}$$

$$= \int_\Omega d\Lambda^1_{\vec{u_\alpha}} \wedge i(\vec{v_G}) d\Lambda^1_{\vec{\psi_F}} + \int_\Omega d\Lambda^1_{\vec{u_\alpha}} \wedge d(i(\vec{v_G})\Lambda^1_{\vec{\psi_F}}) \tag{1.72}$$

$$= \int_\Omega d\Lambda^1_{\vec{u_\alpha}} \wedge i(\vec{v_G}) \Lambda^2_{\vec{v_F}} \tag{1.73}$$

$$= \int_\Omega d\Lambda^1_{\vec{u_\alpha}} \wedge \Lambda^1_{\vec{v_F} \times \vec{v_G}} \tag{1.74}$$

$$= \int_\Omega \vec{\omega} \cdot (\vec{\nabla} \times \frac{\delta F}{\delta \vec{\omega}}) \times (\vec{\nabla} \times \frac{\delta G}{\delta \vec{\omega}}) dV \tag{1.75}$$

where $\vec{\omega}$ is the vector associated with the two-form $d\alpha$.

We have Jacobi's identity.

Proposition 1.4.1

For any F, G, H in $C^\infty(\mathit{\Delta}^*_{vol})$ we have that

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \tag{1.76}$$

Proof:

$$\{F, \{G, H\}\}([\alpha]) = \int_\Omega ([\frac{\delta F}{\delta[\alpha]}, \frac{\delta\{G, H\}}{\delta[\alpha]}], [\alpha]) dV \tag{1.77}$$

and $\dfrac{\delta\{G, H\}}{\delta[\alpha]}$ is such that

$$\int_\Omega (\frac{\delta\{G, H\}}{\delta[\alpha]}, [\beta]) dV = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \{\{G, H\}([\alpha] + \epsilon[\beta]) - \{G, H\}([\alpha])\}. \tag{1.78}$$

And we have that

$$\{G, H\}([\alpha] + \epsilon[\beta]) = \int_\Omega ([\frac{\delta G}{\delta([\alpha] + \epsilon[\beta])}, \frac{\delta H}{\delta([\alpha] + \epsilon[\beta])}], [\alpha] + \epsilon[\beta]) dV. \tag{1.79}$$

To compute $\dfrac{\delta G}{\delta([\alpha] + \epsilon[\beta])}$ we use the definition, namely

$$DG([\alpha] + \epsilon[\beta]) \cdot [\gamma] = \int_\Omega (\frac{\delta G}{\delta([\alpha] + \epsilon[\beta])}, [\gamma]) dV. \tag{1.80}$$

So we have

$$
\begin{aligned}
DG([\alpha]+\epsilon[\beta])\cdot[\gamma\,] &= \lim_{\tau\to 0}\frac{1}{\tau}\{G([\alpha]+\epsilon[\beta]+\tau[\gamma\,])-G([\alpha]+\epsilon[\beta])\} && (1.81)\\
&= \lim_{\tau\to 0}\frac{1}{\tau}\{DG([\alpha]+\epsilon[\beta])\cdot\tau[\gamma\,]+o(\tau)\} && (1.82)\\
&= \lim_{\tau\to 0}\frac{1}{\tau}\{(DG([\alpha]+\epsilon[\beta])-DG([\alpha]))\cdot\tau[\gamma\,]\\
&\quad +DG([\alpha])\cdot\tau[\gamma\,]+o(\tau)\} && (1.83)\\
&= \lim_{\tau\to 0}\frac{1}{\tau}\{\tau\epsilon(D^2G([\alpha])\cdot[\beta])\cdot[\gamma\,]+o(\epsilon)\tau\\
&\quad +\tau DG([\alpha])\cdot[\gamma\,]+o(\tau)\} && (1.84)\\
&= \epsilon(D^2G([\alpha])\cdot[\beta])\cdot[\gamma\,]+DG([\alpha])\cdot[\gamma\,]+o(\epsilon). && (1.85)
\end{aligned}
$$

Therefore, if $\frac{\delta^2 G}{\delta[\alpha]^2}[\beta]$ stands for the element of Δ_{vol} that depends linearly on $[\beta]$ and such that

$$\int_\Omega\left(\frac{\delta^2 G}{\delta[\alpha]^2}[\beta],[\gamma\,]\right)dV=(D^2G([\alpha])\cdot[\beta])\cdot[\gamma\,], \qquad (1.86)$$

we have that

$$\frac{\delta G}{\delta([\alpha]+\epsilon[\beta])}=\epsilon\frac{\delta^2 G}{\delta[\alpha]^2}[\beta]+\frac{\delta G}{\delta[\alpha]}+o(\epsilon). \qquad (1.87)$$

Hence

$$
\begin{aligned}
\{G,H\}([\alpha]+\epsilon[\beta]) &= \epsilon^2\int_\Omega\left(\left[\frac{\delta^2 G}{\delta[\alpha]^2}[\beta],\frac{\delta^2 H}{\delta[\alpha]^2}[\beta]\right],[\alpha]+\epsilon[\beta]\right)dV\\
&\quad +\epsilon\int_\Omega\left(\left[\frac{\delta^2 G}{\delta[\alpha]^2}[\beta],\frac{\delta H}{\delta[\alpha]}\right],[\alpha]+\epsilon[\beta]\right)dV\\
&\quad +\epsilon\int_\Omega\left(\left[\frac{\delta G}{\delta[\alpha]},\frac{\delta^2 H}{\delta[\alpha]^2}[\beta]\right],[\alpha]+\epsilon[\beta]\right)dV\\
&\quad +\int_\Omega\left(\left[\frac{\delta G}{\delta[\alpha]},\frac{\delta H}{\delta[\alpha]}\right],[\alpha]+\epsilon[\beta]\right)dV+o(\epsilon). && (1.88)
\end{aligned}
$$

Integrating by parts we have that

$$\int_\Omega\left(\frac{\delta\{G,H\}}{\delta[\alpha]},[\beta]\right)dV = \int_\Omega\left(\left[\frac{\delta^2 G}{\delta[\alpha]^2}[\beta],\frac{\delta H}{\delta[\alpha]}\right],[\alpha]\right)dV$$

$$
\begin{aligned}
&+\int_\Omega([\frac{\delta G}{\delta[\alpha}, \frac{\delta^2 H}{\delta[\alpha]^2}[\beta]], [\alpha])dV \\
&+\int_\Omega([\frac{\delta G}{\delta[\alpha]}, \frac{\delta H}{\delta[\alpha]}], [\beta])dV \qquad (1.89)
\end{aligned}
$$

$$
\begin{aligned}
= &\int_\Omega(L_{\frac{\delta H}{\delta[\alpha]}}\frac{\delta^2 G}{\delta[\alpha]^2}[\beta], [\alpha])dV \\
&-\int_\Omega(L_{\frac{\delta G}{\delta[\alpha]}}\frac{\delta^2 H}{\delta[\alpha]^2}[\beta], [\alpha])dV \\
&+\int_\Omega([\frac{\delta G}{\delta[\alpha]}, \frac{\delta H}{\delta[\alpha]}], [\beta])dV \qquad (1.90)
\end{aligned}
$$

$$
\begin{aligned}
= &-\int_\Omega(\frac{\delta^2 G}{\delta[\alpha]^2}[\beta], L_{\frac{\delta H}{\delta[\alpha]}}[\alpha])dV \\
&+\int_\Omega(\frac{\delta^2 H}{\delta[\alpha]^2}[\beta], L_{\frac{\delta G}{\delta[\alpha]}}[\alpha])dV \\
&+\int_\Omega([\frac{\delta G}{\delta[\alpha]}, \frac{\delta H}{\delta[\alpha]}], [\beta])dV \qquad (1.91)
\end{aligned}
$$

$$
\begin{aligned}
= &-\int_\Omega(\frac{\delta^2 G}{\delta[\alpha]^2}(L_{\frac{\delta H}{\delta[\alpha]}}[\alpha]), [\beta])dV \\
&+\int_\Omega(\frac{\delta^2 H}{\delta[\alpha]^2}(L_{\frac{\delta G}{\delta[\alpha]}}[\alpha]), [\beta])dV \\
&+\int_\Omega([\frac{\delta G}{\delta[\alpha]}, \frac{\delta H}{\delta[\alpha]}], [\beta])dV. \qquad (1.92)
\end{aligned}
$$

Therefore

$$
\frac{\delta\{G,H\}}{\delta[\alpha]} = -\frac{\delta^2 G}{\delta[\alpha]^2}(L_{\frac{\delta H}{\delta[\alpha]}}[\alpha]) + \frac{\delta^2 H}{\delta[\alpha]^2}(L_{\frac{\delta G}{\delta[\alpha]}}[\alpha]) + [\frac{\delta G}{\delta[\alpha]}, \frac{\delta H}{\delta[\alpha]}]. \qquad (1.93)
$$

In consequence

$$
\begin{aligned}
\{F,\{G,H\}\}([\alpha]) &= \int_\Omega([\frac{\delta F}{\delta[\alpha]}, \frac{\delta\{G,H\}}{\delta[\alpha]}], [\alpha])dV \qquad (1.94) \\
&= -\int_\Omega([\frac{\delta F}{\delta[\alpha]}, \frac{\delta^2 G}{\delta[\alpha]^2}(L_{\frac{\delta H}{\delta[\alpha]}}[\alpha])], [\alpha])dV
\end{aligned}
$$

$$
\begin{aligned}
& + \int_\Omega ([\frac{\delta F}{\delta[\alpha]}, \frac{\delta^2 H}{\delta[\alpha]^2}(L_{\frac{\delta G}{\delta[\alpha]}}[\alpha])], [\alpha]) dV \\
& + \int_\Omega ([\frac{\delta F}{\delta[\alpha]}, [\frac{\delta G}{\delta[\alpha]}, \frac{\delta H}{\delta[\alpha]}], [\alpha]) dV \qquad (1.95) \\
= & - \int_\Omega (\frac{\delta^2 G}{\delta[\alpha]^2}(L_{\frac{\delta H}{\delta[\alpha]}}[\alpha]), L_{\frac{\delta F}{\delta[\alpha]}}[\alpha]) dV \\
& + \int_\Omega (\frac{\delta^2 H}{\delta[\alpha]^2}(L_{\frac{\delta G}{\delta[\alpha]}}[\alpha]), L_{\frac{\delta F}{\delta[\alpha]}}[\alpha]) dV \\
& + \int_\Omega ([\frac{\delta F}{\delta[\alpha]}, [\frac{\delta G}{\delta[\alpha]}, \frac{\delta H}{\delta[\alpha]}]], [\alpha]) dV \qquad (1.96) \\
= & -(D^2 G([\alpha]) \cdot L_{\frac{\delta H}{\delta[\alpha]}}[\alpha]) \cdot L_{\frac{\delta F}{\delta[\alpha]}}[\alpha] \\
& +(D^2 H([\alpha]) \cdot L_{\frac{\delta G}{\delta[\alpha]}}[\alpha]) \cdot L_{\frac{\delta F}{\delta[\alpha]}}[\alpha] \\
& + \int_\Omega ([\frac{\delta F}{\delta[\alpha]}, [\frac{\delta G}{\delta[\alpha]}, \frac{\delta H}{\delta[\alpha]}]], [\alpha]) dV. \qquad (1.97)
\end{aligned}
$$

Since the Lie bracket of Δ_{vol} satisfies Jacobi's identity and the second derivative of a smooth function at a point is a bilinear symmetric form we are done.

Q.E.D.

Chapter 2

The Hamiltonian structure of the Euler equations

We describe the Hamiltonian structure of the Euler equations via the Lie-Poisson bracket of $\mathit{\Delta}^*_{vol}$. In his seminal paper of 1966, Arnold [?] started a geometrical way of looking at the Euler equations. In 1970, Ebin and Marsden [?] described precisely the manifold where Arnold saw the geometry. And in 1983, Marsden and Weinstein [?] formalized the geometrical approach to the Hamiltonian structure using the theory of Poisson manifolds.

2.1 The kinetic energy.

We consider the kinetic energy as a function

$$H\colon \mathit{\Delta}^*_{vol} \longrightarrow \mathbf{R} \tag{2.1}$$

defined on $\mathit{\Lambda}^1(\Omega)/E^1(\Omega)$ by the formula

$$H([\alpha]) = \frac{1}{2}\langle \Delta^{-1} d\alpha, d\alpha\rangle_2. \tag{2.2}$$

This function agrees with the classic kinetic energy because

$$\begin{aligned} \frac{1}{2}\langle\Delta^{-1}d\alpha, d\alpha\rangle_2 &= \frac{1}{2}\langle\delta\Delta^{-1}d\alpha, \alpha\rangle_1 && (2.3)\\ &= \frac{1}{2}\langle\Lambda^1_{\vec{u}_\alpha}, \Lambda^1_{\vec{u}_\alpha}\rangle_1 && (2.4)\\ &= \frac{1}{2}\int_\Omega \Lambda^1_{\vec{u}_\alpha} \wedge \star\Lambda^1_{\vec{u}_\alpha} && (2.5)\\ &= \frac{1}{2}\int_\Omega \Lambda^1_{\vec{u}_\alpha} \wedge \Lambda^2_{\vec{u}_\alpha} && (2.6)\\ &= \frac{1}{2}\int_\Omega \vec{u}_\alpha \cdot \vec{u}_\alpha dV. && (2.7) \end{aligned}$$

We compute the variational derivative of H in the following way:

$$\begin{aligned} H([\alpha]+\epsilon[\beta]) - H([\alpha]) &= \frac{1}{2}\langle\Delta^{-1}d\alpha + \epsilon\Delta^{-1}d\beta, d\alpha + \epsilon d\beta\rangle_2 \\ &\quad -\frac{1}{2}\langle\Delta^{-1}d\alpha, d\alpha\rangle_2 && (2.8)\\ &= \epsilon\{\frac{1}{2}\langle\Delta^{-1}d\alpha, d\beta\rangle_2 + \frac{1}{2}\langle\Delta^{-1}d\beta, d\alpha\rangle_2\} + o(\epsilon). && (2.9) \end{aligned}$$

Hence

$$\begin{aligned} DH([\alpha])\cdot[\beta] &= \langle\Delta^{-1}d\alpha, d\beta\rangle_2 && (2.10)\\ &= \langle\delta\Delta^{-1}d\alpha, \beta\rangle_1 && (2.11)\\ &= \int_\Omega (\frac{\delta H}{\delta[\alpha]}, [\beta])dV && (2.12) \end{aligned}$$

so we see that

$$\delta\Delta^{-1}d\alpha = \Lambda^1_{\vec{v}_H}, \quad (2.13)$$

i.e. the variational derivative of the kinetic energy produces the one-form associated with the velocity field of the vorticity $d\alpha$.

The induced function

$$\begin{aligned} \tilde{H}\colon E^2(\Omega) &\longrightarrow \mathbf{R} \\ \omega &\longmapsto H([\delta\Delta^{-1}\omega)] \end{aligned} \quad (2.14)$$

has associated with its derivative at ω the element $\frac{\delta \tilde{H}}{\delta \omega} \in E^2(\Omega)$. And as we show in 1.3 such element is none but the stream two-form corresponding to the velocity created by the vorticity ω. This is Proposition 1.4.1 cast in our geometrical language.

2.2 The Hamiltonian structure.

We describe the Hamiltonian character of the Euler equations via the Lie-Poisson bracket in Δ^*_{vol}. We formulate Theorem 1.1.2 in the Poisson manifold Δ^*_{vol} and reduce its proof to integration by parts and the use of some vector identities.

If M is a symplectic manifold with symplectic form Υ we say that X, a vector field on M, is Hamiltonian iff there exists a smooth function

$$H\colon M \longrightarrow R \tag{2.15}$$

such that

$$i(X)\Upsilon = dH \tag{2.16}$$

So the Hamiltonian vector fields are the ones that make the following diagram commutative:

$$\begin{array}{ccc} Vec(M) & \stackrel{i(.)\Upsilon}{\longrightarrow} & \Lambda^1(M) \\ & \nwarrow & \uparrow_d \\ & & C^\infty(M) \end{array} \tag{2.17}$$

If $(P, \{\ ,\ \})$ is a Poisson manifold then we say that X, a vector field on P, is Hamiltonian iff there exists a smooth function

$$H\colon P \longrightarrow R \tag{2.18}$$

such that

$$X = \{\ , H\} \tag{2.19}$$

The Poisson structure on P induces a pairing between functions and certain vector fields; the Hamiltonian vector fields are precisely those that are paired with a function through the Lie-Poisson bracket.

$$\begin{array}{ccc} Vec(M) & & \\ & \nwarrow & \\ & & C^\infty(M) \end{array} \tag{2.20}$$

Theorem 2.1

The Euler equations are equivalent to the Hamiltonian system

$$F_t = \{F, H\} \tag{2.21}$$

where $F\colon \Delta^*_{vol} \to \mathbf{R}$ is any smooth function on Δ^*_{vol}.

Proof:

By the Chain Rule we have that

$$\frac{\partial F}{\partial t}([\alpha]) = DF([\alpha]) \cdot [\alpha]_t. \tag{2.22}$$

On the other hand,

$$\begin{aligned} \{F,H\}([\alpha]) &= \int_\Omega ([\frac{\delta F}{\delta[\alpha]}, \frac{\delta H}{\delta[\alpha]}], [\alpha]) dV && (2.23)\\ &= \int_\Omega (L_{\frac{\delta H}{\delta[\alpha]}} \frac{\delta F}{\delta[\alpha]}, [\alpha]) dV && (2.24)\\ &= -\int_\Omega (\frac{\delta F}{\delta[\alpha]}, L_{\frac{\delta H}{\delta[\alpha]}}[\alpha]) dV && (2.25)\\ &= -DF([\alpha]) \cdot L_{\frac{\delta H}{\delta[\alpha]}}[\alpha]. && (2.26) \end{aligned}$$

Hence (2.13) is equivalent to

$$DF([\alpha]) \cdot \{[\alpha]_t + L_{\frac{\delta H}{\delta[\alpha]}}[\alpha]\} = 0. \tag{2.27}$$

Since this is true for all F we have that

$$[\alpha]_t + L_{\frac{\delta H}{\delta[\alpha]}}[\alpha] = [0]. \tag{2.28}$$

But

$$[\alpha]_t + L_{\frac{\delta H}{\delta[\alpha]}}[\alpha] = [\Lambda^1_{\vec{u}_\alpha}]_t + [L_{\vec{u}_\alpha}\Lambda^1_{\vec{u}_\alpha}] \tag{2.29}$$

$$= [\Lambda^1_{\vec{u}_\alpha}]_t + [i(\vec{u}_\alpha)d\Lambda^1_{\vec{u}_\alpha}] + [d(i(\vec{u}_\alpha)\Lambda^1_{\vec{u}_\alpha})] \tag{2.30}$$

$$= [\Lambda^1_{\vec{u}_\alpha}]_t + [\Lambda^1_{(\vec{\nabla}\times\vec{u}_\alpha)\times\vec{u}_\alpha}] + [\Lambda^1_{\vec{\nabla}u^2_\alpha}]. \tag{2.31}$$

Therefore 2.28 is equivalent to

$$\vec{u}_{\alpha_t} + (\vec{\nabla}\times\vec{u}_\alpha)\times\vec{u}_\alpha + \vec{\nabla}u^2_\alpha = \vec{\nabla}P. \tag{2.32}$$

Q.E.D.

2.3 The coadjoint representation.

We consider the map

$$\begin{array}{ccc} \mathcal{D}_{vol} & \longrightarrow & \mathcal{D}_{vol} \\ \eta & \longmapsto & \eta\gamma\eta^{-1}. \end{array} \tag{2.33}$$

Its pull back at the identity $I_\Omega \in \mathcal{D}_{vol}$ is the map

$$\mathit{\Delta}^*_{vol} \stackrel{(R_{\gamma^{-1}}L_\gamma)^*_{I_\Omega}}{\longleftarrow} \mathit{\Delta}^*_{vol} \tag{2.34}$$

that we denote by Ad^*_γ. In this way we obtain a map

$$\begin{array}{ccc} \mathcal{D}_{vol} & \stackrel{Ad^*}{\longrightarrow} & Aut(\mathit{\Delta}^*_{vol}) \\ \gamma & \longmapsto & Ad^*_\gamma. \end{array} \tag{2.35}$$

The map Ad^* is called the coadjoint representation of $\mathcal{D}_{vol}$ on $\mathit{\Delta}^*_{vol}$. Associated with the coadjoint representation we have the coadjoint action :

$$\begin{array}{ccc} \mathcal{D}_{vol}\times\mathit{\Delta}^*_{vol} & \longrightarrow & \mathit{\Delta}^*_{vol} \\ (\gamma,\omega) & \longmapsto & Ad^*_\gamma(\omega). \end{array} \tag{2.36}$$

It turns out that

$$Ad^*_\gamma(\omega) = \gamma^*\omega. \tag{2.37}$$

Therefore the coadjoint orbit through $\omega \in \Delta^*_{vol}$ is the set

$$\mathcal{O}_\omega = \{\eta^*\omega : \eta \in \mathcal{D}_{vol}\}. \tag{2.38}$$

Since we know by Kelvin's theorem that the vorticity is transported by the flow, we see that a vorticity that starts in $\mathcal{O}_\omega$ remains in it.

We know from the Kirillov theory of orbits [**?**, **?**] that the coadjoint orbits are symplectic manifolds and we have an explicit definition of their symplectic forms as follows. Let $\omega \in \Delta^*_{vol}$ and let ω_1 be a vector tangent the orbit of ω at ω. Since Δ^*_{vol} is a vector space we identify its tangent space at ω with Δ^*_{vol} , so $\omega_1 \in \Delta^*_{vol}$. The vector ω_1 can be represented (in many ways) as the velocity vector of the motion of ω under the coadjoint action of the one-parameter group $\exp_{\mathcal{D}_{vol}}(\vec{v}_1 t)$ with velocity vector $\vec{v}_1 \in \Delta_{vol}$. In other words, every vector tangent to the orbit of ω in the coadjoint representation of $\mathcal{D}_{vol}$ can be expressed in terms of a suitable vector $\vec{v}_1 \in \Delta_{vol}$ by the formula

$$\omega_1 = ad^*_{\vec{v}_1}(\omega). \tag{2.39}$$

Then the value of the symplectic two-form Υ_ω on a pair of vectors ω_1, ω_2 tangent to the orbit of ω is obtained in the following way: we express ω_1 and ω_2 in terms of the algebra elements $\vec{v}_1$ and $\vec{v}_2$ by the formula 2.39 , and then get the scalar

$$\Upsilon_\omega(\omega_1, \omega_2) = \langle \omega, [\vec{v}_1, \vec{v}_2] \rangle. \tag{2.40}$$

Proposition 2.3.1

The symplectic structure Υ_ω on $T_\omega \mathcal{O}_\omega$ is given by

$$\Upsilon_\omega(L_{u_1}\omega, L_{u_2}\omega) = \int \omega(u_1, u_2) dV \tag{2.41}$$

Proof:

We have that

$$ad^*_{\vec{v}_1}(\omega) = L_{\vec{v}_1}\omega. \tag{2.42}$$

and

$$\langle \omega, [\vec{v}_1, \vec{v}_2] \rangle = -\int \vec{v}_\alpha \cdot [\vec{v}_1, \vec{v}_2] dV \tag{2.43}$$

where $d\alpha = \omega$. Since $[\vec{v}_1, \vec{v}_2] = L_{\vec{v}_1}\vec{v}_2$ we get integrating by parts:

$$\Upsilon_\omega(L_{\vec{v}_1}\omega, L_{\vec{v}_2}\omega) = \int L_{\vec{v}_1}\vec{v}_\alpha \cdot \vec{v}_2 dV \tag{2.44}$$

$$= \int [i(\vec{v}_1)d\alpha + di(\vec{v}_1)\alpha] \cdot \vec{v}_2 dV \tag{2.45}$$

$$= \int i(\vec{v}_1)d\alpha \cdot \vec{v}_2 dV \tag{2.46}$$

$$= \int \omega(\vec{v}_1, \vec{v}_2) dV. \tag{2.47}$$

Q.E.D.

Chapter 3

The vortices

The point vortices were known to be a Hamiltonian system since last century [?, ?, ?]. Such a system has some constants of the motion that can be obtained by noticing certain symmetries of the Hamiltonian function. When the number N of vortices is at most three the system of point vortices is Liouville integrable. For $N = 4$ some extra conditions, like zero total circulation, are to be satisfied in order to get integrability.

3.1 Systems of point vortices.

The motion of a system of N point vortices on the plane is described by the $2N$-dimensional system

$$\begin{cases} \dfrac{dX_m}{dt} = -\dfrac{1}{2\pi} \displaystyle\sum_{\substack{j=1 \\ j\neq m}}^{N} \Gamma_j \dfrac{Y_m - Y_j}{(X_m - X_j)^2 + (Y_m - Y_j)^2}, \\ \dfrac{dY_m}{dt} = \dfrac{1}{2\pi} \displaystyle\sum_{\substack{j=1 \\ j\neq m}}^{N} \Gamma_j \dfrac{X_m - X_j}{(X_m - X_j)^2 + (Y_m - Y_j)^2}, \end{cases} \tag{3.1}$$

where (X_m, Y_m) is the position of the mth vortex and Γ_m is its intensity, and $1 \leq m \leq N$. This system will be refered to from here on as the Kirchoff vector field. If

$$H_{\mathrm{v}} = -\frac{1}{4\pi} \sum_{\substack{m,j=1 \\ j\neq m}}^{N} \Gamma_m \Gamma_j \log \sqrt{(X_m - X_j)^2 + (Y_m - Y_j)^2}, \tag{3.2}$$

the system above can be written as

$$\begin{cases} \Gamma_m \dfrac{dX_m}{dt} = \dfrac{\partial H_{\mathrm{v}}}{\partial Y_m} \\ \Gamma_m \dfrac{dY_m}{dt} = -\dfrac{\partial H_{\mathrm{v}}}{\partial X_m}, \end{cases} \tag{3.3}$$

which is a Hamiltonian system in the symplectic manifold $\mathbf{R}^{2N}$ with symplectic form

$$\sum_{m=1}^{N} \Gamma_m \, dY_m \wedge dX_m. \tag{3.4}$$

The velocity of each point vortex is computed by adding the velocity fields due to all the other point vortices and assuming that the point vortex does not act on itself. This renormalization assumption needs to be justified. Without it the velocity becomes singular. Namely, if we consider the system of point vortices as a singular solution of the Euler equations corresponding to an initial vorticity

$$\omega(\vec{x},0) = \sum_{m=1}^{N} \Gamma_m \delta(\vec{x} - \vec{X}_m(0)) \tag{3.5}$$

that at time t becomes

$$\omega(\vec{x},t) = \sum_{m=1}^{N} \Gamma_m \delta(\vec{x} - \vec{X}_m(t)) \tag{3.6}$$

we encounter two singularities: one in the velocity and one in the energy. More precisely, the expression of the velocity in terms of the vorticity in two dimensions is the following:

$$\vec{v}(\vec{x}) = -\frac{1}{2\pi} \int_{\Omega} [\frac{y - y_0}{\| \vec{x} - \vec{x}_0 \|^2} \, \vec{\mathbf{i}} - \frac{x - x_0}{\| \vec{x} - \vec{x}_0 \|^2} \, \vec{\mathbf{j}} \,] \omega(\vec{x}_0) dA(\vec{x}_0). \tag{3.7}$$

To obtain the Kirchoff vector field from the Euler velocity we proceed as follows. We consider the functional $F_{\vec{x}}$ defined by

$$F_{\vec{x}}[\vec{v}\,] = \lim_{\epsilon \to 0} \int_{\Omega} d_{\epsilon}(\vec{x}_0 - \vec{x}) \vec{v}(\vec{x}_0) dA(\vec{x}_0) \tag{3.8}$$

where d_ϵ is such that

$$d_\epsilon(\vec{x}) \quad = \quad \begin{cases} \epsilon^{-2} \ \text{ if } \ x^2+y^2 \leq \epsilon^2, \\ \\ 0 \ \text{ otherwise.} \end{cases} \tag{3.9}$$

Using (3.6), the expression (3.7) for the velocity gives

$$\vec{v}(\vec{x}) = -\frac{1}{2\pi}\sum_{m=1}^{N}\Gamma_m[\frac{y-Y_m(t)}{\| \vec{x}-\vec{X}_m(t) \|^2}\,\vec{\mathbf{i}} - \frac{x-X_m(t)}{\| \vec{x}-\vec{X}_m(t) \|^2}\,\vec{\mathbf{j}}\,]. \tag{3.10}$$

Hence we see that the velocity is singular at $\vec{X}_m(t)$ for $1 \leq m \leq N$. But if we evaluate the velocity at $\vec{X}_m(t)$ through the functional $F_{\vec{X}_m(t)}$ we obtain:

$$\begin{aligned}
\vec{v}(\vec{X}_m(t)) \quad &= \quad -\frac{1}{2\pi}\sum_{\substack{j=1\\ j\neq m}}^{N}\Gamma_j[\frac{Y_m(t)-Y_j(t)}{\| \vec{X}_m(t)-\vec{X}_j(t) \|^2}\,\vec{\mathbf{i}} - \frac{X_m(t)-X_j(t)}{\| \vec{X}_m(t)-\vec{X}_j(t) \|^2}\,\vec{\mathbf{j}}\,] \\
&\quad -\frac{\Gamma_m}{2\pi}\lim_{\epsilon\to 0}\int_\Omega d_\epsilon(\vec{x}_0-\vec{X}_m(t))[\frac{y_0-Y_m(t)}{\| \vec{x}_0-\vec{X}_m(t) \|^2}\,\vec{\mathbf{i}} \\
&\quad -\frac{x_0-X_m(t)}{\| \vec{x}_0-\vec{X}_m(t) \|^2}\,\vec{\mathbf{j}}\,]dA(\vec{x}_0) \qquad (3.11) \\
&= \quad -\frac{1}{2\pi}\sum_{\substack{j=1\\ j\neq m}}^{N}\Gamma_j[\frac{Y_m(t)-Y_j(t)}{\| \vec{X}_m(t)-\vec{X}_j(t) \|^2}\,\vec{\mathbf{i}} - \frac{X_m(t)-X_j(t)}{\| \vec{X}_m(t)-\vec{X}_j(t) \|^2}\,\vec{\mathbf{j}}\,] \\
&\quad -\frac{\Gamma_m}{2\pi}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\underbrace{\int_0^{2\pi}(\sin\theta\vec{\mathbf{i}}-\cos\theta\vec{\mathbf{j}})d\theta}_{0}\,. \qquad (3.12)
\end{aligned}$$

This procedure we have just described amounts to first evaluating the velocity field for a smooth vorticity supported in circular regions of radius ϵ and then letting $\epsilon \to 0$. Although it produces what we want, we cannot infer anything about the δ function vorticity distribution because to do so the limit process should not depend on how we approximate the δ functions. Moreover (3.6) is an assumption that needs to be proved; we do so, for an interval of time, below. Note also that the singularity in the Hamiltonian cannot be avoided using this idea.

3.2 The constants of the motion.

We have three integrals of the motion besides the Hamiltonian.

$$T_X \stackrel{\text{def}}{=} \sum_{m=1}^{N} \Gamma_m Y_m, \tag{3.13}$$

$$T_Y \stackrel{\text{def}}{=} \sum_{m=1}^{N} \Gamma_m X_m, \tag{3.14}$$

$$R \stackrel{\text{def}}{=} \sum_{m=1}^{N} \Gamma_m (X_m^2 + Y_m^2). \tag{3.15}$$

They can be obtained, from the finite dimensional point of view, by observing that the Hamiltonian is invariant, respectively, under translations parallel to the X-axis ,translations parallel to the Y-axis and rotations.

The Lie-Poisson bracket associated with the symplectic form

$$\sum_{m=1}^{N} \Gamma_m \, dY_m \wedge dX_m, \tag{3.16}$$

is the following:

$$\{P, Q\} = \sum_{m=1}^{N} \Gamma_m^{-1} [\frac{\partial P}{\partial X_m} \frac{\partial Q}{\partial Y_m} - \frac{\partial Q}{\partial Y_m} \frac{\partial P}{\partial X_m}]. \tag{3.17}$$

We have the following double-entry table of values for the Lie-Poisson bracket on H, T_X, T_Y, R and $T_X^2 + T_Y^2$:

$\{\ ,\ \}$	R	H	$T_X^2 + T_Y^2$	T_X	T_Y
R	0	0	0	$2T_Y$	$-2T_X$
H	0	0	0	0	0
$T_X^2 + T_Y^2$	0	0	0	$2T_Y \sum_m \Gamma_m$	$2T_X \sum_m \Gamma_m$
T_X	$-2T_Y$	0	$2T_Y \sum_m \Gamma_m$	0	$-\sum_m \Gamma_m$
T_Y	$2T_X$	0	$2T_X \sum_m \Gamma_m$	$\sum_m \Gamma_m$	0

Table 3.1: The Lie-Poisson bracket.

Therefore the problem of the motion of N point vortices in the plane is Liouville integrable for $N \leq 3$, and for $N = 4$ if $\sum_m \Gamma_m = 0$. For some special cases of $N = 4$ there is a proof of nonintegrability due to Ziglin [**?**], but its validity has been questioned by Holmes and Marsden [**?**].

Note that the some of the constants of the motion described above come from integrals of the Euler equations, namely the following table indicates such correspondence:

$\int \omega dA$	$\sum_m \Gamma_m$
$\int \omega^2 dA$	$\sum_m \Gamma_m^2$
$\int x\omega dA$	T_Y
$\int y\omega dA$	T_X
$\int (x^2+y^2)\omega dA$	R

Table 3.2: The constants of the motion.

The Hamiltonian is the only constant of the motion that we cannot obtain from the corresponding integral of the Euler equations. If we use the expression for the energy in terms of the vorticity we face the problem of multiplication of δ functions. In the next chapter we provide an asymptotic way of getting rid of the infinity in the energy.

Chapter 4

The desingularization of the point vortices

In this chapter we present a desingularization of the system of N point vortices. We understand the term "desingularization" when applied to the system of N point vortices to mean that there exists a family of solutions of Euler equations parametrized by $\epsilon \ll 1$ such that their Hamiltonians as $\epsilon \to 0$ converge to a Hamiltonian that is a sum of the Hamiltonian of the system of N point vortices and a function that is independent of the locations of the point vortices. Such a family of solutions is constructed by considering N vortex patches of measure ϵ^2 as an approximation to the N point vortices. And for each one of these solutions we take its kinetic energy plus a linear combination of other integrals of the motion as its Hamiltonian. As $\epsilon \to 0$ the coefficients of this linear combination of integrals of the motion blow up as the negative of the kinetic energy so that the Hamiltonian has a finite limit.

The Poisson structure of the Euler equations converges as $\epsilon \to 0$ to a Poisson structure whose associated symplectic structure agrees with the symplectic structure on the non-smooth coadjoint orbit of N point vortices . The part of the finite limit of the Hamiltonian depending on the locations of the point vortices gives through this symplectic structure the Kirchoff vector field.

4.1 The vortex patch approximation.

We use the following approximation to the system of N point vortices. We consider N patches of vorticity initially supported in N disjoint open regions $\{\Lambda_\epsilon^m\}_{m=1}^N$ such that $\operatorname{meas}\Lambda_\epsilon^m = \epsilon^2$, i.e.

$$\omega_\epsilon(\vec{x}, 0) = \sum_{m=1}^{N} \epsilon^{-2}\Gamma_m \chi_{\Lambda_\epsilon^m}(\vec{x}), \tag{4.1}$$

where χ_Λ stands for the characteristic function of the set Λ.

We introduce, for $1 \leq m \leq N$, the center of vorticity, $\vec{M}_\epsilon^m(t)$, and the second moment, $I_\epsilon^m(t)$, by

$$\vec{M}_\epsilon^m(t) \stackrel{\text{def}}{=} \epsilon^{-2}\int_{\Lambda_{\epsilon,t}^m} \vec{x}dA(\vec{x}), \tag{4.2}$$

$$I_\epsilon^m(t) \stackrel{\text{def}}{=} \epsilon^{-2}\Gamma_m \int_{\Lambda_{\epsilon,t}^m} \| \vec{x} - \vec{M}_\epsilon^m(t) \|^2 \, dA(\vec{x}), \tag{4.3}$$

where $\Lambda_{\epsilon,t}^m$ is the evolution of the region Λ_ϵ^m under the Euler flow. The Euler flow is the one whose characteristics satisfy

$$\begin{cases} \dfrac{d\vec{x}_\epsilon}{dt} = \vec{v}(\vec{x}_\epsilon(t, \vec{x}_0), t), \\ \vec{v}(\vec{x}, t) = -\dfrac{\epsilon^{-2}}{2\pi}\displaystyle\sum_{j=1}^{N}\Gamma_j \int_{\Lambda_{\epsilon,t}^j} \vec{\nabla}_{\vec{x}} \times (\log \| \vec{x} - \vec{y} \| \, \vec{\mathbf{k}}) \, dA(\vec{y}), \\ \vec{x}_\epsilon(0, \vec{x}_0) = \vec{x}_0. \end{cases} \tag{4.4}$$

We prove that for an interval of time $(0, T)$ the flow and the limit process as $\epsilon \to 0$ commute, i.e. $\omega_\epsilon(\vec{x}, t)$ converges as $\epsilon \to 0$ to a linear combination of δ functions at $\vec{X}_m(t)$ for $1 \leq m \leq N$, where $\vec{X}_m(t)$ solves the Kirchoff problem.

Theorem 4.1.1

Let $\vec{X}_m$ stand for $\lim_{\epsilon\to 0} \vec{M}_\epsilon^m(0)$ for $1 \leq m \leq N$. Suppose that

$$\lim_{\epsilon\to 0}\int f(\vec{x})\omega_\epsilon(\vec{x}, 0)dA = \sum_{m=1}^{N}\Gamma_m f(\vec{X}_m), \tag{4.5}$$

for any continuous, bounded function $f: \mathbf{R^2} \to \mathbf{R}$. And suppose that

$$\min_{i\neq j} \inf\{\| \vec{x}-\vec{y} \|: \vec{x}\in \Lambda^i_\epsilon, \vec{y}\in \Lambda^j_\epsilon\} \geq \frac{1}{2}\min_{i\neq j} \| \vec{X}_i - \vec{X}_j \| > 0, \tag{4.6}$$

and

$$I^m_\epsilon(0) \leq D_m \epsilon^2 \tag{4.7}$$

for some D_m and $1 \leq m \leq N$. Then there exists a time T such that for all $t \in [0, T)$

$$\lim_{\epsilon\to 0} \vec{M}^m_\epsilon(t) = \vec{X}_m(t). \tag{4.8}$$

Moreover,

$$\lim_{\epsilon\to 0} \int f(\vec{x})\omega_\epsilon(\vec{x},t)dA = \sum_{m=1}^{N} \Gamma_m f(\vec{X}_m(t)). \tag{4.9}$$

Proof:

We consider the motion of the initial patch Λ^m_ϵ under a velocity that is the sum of the self-induced velocity of the patch itself, i.e.

$$-\frac{\Gamma_m \epsilon^{-2}}{2\pi}\int_{\hat{\Lambda}^m_{\epsilon,t}} \vec{\nabla}_{\vec{x}} \times (\log \| \vec{x}-\vec{y} \| \,\vec{\mathbf{k}})\, dA(\vec{y}), \tag{4.10}$$

plus a field $\vec{F}$ that remains regular as $\epsilon \to 0$, i.e.

$$\vec{F}(\vec{x},t) \stackrel{\text{def}}{=} \epsilon^{-2} \sum_{\substack{j=1\\ j\neq m}}^{N} \Gamma_j \int_{\hat{\Lambda}^j_{\epsilon,t}} \vec{K}_\eta(\vec{x}-\vec{y})dA(\vec{y}). \tag{4.11}$$

More precisely, we introduce the regularized Euler evolution and its corresponding vortex model in the following way. The characteristics or particle paths of the regularized Euler evolution satisfy:

$$\frac{d\vec{z}_\epsilon}{dt} = \vec{v}_\epsilon(\vec{z}_\epsilon(t,\vec{x}_0),t), \quad \vec{x}_0 \in \Lambda^m_\epsilon, \tag{4.12}$$

$$\begin{aligned} \vec{v}_\epsilon(\vec{x},t) &= -\frac{\Gamma_m \epsilon^{-2}}{2\pi}\int_{\hat{\Lambda}^m_{\epsilon,t}} \vec{\nabla}_{\vec{x}} \times (\log \| \vec{x}-\vec{y} \| \,\vec{\mathbf{k}})\, dA(\vec{y}) \\ &\quad + \epsilon^{-2} \sum_{\substack{j=1\\ j\neq m}}^{N} \Gamma_j \int_{\hat{\Lambda}^j_{\epsilon,t}} \vec{K}_\eta(\vec{x}-\vec{y})dA(\vec{y}), \end{aligned} \tag{4.13}$$

$$\vec{z}_\epsilon(0,\vec{x}_0) = \vec{x}_0, \tag{4.14}$$

where $\hat{\Lambda}^j_{\epsilon,t}$ stands for the image at time t of the initial region Λ^j_ϵ under the regularized Euler evolution; and where the regularized kernel $\vec{K}_\eta$ is defined by

$$\vec{K}_\eta(\vec{x}) = \vec{\nabla} \times (g_\eta(\vec{x})\vec{\mathbf{k}}), \tag{4.15}$$

where

$$g_\eta(\vec{x}) = -\frac{1}{2\pi}\log \parallel \vec{x} \parallel \quad \text{if} \quad \parallel \vec{x} \parallel \geq \eta, \tag{4.16}$$

and such that

$$\parallel D\vec{K}_\eta(\vec{x}) \parallel \leq \frac{Q}{\eta^2} \quad \text{for all} \quad \vec{x} \in \mathbf{R}^2, \tag{4.17}$$

where Q is a positive constant. Note that, so long as the distances between patches remain sufficiently large the regularized Euler flow agrees with the Euler flow. And the regularized vortex model is the following:

$$\begin{cases} \dfrac{d\vec{z}_m}{dt} = \sum\limits_{\substack{j=1\\ j\neq m}}^{N} \Gamma_j \vec{K}_\eta(\vec{z}_m(t) - \vec{z}_j(t)), \\ \vec{z}_m(0) = \vec{X}_m. \end{cases} \tag{4.18}$$

We denote by $\vec{\hat{M}}^m_\epsilon(t)$ and $\hat{I}^m_\epsilon(t)$ the same quantities as $\vec{M}^m_\epsilon(t)$ and $I^m_\epsilon(t)$ with $\Lambda^m_{\epsilon,t}$ replaced by $\hat{\Lambda}^m_{\epsilon,t}$. We prove first the following lemma:

Lemma 4.1

Let $\vec{X}_m$ stand for $\lim\limits_{\epsilon\to 0} \vec{M}^m_\epsilon(0)$ for $1 \leq m \leq N$. Suppose that

$$\lim_{\epsilon\to 0} \int f(\vec{x})\omega_\epsilon(\vec{x}, 0)dA(\vec{x}) = \sum_{m=1}^{N} \Gamma_m f(\vec{X}_m), \tag{4.19}$$

for any continuous, bounded function $f\colon \mathbf{R^2} \to \mathbf{R}$. Then for all $t > 0$

$$\lim_{\epsilon\to 0} \vec{\hat{M}}^m_\epsilon(t) = \vec{z}_m(t), \tag{4.20}$$

where $\vec{z}_m(t)$ for $1 \leq m \leq N$ solves the regularized vortex problem. Moreover,

$$\lim_{\epsilon\to 0} \int f(\vec{x})\hat{\omega}_\epsilon(\vec{x}, t)dA(\vec{x}) = \sum_{m=1}^{N} \Gamma_m f(\vec{z}_m(t)). \tag{4.21}$$

where

$$\hat{\omega}_\epsilon(\vec{x},t) = \sum_{m=1}^{N} \epsilon^{-2}\Gamma_m \chi_{\hat{\Lambda}^m_{\epsilon,t}}(\vec{x}). \tag{4.22}$$

Proof of Lemma 4.1:

We have that

$$\begin{aligned} \frac{d}{dt}\vec{\hat{M}}^m_\epsilon(t) &= \epsilon^{-2}\int_{\Lambda^m_\epsilon} \frac{d\vec{z}_\epsilon}{dt}(\vec{x},t)dA(\vec{x}) \\ &= \epsilon^{-2}\int_{\Lambda^m_\epsilon} \vec{F}(\vec{z}_\epsilon(\vec{x},t),t)dA(\vec{x}), \end{aligned} \tag{4.23}$$

because

$$\int_{\hat{\Lambda}^m_{\epsilon,t}} \int_{\hat{\Lambda}^m_{\epsilon,t}} \vec{\nabla}_{\vec{x}} \times (\log \| \vec{x}-\vec{y} \| \vec{\mathbf{k}})dA(\vec{y})dA(\vec{x}) = 0. \tag{4.24}$$

Because of the area-preserving character of the flow we have exchanged the integration over Λ^m_ϵ with the integration over $\hat{\Lambda}^m_\epsilon(t)$. For the same reason we have that

$$\frac{d}{dt}\vec{\hat{M}}^m_\epsilon(t) = \epsilon^{-2}\int_{\hat{\Lambda}^m_{\epsilon,t}} \vec{F}(\vec{x},t)dA(\vec{x}). \tag{4.25}$$

Note that

$$\| \vec{F}(\vec{x},t) - \vec{F}(\vec{y},t) \| \leq \frac{\hat{Q}}{\eta^2} \| \vec{x}-\vec{y} \|, \tag{4.26}$$

i.e. $\vec{F}$ satisfies a Lipschitz condition in its first variable, where $\hat{Q} = (N-1)\max_m |\Gamma_m|$.

We want to prove that $\hat{\Lambda}^m_{\epsilon,t}$ is in a small neighborhood of the point $\vec{\hat{M}}^m_\epsilon(t)$. And we use the second moment $\hat{I}^m_\epsilon(t)$ to measure the spread of $\hat{\Lambda}^m_{\epsilon,t}$ around $\vec{\hat{M}}^m_\epsilon(t)$.

We have that

$$\begin{aligned} \frac{d}{dt}\hat{I}^m_\epsilon(t) &= \epsilon^{-2}\Gamma_m \frac{d}{dt}\int_{\hat{\Lambda}^m_{\epsilon,t}} \| \vec{x} - \vec{\hat{M}}^m_\epsilon(t) \|^2 \, dA(\vec{x}) \\ &= 2\epsilon^{-2}\Gamma_m \int_{\Lambda^m_\epsilon} (\vec{z}_\epsilon(\vec{x},t) - \vec{\hat{M}}^m_\epsilon(t)) \cdot \vec{F}(\vec{z}_\epsilon(\vec{x},t),t)dA(\vec{x}), \end{aligned} \tag{4.27}$$

because

$$\int_{\Lambda^m_{\epsilon,t}} \vec{x} \cdot (\int_{\Lambda^m_{\epsilon,t}} \vec{\nabla}_{\vec{x}} \times (\log \| \vec{x}-\vec{y} \| \vec{\mathbf{k}} dA(\vec{y}))\, dA(\vec{x}) = 0. \tag{4.28}$$

Hence

$$
\begin{aligned}
|\frac{d}{dt}\hat{I}^m_\epsilon(t)| &\leq 2\epsilon^{-2}|\Gamma_m|\int_{\Lambda^m_\epsilon} \| \vec{z}_\epsilon(\vec{x},t) - \vec{\hat{M}}^m_\epsilon(t) \| \| \vec{F}(\vec{z}_\epsilon(\vec{x},t),t) - \vec{F}(\vec{\hat{M}}^m_\epsilon(t),t) \| dA(\vec{x}) \\
&\leq \frac{2\hat{Q}}{\eta^2}|\Gamma_m|\epsilon^{-2}\int_{\Lambda^m_\epsilon} \| \vec{z}_\epsilon(\vec{x},t) - \vec{\hat{M}}^m_\epsilon(t) \|^2 \, dA(\vec{x}) \\
&= \frac{2\hat{Q}}{\eta^2}|\hat{I}^m_\epsilon(t)|,
\end{aligned}
\tag{4.29}
$$

so that

$$
|\hat{I}^m_\epsilon(t)| \leq |I^m_\epsilon(0)|e^{\frac{c}{\eta^2}t}, \tag{4.30}
$$

where $c = 2\hat{Q}$.

We proceed to estimate the distance between $\vec{z}_m(t)$ and $\vec{\hat{M}}^m_\epsilon(t)$:

$$
\begin{aligned}
\| \vec{\hat{M}}^m_\epsilon(t) - \vec{z}_m(t) \| \;\leq\; & \| \vec{M}^m_\epsilon(0) - \vec{X}_m \| \\
& + \int_0^t \sum_{\substack{j=1 \\ j\neq m}}^N |\Gamma_j| \, \| \vec{K}_\eta(\vec{\hat{M}}^m_\epsilon(s) - \vec{z}_j(s)) \\
& - \vec{K}_\eta(\vec{z}_m(s) - \vec{z}_j(s)) \| \, ds \\
& + \int_0^t \| \sum_{\substack{j=1 \\ j\neq m}}^N \Gamma_j \vec{K}_\eta(\vec{\hat{M}}^m_\epsilon(s) - \vec{z}_j(s)) \\
& - \epsilon^{-2}\int_{\hat{\Lambda}^m_{\epsilon,s}} \vec{F}(\vec{x},s)dA(\vec{x}) \| \, ds
\end{aligned}
\tag{4.31}
$$

since

$$
\vec{\hat{M}}^m_\epsilon(t) = \vec{M}^m_\epsilon(0) + \int_0^t \frac{d\vec{\hat{M}}^m_\epsilon}{dt}(s)ds, \tag{4.32}
$$

and

$$
\vec{z}_m(t) = \vec{X}_m + \int_0^t \sum_{\substack{j=1 \\ j\neq m}}^N \Gamma_j \vec{K}_\eta(\vec{z}_m(s) - \vec{z}_j(s))ds. \tag{4.33}
$$

And since

$$
\| \vec{K}_\eta(\vec{x}) - \vec{K}_\eta(\vec{y}) \| \leq \frac{Q}{\eta^2} \| \vec{x} - \vec{y} \| \tag{4.34}
$$

it follows that

$$\begin{aligned}
\| \vec{\hat{M}}_\epsilon^{\,m}(t) - \vec{z}_m(t) \| &\leq \| \vec{M}_\epsilon^m(0) - \vec{X}_m \| \\
&+ (\sum_{\substack{j=1 \\ j\neq m}}^N |\Gamma_j|) \frac{Q}{\eta^2} \int_0^t \| \vec{\hat{M}}_\epsilon^{\,m}(s) - \vec{z}_m(s) \| \, ds \\
&+ \epsilon^{-2} (\sum_{\substack{j=1 \\ j\neq m}}^N |\Gamma_j|) \frac{Q}{\eta^2} \int_0^t \int_{\hat{\Lambda}^m_{\epsilon,s}} \| \vec{\hat{M}}_\epsilon^{\,m}(s) - \vec{x} \| \, dA(\vec{x}) ds \\
&+ \frac{Q}{\eta^2} \sum_{\substack{j=1 \\ j\neq m}}^N |\Gamma_j| \int_0^t \| \vec{z}_j(s) - \vec{\hat{M}}_\epsilon^{\,j}(s) \| \, ds \\
&+ \epsilon^{-2} \frac{Q}{\eta^2} \sum_{\substack{j=1 \\ j\neq m}}^N |\Gamma_j|) \int_0^t \int_{\hat{\Lambda}^j_{\epsilon,s}} \| \vec{\hat{M}}_\epsilon^{\,j}(s) - \vec{y} \| \, dA(\vec{y}) ds. \qquad (4.35)
\end{aligned}$$

And using the Cauchy-Schwarz inequality we obtain that

$$\epsilon^{-2} \int_{\hat{\Lambda}^m_{\epsilon,t}} \| \vec{\hat{M}}_\epsilon^{\,m}(t) - \vec{x} \| \, dA(\vec{x}) \leq \frac{1}{\sqrt{|\Gamma_m|}} \sqrt{|\hat{I}^m_\epsilon(t)|}. \qquad (4.36)$$

Therefore

$$\begin{aligned}
\| \vec{\hat{M}}_\epsilon^{\,m}(t) - \vec{z}_m(t) \| &\leq \| \vec{M}_\epsilon^m(0) - \vec{X}_m \| \\
&+ \frac{\Gamma Q(N-1)}{\eta^2} \sum_{j=1}^N \int_0^t \| \vec{\hat{M}}_\epsilon^{\,j}(s) - \vec{z}_j(s) \| \, ds \\
&+ \sum_{j=1}^N \frac{1}{\sqrt{|\Gamma_j|}} \int_0^t \sqrt{|\hat{I}^j_\epsilon(s)|} ds, \qquad (4.37)
\end{aligned}$$

where $\Gamma = \max_k |\Gamma_k|$. So we have that

$$\begin{aligned}
\max_k \| \vec{\hat{M}}_\epsilon^{\,k}(t) - \vec{z}_k(t) \| &\leq \max_k \| \vec{M}_\epsilon^k(0) - \vec{X}_k \| \\
&+ \frac{P}{\eta^2} \int_0^t \max_k \| \vec{\hat{M}}_\epsilon^{\,k}(s) - \vec{z}_k(s) \| \, ds \\
&+ \frac{Pt}{\eta^2} \sum_{j=1}^N \frac{1}{\sqrt{|\Gamma_j|}} \sup\{ \sqrt{|\hat{I}^j_\epsilon(s)|} : \; s \in (0,t) \}, \qquad (4.38)
\end{aligned}$$

where $P = \Gamma Q(N-1)N$.

Finally, Gronwall's inequality gives that

$$\max_k \| \vec{\hat{M}}^k_\epsilon(t) - \vec{z}_k(t) \| \leq e^{\frac{Pt}{\eta^2}} \max_k \| \vec{M}^k_\epsilon(0) - \vec{X}_k \|$$

$$+\frac{Pt}{\eta^2} \sum_{j=1}^{N} \frac{1}{\sqrt{|\Gamma_j|}} \sup\{\sqrt{|\hat{I}^j_\epsilon(s)|} : \ s \in (0,t)\} e^{\frac{Pt}{\eta^2}}, \quad (4.39)$$

therefore

$$\max_k \| \vec{\hat{M}}^k_\epsilon(t) - \vec{z}_k(t) \| \rightarrow 0 \ \text{ as } \ \epsilon \rightarrow 0 \quad (4.40)$$

because by hypothesis

$$\| \vec{M}^k_\epsilon(0) - \vec{X}_k \| \rightarrow 0 \ \text{ and } \ I^k_\epsilon(0) \rightarrow 0. \quad (4.41)$$

This finishes the proof of Lemma 4.1.

To finish the proof of the theorem we note that the regularized vortex model coincides with the Kirchoff point vortices up to the time

$$T_\eta \stackrel{\text{def}}{=} \inf\{t : \ \min_{i,j} \| \vec{z}_i(t) - \vec{z}_j(t) \| \leq \eta\}. \quad (4.42)$$

And the regularized Euler evolution coincides with the real one up to the time

$$T_{\epsilon,\eta} \stackrel{\text{def}}{=} \inf\{t : \ \min_{i,j} \text{dist}(\Lambda^i_{\epsilon,t}, \Lambda^j_{\epsilon,t}) \leq \eta\}. \quad (4.43)$$

Hence we are done once we show the existence of a positive η for which

$$\min(T_\eta, \inf_\epsilon T_{\epsilon,\eta}) > 0. \quad (4.44)$$

Given $\vec{X}_m$ for $1 \leq m \leq N$, we can find a time T^* for which

$$\min_{i \neq j} \inf\{\| \vec{X}_i(t) - \vec{X}_j(t) \| : \ 0 \leq t \leq T^*\} \geq b > 0, \quad (4.45)$$

and, hence, choosing $\eta = \dfrac{b}{3}$, we have that $\vec{X}_m(t) = \vec{z}_m(t)$ for $t \in [0, T^*]$.

We proceed to estimate the velocity field in order to estimate how patches approach each other.

Let $C_d^m(t)$ be the circle of radius d centered at $\vec{\hat{M}}_\epsilon^m(t)$. Then defining $A_\epsilon^m(t)$ by

$$A_\epsilon^m(t) \stackrel{\text{def}}{=} \hat{\Lambda}_{\epsilon,t}^m \bigcap \mathbf{R}^2 \setminus C_d^m(t) \tag{4.46}$$

we have that

$$\begin{aligned} |\hat{I}_\epsilon^m(t)| &\geq |\Gamma_m|\epsilon^{-2} \int_{A_\epsilon^m(t)} \| \vec{x} - \vec{\hat{M}}_\epsilon^m(t) \|^2 \, dA(\vec{x}) \\ &\geq |\Gamma_m|\epsilon^{-2} d^2 \text{meas}\, A_\epsilon^m(t). \end{aligned} \tag{4.47}$$

From the proof of Lemma 4.1 we know that

$$|\hat{I}_\epsilon^m(t)| \leq |I_\epsilon^m(0)| e^{\frac{c}{\eta^2}t}. \tag{4.48}$$

This together with the hypothesis

$$|I_\epsilon^m(0)| \leq D_m \epsilon^2 \tag{4.49}$$

gives

$$|\hat{I}_\epsilon^m(t)| \leq D_m \epsilon^2 \, e^{\frac{c}{\eta^2}t}. \tag{4.50}$$

Therefore it follows that

$$\text{meas}\, A_\epsilon^m(t) \leq \frac{D_m}{d^2 |\Gamma_m|} \epsilon^4 e^{\frac{c}{\eta^2}t}, \tag{4.51}$$

for $t < T^*$.

The velocity field at any point $\vec{x}$ outside $C_{2d}^m(t)$ and inside $C_\eta^m(t)$ is the sum of three fields: the first field is the one generated by the vorticity inside $C_d^m(t)$, i.e.

$$\vec{v}_1(\vec{x}, t) = -\frac{\Gamma_m \epsilon^{-2}}{2\pi} \int_{C_d^m(t)} \vec{\nabla}_{\vec{x}} \times (\log \| \vec{x} - \vec{y} \| \, \vec{\mathbf{k}}) dA(\vec{y}); \tag{4.52}$$

the second field is the one due to the other patches, i.e.

$$\vec{v}_2(\vec{x}, t) = -\frac{\epsilon^{-2}}{2\pi} \sum_{\substack{j=1 \\ j \neq m}}^{N} \Gamma_j \int_{\Lambda_{\epsilon,t}^j} \vec{\nabla}_{\vec{x}} \times (\log \| \vec{x} - \vec{y} \| \, \vec{\mathbf{k}}) dA(\vec{y}); \tag{4.53}$$

and the third field is the one due to the vorticity outside $C_d^m(t)$, i.e.

$$\vec{v}_3(\vec{x},t) = -\frac{\Gamma_m \epsilon^{-2}}{2\pi} \int_{A_\epsilon^m(t)} \vec{\nabla}_{\vec{x}} \times (\log \parallel \vec{x} - \vec{y} \parallel \vec{\mathbf{k}}) dA(\vec{y}). \tag{4.54}$$

The first field $\vec{v}_1$ is estimated in the following way:

$$\begin{aligned} \parallel \vec{v}_1(\vec{x},t) \parallel &\leq \frac{|\Gamma_m|\epsilon^{-2}}{2\pi} \int_{C_d^m(t)} \frac{1}{\parallel \vec{x} - \vec{y} \parallel} dA(\vec{y}) \\ &\leq \frac{|\Gamma_m|\epsilon^{-2}}{2\pi d} \int_{\Lambda_{\epsilon,t}^m} dA(\vec{y}) \\ &\leq \frac{|\Gamma_m|}{d}. \end{aligned} \tag{4.55}$$

The second field $\vec{v}_2$ is estimated in the following way:

$$\begin{aligned} \parallel \vec{v}_2(\vec{x},t) \parallel &\leq \sum_{\substack{j=1 \\ j\neq m}}^{N} \frac{|\Gamma_j|\epsilon^{-2}}{2\pi} \int_{\Lambda_{\epsilon,t}^j} \frac{1}{\parallel \vec{x} - \vec{y} \parallel} dA(\vec{y}) \\ &\leq \sum_{\substack{j=1 \\ j\neq m}}^{N} \frac{|\Gamma_j|}{\eta}. \end{aligned} \tag{4.56}$$

The third field $\vec{v}_3$ is estimated in the following way :

$$\begin{aligned} \parallel \vec{v}_3(\vec{x},t) \parallel &\leq \frac{|\Gamma_m|\epsilon^{-2}}{2\pi} \int_{A_\epsilon^m(t)} \frac{1}{\parallel \vec{x} - \vec{y} \parallel} dA(\vec{y}) \\ &\leq \frac{|\Gamma_m|\epsilon^{-2}}{2\pi} \int_{C_{r_0}^0(t)} \frac{1}{\parallel \vec{y} \parallel} dA(\vec{y}) \\ &\leq |\Gamma_m|\epsilon^{-2}\sqrt{\text{meas } A_\epsilon^m(t)}, \end{aligned} \tag{4.57}$$

where $C_{r_0}^0$ stands for the circle centered at the origin and of radius

$$r_0 = \sqrt{\frac{\text{meas } A_\epsilon^m(t)}{\pi}}. \tag{4.58}$$

Using the inequalities obtained above we get

$$\parallel \vec{v}_3(\vec{x},t) \parallel \leq \frac{K_1}{d} e^{\frac{c}{\eta^2}T^*}, \tag{4.59}$$

where K_1 stands for $\max_m \sqrt{D_m|\Gamma_m|}$.

Hence the velocity field outside $C_{2d}^m(t)$ is estimated in the following way:

$$\| \vec{v}(\vec{x},t) \| \leq \frac{K_1}{d} e^{\frac{c}{\eta^2}T^*} + \sum_{\substack{j=1 \\ j\neq m}}^{N} \frac{|\Gamma_j|}{\eta} + \frac{|\Gamma_m|}{d}. \tag{4.60}$$

We are now able to estimate the minimum distance between patches after a time t: two points in two different patches will get closer in time t by less than twice the estimate of the velocity times t, and we still have to take away a term due to rotation of each patch around its center of mass. More exactly, choosing ϵ so small that

$$\max_m \sup\{\| \vec{z}_m(t) - \vec{\hat{M}}_\epsilon^m(t) \| : t \in (0,T)\} \leq \frac{d}{2}, \tag{4.61}$$

and putting $d = \frac{b}{100}$, we conclude that for ϵ sufficiently small

$$\min_{i\neq j} \inf\{\text{dist } (\hat{\Lambda}^i_{\epsilon,s}, \hat{\Lambda}^j_{\epsilon,s}) : \; 0 \leq s \leq t\} \geq \frac{b}{2} - 5d - 2t(\frac{K_1}{d} e^{\frac{c}{\eta^2}T^*} + \frac{(N-1)\Gamma}{\eta} + \frac{\Gamma}{d}), \tag{4.62}$$

where $\Gamma = \max_m |\Gamma_m|$. Hence we can choose t so small that the right hand side of the last inequality is equal or greater than $\frac{b}{3} = \eta$.

Q.E.D.

This proof is due to C. Marchioro and M. Pulvirenti [**?**]

4.2 The desingularization of the energy.

We consider at each time $t \in (0,T)$ a vorticity distribution consisting of N circular patches of radius $\frac{\epsilon}{\sqrt{\pi}}$ centered at $\vec{M}_\epsilon^m(t)$. This vorticity is close to the one with arbitrary patches, so it is known that their respective energies are close to each other [**?**]. We obtain the asymptotic form of the energy for the circular patches and "substract the infinities " to get the Hamiltonian.

More precisely, if the vorticity ω is a linear combination of circular vortex patches, i.e.

$$\omega = \sum_{m=1}^{N} \epsilon^{-2} \Gamma_m \chi_{C_\epsilon^m(t)}, \tag{4.63}$$

where $C_\epsilon^m(t)$ stands for the circle of radius $\frac{\epsilon}{\sqrt{\pi}}$ and centered at $\vec{M}_\epsilon^m(t)$, we have that the stream function ψ being a solution of the Poisson equation

$$\nabla^2\psi = -\omega, \tag{4.64}$$

is itself a sum

$$\psi = \sum_{m=1}^{N} \psi_m, \tag{4.65}$$

where ψ_m satisfies for $1 \leq m \leq N$

$$\nabla^2\psi_m = -\epsilon^{-2}\Gamma_m \chi_{C_\epsilon^m(t)}. \tag{4.66}$$

In consequence, we have that

$$\vec{u} = \sum_{m=1}^{N} \vec{u}_m, \tag{4.67}$$

where for $1 \leq m \leq N$

$$\vec{u}_m = \vec{\nabla} \times \psi_m \vec{\mathbf{k}}. \tag{4.68}$$

An asymptotic expansion for the kinetic energy of the fluid contained in a ball C_R of radius R and centered at the origin describes the blow-up of the kinetic energy at infinity:

$$\begin{aligned} \frac{1}{2}\int_{C_R} u^2 dA &= \frac{1}{2}\int_{C_R} (u\frac{\partial\psi}{\partial y} - v\frac{\partial\psi}{\partial x})dA \\ &= \frac{1}{2}\int_{C_R} (\psi\omega + \vec{\nabla} \times \psi\vec{u})dA \\ &= \frac{1}{2}\int_{C_R} \psi\omega dA - \frac{1}{2}\oint_{\partial C_R} \psi\vec{u} \cdot d\vec{l}. \end{aligned} \tag{4.69}$$

As a consequence of (4.59) we have that

$$\psi(x,y) = -\frac{1}{2\pi}[\int \omega dA] \log r + O(r^{-1}) \tag{4.70}$$

as $r = \sqrt{x^2+y^2} \to \infty$. Therefore

$$-\frac{1}{2}\oint_{\partial C_R} \psi\vec{u} \cdot d\vec{l} = \frac{1}{4\pi}[\int \omega dA]^2 \log R + O(R^{-1}) \tag{4.71}$$

as $R \to \infty$.

If we further remove N small disks of radius $\dfrac{\epsilon}{\sqrt{\pi}}$ centered at $\vec{M}_\epsilon^m(t)$ for $1 \le m \le N$, we have that the kinetic energy in the perforated disk $\tilde{C}_R = C_R \setminus \bigcup_{m=1}^N C_\epsilon^m(t)$ is:

$$\begin{aligned} H_\epsilon[\vec{u}\,] \stackrel{\text{def}}{=} \frac{1}{2}\int_{\tilde{C}_R} u^2 dA &= \frac{1}{2}\int_{\tilde{C}_R} \psi\omega dA - \frac{1}{2}\oint_{\partial\tilde{C}_R} \psi\vec{u}\cdot d\vec{l} \\ &= -\frac{1}{2}\oint_{\partial C_R} \psi\vec{u}\cdot d\vec{l} + \frac{1}{2}\sum_{m=1}^N \oint_{\partial C_\epsilon^m(t)} \psi\vec{u}\cdot d\vec{l}. \end{aligned} \tag{4.72}$$

We have that

$$\begin{aligned} \oint_{\partial C_\epsilon^m(t)} \psi\vec{u}\cdot d\vec{l} &= \oint_{\partial C_\epsilon^m(t)} (\sum_{j=1}^N \psi_j)(\sum_{k=1}^N \vec{u}_k)\cdot d\vec{l} \\ &= \oint_{\partial C_\epsilon^m(t)} \psi_m\vec{u}_m\cdot d\vec{l} \\ &\quad + \sum_{\substack{k=1\\k\neq m}}^N \oint_{\partial C_\epsilon^m(t)} \psi_m\vec{u}_k\cdot d\vec{l} \\ &\quad + \sum_{\substack{j=1\\j\neq m}}^N \oint_{\partial C_\epsilon^m(t)} \psi_j\vec{u}_m\cdot d\vec{l} \\ &\quad + \sum_{\substack{k=1\\k\neq m}}^N \sum_{\substack{j=1\\j\neq m}}^N \oint_{\partial C_\epsilon^m(t)} \psi_j\vec{u}_k\cdot d\vec{l} \end{aligned} \tag{4.73}$$

And we have that

$$\oint_{\partial C_\epsilon^m(t)} \psi_m\vec{u}_m\cdot d\vec{l} = \phi_m(\epsilon)\oint_{\partial C_\epsilon^m(t)} \vec{u}_m\cdot d\vec{l} = -\frac{\Gamma_m^2}{2\pi}\log\epsilon + O(\epsilon). \tag{4.74}$$

$$\oint_{\partial C_\epsilon^m(t)} \psi_m\vec{u}_k\cdot d\vec{l} = \phi_m(\epsilon)\underbrace{\oint_{\partial C_\epsilon^m(t)} \vec{u}_k\cdot d\vec{l}}_{0} = 0. \tag{4.75}$$

$$\begin{aligned} \oint_{\partial C_\epsilon^m(t)} \psi_j\vec{u}_m\cdot d\vec{l} &= \phi_j(\| \vec{M}_\epsilon^m(t) - \vec{M}_\epsilon^j(t) \|)\oint_{\partial C_\epsilon^m(t)} \vec{u}_m\cdot d\vec{l} + O(\epsilon) \\ &= -\frac{\Gamma_m\Gamma_j}{2\pi}\log\| \vec{M}_\epsilon^m(t) - \vec{M}_\epsilon^j(t) \| + O(\epsilon). \end{aligned} \tag{4.76}$$

$$\oint_{\partial C_\epsilon^m(t)} \psi_j\vec{u}_k\cdot d\vec{l} = \phi_j(\| \vec{M}_\epsilon^m(t) - \vec{M}_\epsilon^j(t) \|)\underbrace{\oint_{\partial C_\epsilon^m(t)} \vec{u}_k\cdot d\vec{l}}_{0} + O(\epsilon) = O(\epsilon). \tag{4.77}$$

Therefore

$$\frac{1}{2}\sum_{m=1}^{N}\oint_{\partial C_\epsilon^m(t)}\psi\vec{u}\cdot d\vec{l} = \frac{1}{4\pi}[\sum_{m=1}^{N}\Gamma_m^2]\log\epsilon - \frac{1}{4\pi}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\Gamma_i\Gamma_j\log\parallel \vec{M}_\epsilon^i(t)-\vec{M}_\epsilon^j(t)\parallel + O(\epsilon) \tag{4.78}$$

as $\epsilon \to 0$, so we have that

$$\begin{aligned}\frac{1}{2}\int_{\tilde{C}_R} u^2 dA \quad = \quad & \left.\frac{1}{4\pi}[\int \omega dA]^2 \log R\right\} \quad Blow-up\ at\ \infty \\ & \left.-\frac{1}{4\pi}[\sum_{m=1}^{N}\Gamma_m^2]\log\epsilon\right\} \quad Self-interaction\ blow-up \\ & -\frac{1}{4\pi}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\Gamma_i\Gamma_j\log\parallel \vec{M}_\epsilon^i(t)-\vec{M}_\epsilon^j(t)\parallel + O(\epsilon)+O(R^{-1})\end{aligned} \tag{4.79}$$

as $R \to \infty$ and $\epsilon \to 0$.

The energy inside a small disk centered at $\vec{M}_\epsilon^m(t)$ is the sum of the following three terms:

$$\int_{C_\epsilon^m(t)} u^2 dA = 2\sum_{j=1}^{N}\int_{C_\epsilon^m(t)}\vec{u}_m\cdot\vec{u}_j dA + \sum_{\substack{j=1\\ j\neq m}}^{N}\int_{C_\epsilon^m(t)} u_j^2 dA + \int_{C_\epsilon^m(t)} u_m^2 dA. \tag{4.80}$$

The first summand is order ϵ because $\vec{u}_m$ is tangent to the boundary of the circle $C_r^m(t)$ for $r \leq \dfrac{\epsilon}{\pi}$ and $\vec{u}_j$ differs from a constant vector by a quantity of order ϵ when evaluated in the circle $C_\epsilon^m(t)$. The second summand is of order $N\epsilon^2$. And the third summand is a constant that is independent of $\vec{M}_\epsilon^m(t)$.

The asymptotic expansion of the kinetic energy for $\epsilon \to 0$ and $R \to \infty$ we have just computed indicates the way to proceed to obtain a desingularization of the Hamiltonian. For each ϵ and each R we consider the function

$$\hat{H}(\epsilon, R) \stackrel{\text{def}}{=} H - V_2\log\epsilon + V_1^2\log R, \tag{4.81}$$

where V_1 and V_2 are respectively

$$V_1 \stackrel{\text{def}}{=} \int \omega dA, \tag{4.82}$$

$$V_2 \stackrel{\text{def}}{=} \int \omega^2 dA. \tag{4.83}$$

Since V_1 and V_2 are constants of the motion, the Hamiltonian flow given by the kinetic energy H is the same that the one given by $\hat{H}(\epsilon, R)$ for each ϵ and R. And this function $\hat{H}(\epsilon, R)$ in the limit as $\epsilon \to 0$ produces the nonsingular Hamiltonian of N point vortices .

Theorem 4.2.1

There is an interval of time $(0, T)$ such that $\hat{H}(\epsilon, R)$ converges to

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{N} \Gamma_i \Gamma_j \log \sqrt{(x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2} + M \tag{4.84}$$

where M is independent of $\vec{X}_m(t)$ for $1 \leq m \leq N$, as $\epsilon \to 0$ and $R \to \infty$.

Proof:

We just need to show that V_2 converges to $\sum_{m=1}^{N} \Gamma_m^2$ as $\epsilon \to 0$ and $R \to \infty$. But this follows from the previous theorem in this chapter.

Q.E.D.

4.3 The convergence of the velocity fields.

The expression for the velocity in terms of the vorticity, namely (3.7), blows up when we use ω_ϵ, the ϵ-patches, for the vorticity and let $\epsilon \to 0$. On the other hand, Theorem 4.1.1 proves that the patches converge as $\epsilon \to 0$ to points that move with the Kirchoff velocity. The reconciliation of these two facts is done through the Hamiltonian structure: if we take the limit of the symplectic structure of the coadjoint orbits of the ϵ-patches as $\epsilon \to 0$ we obtain a symplectic structure on the variables (X_m, Y_m) on $\mathbf{R}^{2N}$ such that the Hamiltonian vector field given by the desingularized energy through it is precisely the Kirchoff vector field.

More precisely, from Proposition 2.3.1 we have the expression for the symplectic structure in the coadjoint orbits, namely in the orbit $\mathcal{O}_\omega$ the symplectic structure Υ_ω is defined by the formula

$$\Upsilon_\omega(L_{u_1}\omega, L_{u_2}\omega) = \int \omega(u_1, u_2)dV. \tag{4.85}$$

Using Theorem 4.1.1 we have that

$$\lim_{\epsilon\to 0} \Upsilon_{\omega_\epsilon}(L_{u_1}\omega_\epsilon, L_{u_2}\omega_\epsilon) = \sum_{m=1}^{N} \Gamma_m(dx \wedge dy)(u_1(X_m, Y_m), u_2(X_m, Y_m)). \tag{4.86}$$

Conclusion

We have shown a way of deriving the Kirchoff vector field for N point vortices and its Hamiltonian structure from the Euler equations and the symplectic structure of the orbits where the vorticities we consider live. This is done by describing an asymptotic decoupling of the evolution of the singularities of the flow from the rest of it, the internal degrees of freedom.

Marsden and Weinstein [**?**] undertook this question and, although their approach explained the way of getting the Hamiltonian structure, it failed to provide a rigorous way of getting the Hamiltonian function for the point vortices from the kinetic energy of the fluid. Marchioro and Pulvirenti [**?**] gave a precise formulation for the derivation of the velocity field of the point vortices but they do not address the question of the Hamiltonian function. We have presented a complete and systematic way of obtaining the Hamiltonian evolution of N point vortices from the evolution of N patches of vorticity. The ingredients we have used were in the literature. What is new is the way they are combined to obtain an asymptotic relation between the Kirchoff system and the Euler equations.

In our future research we intend to apply this asymptotic desingularization process to other singular distributions of vorticity – vortex rings, knotted vortex lines – and to other singular solutions of Hamiltonian partial differential equations.