Multiple-valued complex functions and computer algebra

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Abstract

We first discuss some elementary, but not very well-known facts about multiple-valued complex functions. We then look at how the computer algebra systems Derive, Maple, Mathematica and Reduce handle these problems.

1 Introduction

I recently taught a course on complex analysis. That forced me to think more carefully about branch cuts. Being interested in computer algebra, it was only natural that I wanted to see how such programs dealt with these problems. I was also inspired by a paper by Stoutemyer [4].

At the moment, the use of computers in mathematical research and teaching is getting a lot of attention. Many mathematician seem to have caught a "computer virus". Computers can be as addictive as alcohol to mathematicians, and a lot of us seem to think that computers is a panacea for both research and teaching. The bureaucrats prefer spending money on buying equipment rather than hiring people, and the vendors of the computer algebra systems bombard us with evangelism. The result is that a lot of people, especially outsiders, seem to envision that mathematicians will soon be made redundant by computers.

But computer algebra systems have their fair share of problems, too. In particular, branch cuts are somewhat of an Achilles' heel for them. We will look at some examples that present problems for these programs. It is also interesting to see what the documentation for the different programs have to say about branch cuts. Some of them almost refuse to talk about it, while others try to point out some of the potential problems.

So while computer algebra systems undoubtedly are of great value for both research and teaching, they still need some help. It turns out that what is needed is "esoteric" pure mathematics like branch cuts, demonstrating once again the centrality of pure mathematics.

In Section 2, we will consider some transformation rules for multiple-valued complex functions in painstaking detail, and in Section 3, we will see how Derive, Maple, Mathematica and Reduce handle some of these rules.

Since computer algebra is a commercial activity, I need to make some disclaimers. This brief paper is not intended to be a comprehensive review of computer algebra systems. That would require a broader selection of test problems and looking at more of the

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available programs. The paper does not even attempt to be a review of how the four programs I chose to use handle branch problems. That would require looking at how they cope with branch cuts inside more complicated problems. I merely wanted to see how these programs dealt with some basic transformation rules for multiple-valued complex functions, and to what extent these rules were affected by changing the domain.

It turns out that the programs perform very differently. But the fact that one program is very good at this kind of problems, doesn't necessarily mean that it is a better computer algebra program than the others.

Just for the record, I'm a regular user of Maple and Mathematica, while this was my first exposure to Derive and Reduce.

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2 Basic properties of multiple-valued complex functions

For fear that this section may seem utterly trivial, I will start with the following paradox due to the Danish mathematician Thomas Clausen [1, 2].

Let n be an integer. Then

$$e^{1+2n\pi i} = e. \tag{1}$$

If we write

$$\left(e^{1+2n\pi i}\right)^{1+2n\pi i} = e^{1+2n\pi i} = e, \tag{2}$$

and

$$\left(e^{1+2n\pi i}\right)^{1+2n\pi i} = e^{1+4n\pi i-4n^2\pi^2} = ee^{-4n^2\pi^2}, \tag{3}$$

it follows that

$$e^{-4n^2\pi^2} = 1. \tag{4}$$

There are also a number of paradoxes involving square roots. Let me just give two.

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i\,i = -1, \tag{5}$$

and

$$\begin{aligned}\frac{1}{-1} &= \frac{-1}{1}\\ \frac{1}{\sqrt{-1}} &= \frac{\sqrt{-1}}{1}\\ \frac{1}{i} &= \frac{i}{1}\\ i^2 &= 1.\end{aligned} \tag{6}$$

In order to clarify such problems, we will take a fairly detailed look at some properties of elementary transcendental functions.

For $z = x + iy$, the complex exponential function is defined by

$$e^z = e^x(\cos y + i\sin y).$$

It satisfies the property $e^{(u+v)} = e^u e^v$, but does it satisfy the property $(e^u)^v = e^{(uv)}$? In order to answer this, we must look at the complex logarithm function.

We define the principal argument by $z = |z|e^{i\arg(z)}$ and $\arg(z) \in (-\pi, \pi]$. We do not define the principal argument of 0, and we will from now on assume that z is different from 0. We define the principal argument on the negative axis, too, but it is of course not continuous there. Extending the definition of the principal argument to the negative numbers gives us as a ready supply of counter examples.

We then define $\log(z) = \log|z| + i\arg(z)$, where $\log|z|$ denotes the usual real logarithm of $|z|$. We clearly have $e^{\log(z)} = z$, but do we have $\log(e^z) = z$?

In order to study this, we will introduce the following terminology.

Definition 1 *Define the imaginary remainder* $\operatorname{Imr}(z)$ *and the imaginary quotient* $\operatorname{Imq}(z)$ *by*

$$\operatorname{Im}(z) = \operatorname{Imr}(z) + 2\pi \operatorname{Imq}(z),$$

where $\operatorname{Imr}(z) \in (-\pi, \pi]$ *and* $\operatorname{Imq}(z) \in \mathbb{Z}$.

Notice that if we let $f(x)$ be the left-continuous integer part function ($f(x) = [x]$ if $x \notin \mathbb{Z}$ and $f(x) = x - 1$ if $x \in \mathbb{Z}$), then $\operatorname{Imq}(z) = f((\operatorname{Im}(z) + \pi)/2\pi)$.

We can now prove the following.

Theorem 2 *We have*

$$\log(e^z) = \operatorname{Re}(z) + i \operatorname{Imr}(z).$$

In particular,

$$\log(e^z) = z \quad \textit{if and only if} \quad \operatorname{Im}(z) \in (-\pi, \pi].$$

Proof: We have

$$\log(e^z) = \log|e^z| + i\arg(e^z) = x + i\arg(e^{i\operatorname{Im}(z)}) =$$
$$x + i\arg(e^{i(\operatorname{Imr}(z) + 2\pi\operatorname{Imq}(z))}) = x + i\arg(e^{i\operatorname{Imr}(z)}) = x + i\operatorname{Imr}(z). \quad \square$$

We will next study whether the complex logarithm satisfies the property $\log(uv) = \log(u) + \log(v)$. To this end, we must first study $\arg(uv)$. It is easy to see that

$$\arg(uv) = \arg(u) + \arg(v) + d2\pi, \qquad \text{where } d \text{ is 0 or } \pm 1.$$

We now make the following definitions.

Definition 3 *Define the principal product excess* $\operatorname{ppe}(u, v)$ *of two complex numbers by*

$$\operatorname{ppe}(u, v) = (\arg(uv) - \arg(u) - \arg(v))/(2\pi).$$

Definition 4 *Define the complex sign* $\operatorname{csgn}(z)$ *of a complex number by*

$$\operatorname{csgn}(z) = \begin{cases} 1, & \textit{if } \operatorname{Re}(z) > 0 \textit{ or } (\operatorname{Re}(z) = 0 \textit{ and } \operatorname{Im}(z) > 0) \\ 0, & \textit{if } z = 0 \\ -1, & \textit{if } \operatorname{Re}(z) < 0 \textit{ or } (\operatorname{Re}(z) = 0 \textit{ and } \operatorname{Im}(z) < 0). \end{cases}$$

Definition 5 *Define the right (left) half-plane as the set of points where* $\operatorname{csgn}(z)$ *is positive (negative).*

The next lemma is immediate.

Lemma 6

1. $\mathrm{ppe}(u, v)$ *is always* 0 *or* ± 1.

2. *If either* u *or* v *is positive, then* $\mathrm{ppe}(u, v) = 0$.

3. *If both* u *and* v *lie in the right half-plane, then* $\mathrm{ppe}(u, v) = 0$.

4. *If both* u *and* v *lie in the left half-plane, then* $\mathrm{ppe}(u, v) \neq 0$.

5. $\mathrm{ppe}(z, z) = 0$ *if and only if* z *lies in the right half-plane.*

We can now prove the following.

Theorem 7 *We have*

$$\log(uv) = \log(u) + \log(v) + 2\pi i \,\mathrm{ppe}(u, v).$$

Proof:

$$\begin{aligned} \log(uv) &= \log|uv| + i \arg(uv) \\ &= \log|u| + \log|v| + i(\arg(u) + \arg(v) + 2\pi \,\mathrm{ppe}(u, v)) \\ &= \log(u) + \log(v) + 2\pi i \,\mathrm{ppe}(u, v). \quad \square \end{aligned}$$

Corollary 8 *We have* $\log(z^2) = 2\log(z)$ *if and only if* z *lies in the right half-plane.*

Proof: Follows from Theorem 7 and Lemma 6. $\square$

We are also interested in quotients. We need the following lemma.

Lemma 9 *We have*

$$\arg(1/z) = \begin{cases} -\arg(z), & \text{if } z \text{ is not negative} \\ -\arg(z) + 2\pi, & \text{if } z \text{ is negative.} \end{cases}$$

Proof: We have $1/z = \overline{z}/|z|^2$, but $\arg(\overline{z}) = -\arg z$ unless z is negative, in which case both $\arg(z)$ and $\arg(1/z)$ are equal to π. $\square$

From this we can deduce the next theorem.

Theorem 10 *We have*

$$\log(1/z) = \begin{cases} -\log(z), & \text{if } z \text{ is not negative} \\ -\log(z) + 2\pi i, & \text{if } z \text{ is negative.} \end{cases}$$

We are now ready to define the complex power and exponential functions.

Definition 11 *We define the complex power and exponential functions by*

$$z^a = e^{\log(z) a}, \qquad a^z = e^{\log(a) z}, \quad \text{for } a \neq e.$$

Notice that e^z is a single-valued function, while a^z is multiple-valued for $a \neq e$.

We are now ready to consider whether $(e^u)^v$ equals e^{uv}. The key issue is that the first expression involves the exponential function with base e^u and not just e. So while the right hand side is a single-valued function, the left hand side is a multiple-valued function, depending on the value of $\log(e^u)$.

Theorem 12 *We have*

$$(e^u)^v = e^{uv} e^{-v 2\pi i \,\mathrm{Imq}(u)}.$$

Proof:

$$(e^u)^v = e^{\log(e^u)v} = e^{(\mathrm{Re}(u)+i\,\mathrm{Imr}(u))v} = e^{(u-i2\pi\,\mathrm{Imq}(u))v} = e^{uv}e^{-vi2\pi\,\mathrm{Imq}(u)} \quad \square$$

Using Theorem 12, we can easily resolve Clausen's paradox. In equation (3) we said that

$$(e^{1+2n\pi i})^{1+2n\pi i} = e^{1+4n\pi i-4n^2\pi^2}.$$

This should be replaced by

$$(e^{1+2n\pi i})^{1+2n\pi i} = e^{1+4n\pi i-4n^2\pi^2}e^{-(1+2n\pi i)2\pi i\,\mathrm{Imq}(1+2n\pi i)} =$$
$$e^{1-4n^2\pi^2}e^{-(1+2n\pi i)2\pi in} = e^{1-4n^2\pi^2}e^{-2n\pi i+4n^2\pi^2} = e,$$

which agrees with equation (2).

We can also prove the following corollary.

Corollary 13 *We have*

$$(e^z)^{1/2} = (-1)^{\mathrm{Imq}(z)}e^{z/2}.$$

In particular,

$$(e^z)^{1/2} = e^{z/2} \qquad \textit{if and only if } \mathrm{Im}(z) \in ((4n-1)\pi, (4n+1)\pi],\ n \in \mathbb{Z}.$$

We also have the following immediate generalization.

Theorem 14 *We have*

$$(a^u)^v = a^{uv}e^{-v2\pi i\,\mathrm{Imq}(u\log(a))}.$$

We can now ask similar questions about the power function.

Theorem 15 *We have*

$$(uv)^a = u^a v^a e^{a2\pi i\,\mathrm{ppe}(u,v)}.$$

Proof:

$$(uv)^a = e^{a\log(uv)} = e^{a(\log(u)+\log(v)+2\pi i\,\mathrm{ppe}(u,v))} = u^a v^a e^{a2\pi i\,\mathrm{ppe}(u,v)}. \quad \square$$

We will derive some consequences of Theorem 15.

Theorem 16 *We have*

$$\sqrt{uv} = (-1)^{\mathrm{ppe}(u,v)}\sqrt{u}\sqrt{v}.$$

In particular,

$$\sqrt{u^2} = \mathrm{csgn}(u)u,$$

so

$$\sqrt{u^2} = u \quad \textit{if and only if } u \textit{ lies in the right half-plane.}$$

We will finish this section with the following theorem.

Theorem 17 *We have*

$$\sqrt{1/z} = \begin{cases} 1/\sqrt{z}, & \textit{if } z \textit{ is not negative} \\ -1/\sqrt{z}, & \textit{if } z \textit{ is negative.} \end{cases}$$

In particular, if z is real, then

$$\sqrt{1/z} = \mathrm{sgn}(z)/\sqrt{z},$$

Proof: If z is not negative, we have

$$\sqrt{1/z} = e^{\log(1/z)/2} = e^{-\log(z)/2} = 1/\sqrt{z},$$

while if z is negative, we have

$$\sqrt{1/z} = e^{(\log(1/z))/2} = e^{(-\log(z)+2\pi i)/2} = -e^{-log(z)/2} = -1/\sqrt{z}.$$

The last two results resolve the two square root paradoxes given at the beginning of this section.

3 Computer test

Computer algebra systems are in general much better at reducing the difference between two equivalent expressions to 0, than simplifying an expression to a specific form. We will therefore consider the following 8 tests.

Test 1

(a) $\sqrt{uv} - \sqrt{u}\sqrt{v}$ *should not simplify when* u *and* v *are complex.*

(b) $\sqrt{uv} - \sqrt{u}\sqrt{v}$ *should simplify to* 0 *when* u *and* v *are both positive.*

Test 2

(a) $\sqrt{z^2}$ *should not simplify, or simplify to* $\mathrm{csgn}(z)z$ *when* z *is complex.*

(b) $\sqrt{z^2}$ *should not simplify, or simplify to* $\mathrm{sgn}(z)z = |z|$ *when* z *is real.*

(c) $\sqrt{z^2}$ *should simplify to* z *when* z *is positive.*

Test 3

(a) $\sqrt{1/z} - 1/\sqrt{z}$ *should not simplify when* z *is complex.*

(b) $\sqrt{1/z} - 1/\sqrt{z}$ *should not simplify, or simplify to* $(\mathrm{sgn}(z) - 1)/\sqrt{z}$ *when* z *is real.*

(c) $\sqrt{1/z} - 1/\sqrt{z}$ *should simplify to* 0 *when* z *is positive.*

Test 4

(a) $\sqrt{e^z} - e^{z/2}$ *should not simplify when* z *is complex.*

(b) $\sqrt{e^z} - e^{z/2}$ *should simplify to* 0 *when* z *is real.*

Test 5

(a) $\log(uv) - \log(u) - \log(v)$ *should not simplify when* u *and* v *are complex.*

(b) $\log(uv) - \log(u) - \log(v)$ *should simplify to* 0 *when* u *and* v *are both positive.*

Test 6

(a) $\log(z^2) - 2\log(z)$ *should not simplify when* z *is complex.*

(b) $log(z^2) - 2\log(z)$ *should simplify to* 0 *when* z *is positive.*

Test 7

(a) $\log(1/z) + \log(z)$ *should not simplify when z is complex.*

(b) log$(1/z) + \log(z)$ *should simplify to* 0 *when z is positive.*

Test 8

(a) $\log(e^z)$ *should not simplify when z is complex.*

(b) log(e^z) *should simplify to z when z is real.*

Some people have tried to grade the performance of computer algebra systems. I find this very difficult to do. I'm often not sure what I should consider right or wrong. Some programs simplify expressions automatically, while others do so once you use the `simplify` command. Sometimes you can control the behaviour by using a special option to the `simplify` command, or a different command such as `PowerExpand`. In some programs you can explicitly restrict the domain of a variable, use statements like `on expandlogs` or program your own transformation rules to change the behaviour.

This raises the question of whether I should look at the default behaviour of the program, using just automatic simplification or a generic `simplify` command, or whether I should look at the way an expert can tweak the program. I have chosen to just briefly describe the behaviour of the different programs, and not be judgmental.

Derive 3.0 has a declare variable domain function. The default is to consider variables to be real. As a result, in Tests 2, 3, 4 and 8 it simplifies the expressions as if z were real, but if we declare z to be complex, it doesn't simplify. Otherwise it does everything correctly. Derive has several options for controlling how the program deals with branches, and the manual discusses branches in some depth.

Maple V Release 3 takes a conservative approach in the three root problems. The default is to not simplify in Tests 1, 2 and 3. In 1b the `simplify` command does not transform $\sqrt{uv} - \sqrt{u}\sqrt{v}$ to 0 when u and v are assumed to be positive, but by using the `power` option it works. But if I do the similar problem for the n-th root, `simplify` does transform automatically when the numbers are positive. But in Tests 4 to 8, Maple simplifies right away, even if the variables are assumed to be complex.

Mathematica 2.2 does not have the ability to restrict the domain of a variable. It takes a conservative approach and refuses to do any simplification in any of the tests when using the `Simplify` command, but if we use the `PowerExpand` command, the program simplifies as if the numbers were positive.

Reduce 3.5 has the options `on complex` and `on expandlogs`, but the default is to treat variables as real, but to not expand $\log(ab)$. But it still simplifies $\log e^z$ in Test 8, even if we use `on complex`.

References

[1] Thomas Clausen, Aufgabe, *J. Reine Angew. Math.* **2** (1827), 286–287.

[2] Reinhold Remmert, *Theory of Complex Functions*, Graduate Texts in Math., vol. 122, Springer-Verlag, 1991.

[3] Soft Warehouse, Inc., *Derive, Version 2.51*, Honolulu, Hawaii, 1992.

[4] David R. Stoutemyer, Crimes and misdemeanors in the computer algebra trade, *Notices Amer. Math. Soc.* **38** (1991), 778–785.

[5] Waterloo Maple Software, *Maple V Release 3*, Waterloo, Canada, 1994.

[6] Michael Wester, *A review of CAS mathematical capabilities*, to appear in Proceedings Fourth Pan American Conference of Applied Mechanics, Bueno Aires, 1995.

[7] Wolfram Research, *Mathematica, Version 2.1,* Champaign, Illinois, 1992.