Comparison between the Discrete and Continuous Time Models

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1 Discrete to Continuous

Recall the discrete time model

\[
\dot{I} = AIS
\]
\[
\dot{S} = S - \dot{I}.
\]

These equations tell us how the population changes from one day to the next. We generated a sequence of populations by repeated application of these formulae. We can add an equation to describe the removed population

\( \dot{R} = R + I \). \hspace{1cm} (1)

Let’s number the population on the \( n \)th day with a subscript so that \( I_n \) and \( S_n \) are the number of infective people and the number of susceptible people on the \( n \)th day. The sequence in the table follows the relations

\[
I_{n+1} = AI_n S_n
\]
\[
S_{n+1} = S_n - AI_n S_n
\]
\[
R_{n+1} = R_n + I_n,
\]

with \( I_0, S_0 \) and \( R_0 \) given. Note that the total population size is fixed since \( I_{n+1} + S_{n+1} + R_{n+1} = I_n + S_n + R_n \). In deriving this model, we made the assumption that the sampling period was the same as the period of infectiousness (1 day). We will now relax this assumption.
Suppose, as before, those leaving the susceptible class enter directly into the infective class, but that a proportion, say $B$, of infectives remain infective at the end of the sampling period. Then the equations become

$$\begin{align*}
I_{n+1} &= AI_n S_n + BI_n, \\
S_{n+1} &= S_n - AI_n S_n, \\
R_{n+1} &= R_n + (1 - B)I_n.
\end{align*}$$

The constant $A$ measures the probability of catching the disease during the sampling period.

In order to connect with the continuous time equations, we want to consider the limit of infinitesimal sampling intervals. Let $h$ be a small sampling interval. (We will look a the limit as $h \to 0$.) Let $S(t)$, $I(t)$ and $R(t)$ be smooth functions of $t$ and suppose $S_n = S(nh)$, $I_n = I(nh)$ and $R_n = R(nh)$. The values of the constants $A$ and $B$ depend on the sampling interval and should be rescaled, $A = ah$ and $B = 1 - bh$, for some constants $a$ and $b$. Let $t = nh$, we have

$$\begin{align*}
I(t + h) &= ahI(t)S(t) + (1 - bh)I(t) \to \frac{1}{h} \left( I(t + h) - I(h) \right) = aI(t)S(t) - bI(t), \\
S(t + h) &= S(t) - ahI(t)S(t) \to \frac{1}{h} \left( S(t + h) - S(h) \right) = -aI(t)S(t), \\
R(t + h) &= R(t) + bhI(t) \to \frac{1}{h} \left( R(t + h) - R(h) \right) = bI(t).
\end{align*}$$

In the limit as the sampling interval goes to zero ($h \to 0$), we obtain the SIR model

$$\begin{align*}
\frac{dI}{dt} &= aI(t)S(t) - bI(t), \\
\frac{dS}{dt} &= -aI(t)S(t), \\
\frac{dR}{dt} &= bI(t).
\end{align*}$$

2 Continuous to Discrete

We will look at two different ways in which the continuous-time equations are related to the discrete-time equations. In both approaches, we will start with the continuous-time SIR model and determine related discrete-time equations. In the first approach, let’s start with the equations for $S$ and $I$, divide the $S$ equation by $S$ and the $I$
equation by $I$ to get

$$\frac{1}{S(t)} \frac{dS}{dt} = -aI(t)$$

$$\frac{1}{I(t)} \frac{dI}{dt} = aS(t) - b$$

$$\frac{dR}{dt} = bI(t).$$

Integrate these equations over an interval from $t$ to $t + \Delta t$

$$\int_t^{t+\Delta t} \frac{1}{S(t)} \frac{dS}{dt} \, dt = -\int_t^{t+\Delta t} aI(t) \, dt$$

$$\int_t^{t+\Delta t} \frac{1}{I(t)} \frac{dI}{dt} \, dt = \int_t^{t+\Delta t} (aS(t) - b) \, dt$$

$$\int_t^{t+\Delta t} \frac{dR}{dt} \, dt = \int_t^{t+\Delta t} bI(t) \, dt.$$

The left-hand side of these equations can be integrated exactly to give

$$\ln \left( \frac{S(t + \Delta t)}{S(t)} \right) = -\int_t^{t+\Delta t} aI(t) \, dt$$

$$\ln \left( \frac{I(t + \Delta t)}{I(t)} \right) = \int_t^{t+\Delta t} (aS(t) - b) \, dt$$

$$R(t + \Delta t) - R(t) = \int_t^{t+\Delta t} bI(t) \, dt.$$

Now, take the exponential of both sides of the first two equations

$$S(t + \Delta t) = S(t) \exp \left[ -\int_t^{t+\Delta t} aI(t) \, dt \right]$$

$$I(t + \Delta t) = I(t) \exp \left[ \int_t^{t+\Delta t} (aS(t) - b) \, dt \right]$$

$$R(t + \Delta t) - R(t) = \int_t^{t+\Delta t} bI(t) \, dt.$$

So far, we have not made any approximations. But, we cannot integrate the right-hand sides exactly; so approximate the integrals using a left-hand sum. That is, we
approximate the integrand by a constant value - the value at the lower limit - to obtain

\[ S(t + \Delta t) \approx S(t) \exp \left( -aI(t)\Delta t \right) \]
\[ I(t + \Delta t) \approx I(t) \exp \left( (aS(t) - b)\Delta t \right) \]
\[ R(t + \Delta t) - R(t) = bI(t)\Delta t. \]

If we take \( \Delta t = 1 \), we can write the above equations as

\[ S_{t+1} = S_t \exp \left[ -aI_t \right] \]
\[ I_{t+1} = I_t \exp \left[ aS_t - b \right] \]
\[ R_{t+1} = R_t + bI_t. \]

A biological interpretation of these equations can be given. If we assume that the probability of a susceptible becoming infected is Poisson distributed with mean \( aI_t \), then \( \exp(-aI_t) \) is just the zero term of the Poisson distribution (the probability of not getting infected). Of course, \( \Delta t = 1 \) is taken to be the length of the infectious period. If we expand the exponential terms and neglect higher order terms (assuming infection prevalence is small), we obtain

\[ S_{t+1} = S_t \left[ 1 - aI_t \right] = S_t - aI_t S_t \]
\[ I_{t+1} = I_t \left[ 1 + aS_t - b \right] = aI_t S_t + (1 - b)I_t \]
\[ R_{t+1} = R_t + bI_t. \]

These are the same discrete equations as in equations (2) with \( A = a \) and \( B = 1 - b \) (since \( \Delta t = 1 \)).

In our second approach, let’s now start from the continuous SIR model and examine discretizations that we use when we approximate solutions numerically. Let’s first remind ourselves what Euler’s method is for solving ordinary differential equations. In the simplest case of a scalar equation

\[ \frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0, \quad (3) \]

Euler’s method is

\[ Y_{n+1} = Y_n + hf(t_n, Y_n) \]
\[ t_{n+1} = t_n + h \quad (4) \]
Recall that the solution to a single differential equation with an initial condition is a function, \( y(t) \) with \( y(t_0) = y_0 \). We have used the notation \( Y_n \) as the approximation to \( y(t_n) \). We start this iteration with the given initial conditions \((t_0, Y_0 = y_0)\).

Euler’s method is based on using a tangent line over a finite interval of size \( h \) to approximate the exact function. Let’s call \( \hat{y}(t, t_0) \) the tangent line to the exact solution at the base point \( t = t_0 \), given as a function of \( t \). The equation of the tangent line is

\[
\hat{y}(t, t_0) = y(t_0) + f(t_0, y(t_0))(t - t_0).
\]

Since we start our approximation with the exact initial condition, \( Y_0 = y_0 \) and we can rewrite the above as

\[
\hat{y}(t, t_0) = Y_0 + f(t_0, Y_0)(t - t_0).
\]

If we follow the tangent line from \((t_0, Y_0)\) for a time interval of size \( h \), we obtain

\[
\hat{y}(t_0 + h, t_0) = Y_0 + f(t_0, Y_0)h. \quad (7)
\]

Euler’s method sets \( t_1 = t_0 + h \) and \( Y_1 = \hat{y}(t_1, t_0) = Y_0 + f(t_0, Y_0)h \).

For the next step, we construct the tangent line to the solution of the differential equation that goes through the point \((t_1, Y_1)\). The tangent line is

\[
\hat{y}(t, t_1) = Y_1 + f(t_1, Y_1)(t - t_1). \quad (8)
\]

If we follow the tangent line from \((t_1, Y_1)\) for a time interval of size \( h \), we obtain

\[
\hat{y}(t_1 + h, t_1) = Y_1 + f(t_1, Y_1)h. \quad (9)
\]

Euler’s method sets \( t_2 = t_1 + h \) and \( Y_2 = \hat{y}(t_2, t_1) = Y_1 + f(t_1, Y_1)h \). The process is repeated to give the general formula (4). The process is illustrated in Fig. 1.

The same process can be applied to obtain approximate solutions to systems of equations. Again, the differential equations for the SIR model are

\[
\frac{dS}{dt} = -aI(t)S(t)
\]
\[
\frac{dI}{dt} = aI(t)S(t) - bI(t)
\]
\[
\frac{dR}{dt} = bI(t),
\]

with initial conditions \( S(0) = S_0, I(0) = I_0 \) and \( R(0) = N_0 - S_0 - I_0 \). Apply Euler’s method to the first equation. In this case, \( t_0 = 0 \). The tangent line through \((0, S_0)\) is

\[
\hat{S}(t, 0) = S_0 - aI_0S_0(t - 0). \quad (10)
\]
Figure 1: Illustration of Euler’s method for a scalar equation.

Follow the tangent line from \((0, S_0)\) for a time interval of size \(h\)

\[
\hat{S}(0 + h, 0) = S_0 - aI_0S_0h. \tag{11}
\]

Set \(t_1 = 0 + h = h\) and \(S_1 = \hat{S}(0 + h, 0) = S_0 - aI_0S_0h.\)

Similarly, for the second equation, the tangent line through \((0, I_0)\) is

\[
\hat{I}(t, 0) = I_0 + [aI_0S_0 - bI_0](t - 0). \tag{12}
\]

Follow the tangent line from \((0, I_0)\) for a time interval of size \(h\)

\[
\hat{I}(0 + h, 0) = I_0 + aI_0S_0h - bI_0h. \tag{13}
\]

Set \(I_1 = \hat{I}(0 + h, 0) = I_0 + aI_0S_0h - bI_0h.\)

For the third equation, the tangent line through \((0, R_0)\) is

\[
\hat{R}(t, 0) = R_0 + bI_0(t - 0). \tag{14}
\]

Follow the tangent line from \((0, R_0)\) for a time interval of size \(h\)

\[
\hat{R}(0 + h, 0) = R_0 + bI_0h. \tag{15}
\]
Set $R_1 = R_0 + bI_0h$.

We now have an approximate solution at time $t_1 = h$ that is $S_1$, $I_1$, and $R_1$. We can now construct a tangent line for each equation through the corresponding points $(t_1, S_1)$, $(t_1, I_1)$, and $(t_1, R_1)$. The tangent lines are

\[
\begin{align*}
\hat{S}(t, t_1) &= S_1 - aI_1S_1(t - t_1) \\
\hat{I}(t, t_1) &= I_1 + \left[aI_1S_1 - bI_1\right](t - t_1) \\
\hat{R}(t, t_1) &= bI_1(t - t_1).
\end{align*}
\]  

(16)

Follow the tangent line from the points $(t_1, S_1)$, $(t_1, I_1)$, and $(t_1, R_1)$ for a time interval of size $h$ to obtain

\[
\begin{align*}
\hat{S}(t_1 + h, t_1) &= S_1 - aI_1S_1h \\
\hat{I}(t_1 + h, t_1) &= I_1 + \left[aI_1S_1 - bI_1\right]h \\
\hat{R}(t_1 + h, t_1) &= bI_1h.
\end{align*}
\]  

(17)

Set $t_2 = t_1 + h$ and

\[
\begin{align*}
S_2 &= S_1 - aI_1S_1h \\
I_2 &= I_1 + \left[aI_1S_1 - bI_1\right]h \\
R_2 &= bI_1h.
\end{align*}
\]  

(18)

We now recognize the pattern

\[
\begin{align*}
S_{n+1} &= S_n - ahI_nS_n \\
I_{n+1} &= I_n + ahI_nS_n - bhI_n \\
R_{n+1} &= bhI_n \\
t_{n+1} &= t_n + h.
\end{align*}
\]  

(19)

Since $t_0 = 0$, we have $t_n = nh$, and $S_n$, $I_n$, and $R_n$ are approximations to $S(nh)$, $I(nh)$, $R(nh)$. Obviously, if we set $A = ah$ and $B = 1 - bh$ we return to the discrete-time equations (2).

We have seen that the discrete-time equations and the continuous-time, ordinary differential equations for the SIR model are related. Taking the limit of a small sampling period in the discrete equations gives the ordinary differential equations. The application of Euler’s method is an example of what is called discretizing the ordinary differential equations. Euler’s method gives discrete equations to solve numerically that approximate the solutions to the ordinary differential equations. The approximation gets better as $h$, the time step size, is reduced. Euler’s method is called first order accurate because differences between the approximation and
exact solution to the ordinary differential equations decrease to zero as \( h \) to the first power. This relation is called the convergence rate. More sophisticated numerical methods can be constructed that give rise to different discrete equations with better convergence rates.

Things to try.

1. Solve the continuous-time ordinary differential equations for the SIR model using Euler’s method with different step sizes \( h \). Approximate the error in your solutions. Do you observe first order convergence?

2. Try some other numerical methods for solving the equations. What does the error look like for these methods? What is the convergence rate?