Wavelet characterization of Sobolev norms*

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Here we quickly review the Sobolev space. And then we observe the use of wavelets in defining equivalent norms on L^2-Sobolev spaces. Also we'll discuss about some other spaces.

Sobolev space

Sobolev space is a vector space of functions equipped with a norm that is a combination of L^p norms of the function itself as well as its derivatives up to a given order. The derivatives are understood in a suitable weak sense to make the space complete, thus a Banach space. We begin with the classical definition of Sobolev spaces.

Definition 1. *Let k be a nonnegative integer and let $1 < p < \infty$. The* Sobolev space $W^{k,p}(\mathbb{R}^n)$ *is defined as the space of functions f in $L^p(\mathbb{R}^n)$ all of whose distributional derivatives $\partial^\alpha f$ are also in $L^p(\mathbb{R}^n)$ for all multi-indices α that satisfy $|\alpha| \le k$. This space is normed by the expression*

$$\|f\|_{W^{k,p}} = \sum_{|\alpha|\le k} \|\partial^\alpha f\|_{L^p}$$

where $\partial^{(0,\dots,0)} f = f$.

For the simplicity and convenience of discuss, we will only deal in the case of one dimensional. In the one-dimensional case it is enough to assume $f^{(k-1)}$ is differentiable almost everywhere and is equal almost everywhere to the Lebesgue integral of its derivative. Also, one of the most elegant and useful ways of measuring differentiability properties of functions is in terms of L^2 norms. One of the reason for this is L^2 is a Hilbert space and the other is the Fourier transform is unitary isomorphism on L^2. From now, we only deal with

$$W^k(\mathbb{R}) = W^{k,2}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \,:\, f^{(n)} \in L^2(\mathbb{R}) \ \text{ for all } \ 0 < n \le k\},$$

where the derivatives are in the weak sense; that is, it is a function $f^{(n)}$ such that

$$\int_{\mathbb{R}} f^{(n)}(x)\varphi(x)dx = (-1)^n \int_{\mathbb{R}} f(x)\varphi^{(n)}(x)dx$$

for every test function $\varphi \in \mathcal{S}$. This definition of W^k makes its meaning clear, but there is an equivalent characterization of W^k in terms of the Fourier transform.

Theorem 2. *$f \in W^k$ if and only if $(1+|\xi|^2)^{k/2}\widehat{f} \in L^2$, and the norms*

$$f \to \left[\sum_{n=0}^{k} \|f^{(n)}\|_{L^2}^2\right]^{1/2} \quad \textit{and} \quad f \to \left[\int |\widehat{f}(\xi)|^2(1+|\xi|^2)^k d\xi\right]^{1/2}$$

are equivalent.

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Proof. Since $(f^{(n)})\widehat{\ }(\xi) = (2\pi i\xi)^n \widehat{f}(\xi)$, the Plancherel's theorem implies that

$$\sum_{n=0}^{k} \|f^{(n)}\|_{L^2}^2 = \sum_{n=0}^{k} \int |\widehat{f}(\xi)|^2 |(2\pi\xi)^n|^2 d\xi \,,$$

so the theorem amounts to proving that the quantities $\sum_{n=0}^{k} |\,\xi^n|^2$ and $(1+|\,\xi|^2)^k$ are comparable, i.e., that each is bounded by a constant multiple of the other. But it is clear that $|\,\xi^n| \le 1$ for $|\,\xi| \le 1$, and $|\,\xi^n| \le |\,\xi|^n$ for $|\,\xi| > 1$ and $0 \le n \le k$, so

$$\sum_{n=0}^{k} |\,\xi^n|^2 \le C_1 \max(1, |\,\xi|^{2k}) \le (1+|\,\xi|^2)^k \,.$$

On the other hand, since $|\,\xi|^{2k} \le \sum_{n=1}^{k} |\,\xi^n|^2$,

$$(1+|\,\xi|^2)^k \le 2^k \max(1, |\,\xi|^{2k}) \le 2^k(1+|\,\xi|^{2k}) \le 2^k C_2 \left(1 + \sum_{n=1}^{k} |\,\xi^n|^2\right) = 2^k C_2 \sum_{n=0}^{k} |\,\xi^n|^2 \,.$$

Thus, proof is completed. $\square$

This suggests a generalization of W^k in which k is replaced by an arbitrary real number s. Namely, if $u(\xi)$ is a function on $\mathbb{R}$ such that $(1+|\,\xi|^2)^{s/2}u(\xi) \in L^2$, then $u\phi \in L^1$ for any $\phi \in \mathcal{S}(\mathbb{R})$, so u is a tempered distribution (whose action on $\phi \in \mathcal{S}$ is $\int u\phi$). Since the Fourier transform maps tempered distributions into tempered distributions, we can defined the *Sobolev space of order s*:

$$W^s(\mathbb{R}) = \left\{ f \in \mathcal{S}'(\mathbb{R}) \,:\, \widehat{f} \text{ is a function and } \|f\|_{W^s}^2 \equiv \int |\widehat{f}(\xi)|^2 (1+|\,\xi|^2)^s d\xi < \infty \right\}.$$

The norm $\|\cdot\|_{W^s}$ on W^s thus defined is called the *Sobolev norm of order s*. Theorem 1 shows that this definition agrees with the previous one when s is a nonnegative integer. In particular, $W^0 = L^2$. Now we'll state the very useful lemma.

Lemma 3. (Sobolev Embedding Theorem) *If $s > k + \frac{1}{2}$, then $W^s \subset C^k$ and there is a constant $C = C_{s,k}$ such that*

$$\sup_{n \le k} \sup_{x \in \mathbb{R}} |f^{(n)}(x)| \le C\|f\|_{W^s} \,.$$

Proof. By the Fourier inversion theorem, if $(f^{(n)})\widehat{\ } \in L^1$ then $f^{(n)}$ is continuous and $\sup_x |f^{(n)}(x)| \le \|(f^{(n)})\widehat{\ }\|_{L^1}$. Hence, to complete the proof it is enough to prove that $\|(f^{(n)})\widehat{\ }\|_{L^1} \le \|f\|_{W^s}$ when $n \le k$. But $(f^{(n)})\widehat{\ }(\xi) = (2\pi i\xi)^n \widehat{f}(\xi)$, and by Schwarz inequality, for $n \le k$ we have

$$\begin{aligned} \int |(2\pi i\xi)^n \widehat{f}| d\xi &\le (2\pi)^k \int (1+|\xi|^2)^{k/2} |\widehat{f}(\xi)| d\xi \\ &= (2\pi)^k \int (1+|\xi|^2)^{s/2} |\widehat{f}(\xi)| (1+|\xi|^2)^{(k-s)/2} d\xi \\ &\le (2\pi)^k \left(\int (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi \int (1+|\xi|^2)^{k-s} d\xi \right)^{1/2} \end{aligned}$$

The first integral on the parenthesis is $\|f\|_{W^s}^2$, and the second one is

$$\int (1+|\xi|^2)^{k-s} d\xi = 2\int_0^\infty (1+r^2)^{k-s} dr \,,$$

which is finite if and only if $k - s < -\frac{1}{2}$ since the integral is roughly $r^{2(k-s)}$ for large r. $\square$

Corollary 4. *If $f \in W^s$ for all $s \in \mathbb{R}$, then $f \in C^\infty$.*

Remark A: The Lemma 3. can be sharpened a bit: if $s = k + \alpha + \frac{1}{2}$ where $0 < \alpha < 1$, then $W^s \subset C^{k+\alpha}$.
Remark B: For the case $p = \infty$, the Sobolev space $W^{k,\infty}$ is defined to be the Holder space $C^{n,\alpha}$ where $k = n + \alpha$ and $0 < \alpha \le 1$.

Wavelet characterization of Sobolev norms

Here we discuss the use of wavelets in defining equivalent norms on $W^s(\mathbb{R})$ where $s \in \mathbb{R}$.

Theorem 5. *Suppose that ϕ is the mother wavelet of a wavelet basis for $L^2(\mathbb{R})$, that $\phi \in C^\alpha(\mathbb{R})$ for some $\alpha > s \ge 0$ and that, for some sufficiently large N, $|\phi(x)| \le (1+|x|)^{-N}$. Then there are constants $C_1,\ C_2$ such that $C_1\|f\|^2_{W^s} \le \sum_{j,k}(1+4^{js})|\langle f,\, \phi_{jk}\rangle|^2 \le C_2\|f\|^2_{W^s}$.*

First of all, we'll prove Theorem 5 in the case of *bandlimited wavelet* which means the support of the Fourier transform of this wavelet is contained in a finite interval. Then we'll pass to the case of orthonormal wavelets with a *change of wavelet* argument, showing that changing from one suitably regular wavelet basis to another defines a bounded operator on $W^s(\mathbb{R})$. However, we'll skip the arguments for the biorthogonal case. The arguments for the this case are actually much the same except the notion of change of wavelet is slightly more complicated.

Proof. (*of Theorem 5 in the case of bandlimited wavelet*) We begin with the specific form of the wavelets. Let $b(\xi)$ be a smooth, non-negative function supported inside $[1/3,\, 4/3]$ and decreasing away from $\xi = 1$ such that $\sum_j |b(|\xi|/2^j)|^2 = 1$ for all $\xi \ne 0$. Set $\omega(\xi) = \operatorname{sign}(\xi)e^{\pi i\xi}b(2|\xi|)$. Then we have the functions

$$\omega_{nj}(\xi) = 2^{-j/2}e^{-2\pi in\xi/2^j}\omega(\xi/2^j), \quad (j,\, n \in \mathbb{Z})$$

which form an orthonormal basis basis for $L^2(\mathbb{R})$ and its inverse Fourier transforms are wavelets ϕ_{jn} (see [AWW]). By Plancherel,

$$\langle f,\, \phi_{jn}\rangle = \langle \widehat{f},\, \omega_{nj}\rangle\,.$$

Due to Theorem 2. we can finish this proof by showing that

$$\int |f(\xi)|^2(1+|\xi|^2)^s d\xi < \infty \ \text{ if and only if } \sum_{j\in\mathbb{Z}}(1+4^{js})\sum |\langle f,\, \omega_{nj}\rangle|^2 < \infty\,.$$

Let $A_j = \{2^j/3 \le |\xi| \le 4\cdot 2^j/3\}$. Then 2^j is a wavenumber associated with amplitude of $|A_j|$ and $\sum_j \|f\chi_{A_j}\|^2_{L^2} = 2\|f\|^2_{L^2}$. Thus with Parseval's theorem and the fact that differentiation is equivalent to multiplying in the frequency side. We have $\|f\|^2_{W^s}$ is equivalent in magnitude to $\sum_j(1+(2^{js})^2\|f\chi_{A_j}\|^2_{L^2}$. Since $b(\xi)$ supported only inside $[1/3,\, 4/3]$, $b(|\xi|/2^j)$ survive only for integer $j-1$, j and $j+1$ on A_j. Thus $\sum_{k=j-1}^{j+1}|b(|\xi|/2^k)|^2 = 1$ on A_j, so we can complete this proof by showing there is constants c, C so that

$$c\sum_n |\langle f,\, \omega_{nj}\rangle|^2 \le \|f\chi_{A_j}\|^2_{L^2} \le C\sum_n |\langle f,\, \omega_{nj}\rangle|^2\,.$$

Now, let us assume f is supported in $[0,\infty)$. Then

$$\begin{aligned}\langle f,\, \omega_{nj}\rangle &= \int f(\xi)\overline{2^{-j/2}e^{-2\pi in\xi/2^j}\omega(\xi/2^j)}d\xi \\ &= \int_{2^j/3}^{2^{j+2}/3} e^{-\pi i\xi/2^j}f(\xi)b(\xi/2^j)2^{-j/2}e^{2\pi in\xi/2^j}d\xi\end{aligned}$$

Since $\{2^{-j/2}e^{2\pi in\xi/2^j}\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2([2^{j+1}/3,\, 2^{j-1}/3))$, we have

$$\sum_n |\langle f,\, \omega_{nj}\rangle|^2 = \int |e^{-\pi in\xi/2^j}|^2|f(\xi)|^2|b(\xi/2^j)|^2 d\xi \le C\|f\chi_{A_j}\|_{L^2}^2 .$$

On the other hand,

$$\|f\chi_{A_j}\|_{L^2}^2 \le \sum_{k=j-1}^{j+1} \int |f(\xi)b(\xi/2^k)|^2 d\xi = \sum_{k=j-1}^{j+1} \sum_n |\langle f,\, \omega_{nj}\rangle|^2 .$$

With the above two estimates, we complete the proof when f is supported in $[0,\,\infty)$. Again the same methods apply to f supported in $[-\infty,\, 0]$, then we can get our desired result. $\square$

Actually, for this bandlimited case the restriction $0 < s < 1$ is not necessary condition. Now, we have to pass this result to other wavelets. However, we'll discuss this very roughly. (You can find the detail of this process in [HL].) It will be convenient to return to interval notation $\psi_I = \psi_{jk}$ where $I = I(j,k) = [k/2^j,\, (k+1)/2^j)$. The general case of Theorem 5 change to proving bounds on a *change of wavelet matrix.* This is where we use the hypothesis that $s < 1$.

By expanding the wavelet ψ^1 in terms of the basis ψ_I^2. that is $\psi^1 = \sum_{I\in\mathcal{D}}\langle\psi^1,\, \psi_I^2\rangle\psi_I^2$, any $f \in L^2(\mathbb{R})$ can be expressed as

$$f = \sum_I \langle f,\, \psi_I^1\rangle\psi_I^1 = \sum_I \langle f,\, \psi_I^1\rangle \sum_J \langle\psi_I^1,\, \psi_J^2\rangle\psi_J^2 .$$

Now let $\psi^1 = \psi^b$ be the bandlimited wavelet considered above for which we have the norm equivalence $\|f\|_{W^s}^2 \sim \sum_{I\in\mathcal{D}}(1+|I|^{-2s})|\langle f,\, \psi_I^b\rangle|^2$. Let $\mathcal{H}^2$ be the Hilbert sequence space consisting of those sequence $\{c_I\}$ so that $\sum_{I\in\mathcal{D}}(1+|I|^{-2s})|c_I|^2 < \infty$. Let the matrix $A_{IJ} = \langle\psi_I^b,\, \psi_J\rangle$. To prove that a more general wavelet $\psi = \psi^2$ under consideration also provides a norm equivalence between W^s and $\mathcal{H}^2$, it is enough to show that the matrix A_{IJ} is bounded and continuously invertible on $\mathcal{H}^s$ or equivalently, jointly on $l^2(\mathcal{D})$ and on $\dot{\mathcal{H}}^s$ which is the space of $\{c_I\}$ such that $\sum_{I\in\mathcal{D}}|I|^{-2s}|c_I|^2 < \infty$. Since $\{c_I\} \in \dot{\mathcal{H}}^s$ if and only of $|I|^{-s}c_I \in l^2(\mathcal{D})$, proving that A_{IJ} is bounded on $\dot{\mathcal{H}}^s$ is equivalent to proving that $B_{IJ} = (|I|/|J|)^2 A_{IJ}$ is $l^2(\mathcal{D})$-bounded. Thus, one finds a sufficient condition for l^2-boundedness of A_{IJ} first, then verifies a corresponding condition for B_{IJ}.
Remark C: One can find the characterizations of other function spaces using wavelets. Especially, one can see the characterization of more general case of Sobolev spaces $W^{k,p}(\mathbb{R})$, where $1 < p < \infty,\;\; k = 1,2,3,\cdots$ (see [HW]).

Notes and more

Sobolev norms are often arise in matters of well-posedness for PDEs. The variational form of the Navier-Stokes equations in $\mathbb{R}^n$ is:

$$v(t) = S(t)v_0 - \int_0^t \mathbf{P}S(t-s)\nabla\cdot(v\otimes v)(s)ds .$$

One wished to solve for the velocity field $v = v(x,t)$. Here $S(t) = \exp(t\triangle)$ is the heat semigroup and $\mathbf{P}$ is the singular integral operator that projects a vector field onto its divergence-free component. Cannone and Meyer [CM] developed a method for obtaining so called mild solution of the variational form of the Navier-Stokes equations in $\mathbb{R}^n$(VFNSE) with initial data in certain function spaces including Sobolev spaces. A crucial step was to develop a notion under which a function space X is adapted to the bilinear product estimates required for application of a Picard iteration method to solve for v. Cannone and Meyer then used Littlewood-Paley theory to prove such estimates which can be regarded, in a sense, as estimates for wavelets projections of pointwise products specific to bandlimited wavelets. Extending such estimates to more general wavelets is one step required in adapting the techniques in [CM] to the setting of domain in $\mathbb{R}^n$ with boundary such as half-spaces. Here is an abstraction of the essential role of wavelets in obtaining useful estimates for operators such as $\mathbf{P}S(t-s)\nabla$. As we deal in class, P_j and Q_j denote the orthogonal projection of an orthonormal MRA. One seeks estimates of the form

$$\|A_j(fg)\|_X \leq \eta_j \|f\|_X \|g\|_X \,.$$

Then one can establish estimates of the above form for Sobolev spaces as like following.

Theorem 7. *Let Q_j denote the projection onto the jth wavelet space of multiresolution analysis of $L^2(\mathbb{R})$ where the wavelets are orthogonal and Lipschitz with compact support. Then for $0 \leq \alpha \leq 1$ and $j \geq 1$,*

$$\|Q_k(fg)\|_{W^\alpha} \leq C 2^{j(1/2-\alpha)} \|f\|_{W^\alpha} \|g\|_{W^\alpha} \,.$$

One can find the proof of theorem 7 in [HL]. Several other issues relating the use of wavelets to the study of Navier-Stokes equations are outlined by Katz and Pavlović in [KP].

We've discussed the characterization of Sobolev norms in terms of magnitudes of wavelet coefficients. The change of basis matrix that maps coefficients in one wavelet basis to coefficients in another plays an important role here. Actually the magnitudes of the entries of this matrix depend on the regularity of the the wavelets. Consequently, the wavelet characterization of Sobolev spaces depends only on the regularity of the mother wavelet. This observation also can be extended to Besov spaces which are extensions of Sobolev spaces. Membership in Besov and Sobolev spaces is determined by the "smoothness" of the functions concerned; the Besov norms involve *differences*, the Sobolev norms use *derivatives*.

Let us define the Besov spaces in a classical way. As a first step, we define some operators. For each $h \in \mathbb{R}$, the *translation operator* T_h is defined on functions f on $\mathbb{R}$ by $T_h f(x) = f(x+h)$. The *first-difference* operator $\triangle_h \equiv \triangle_h^1$ is defined by

$$\triangle_h f(x) = (T_h - I) f(x) = f(x+h) - f(x) \,,$$

and higher-order differences are defined inductively by

$$\triangle_h^{r+1} f(x) = \triangle_h (\triangle_h^r f)(x), \quad (r = 1, 2, 3, \ldots).$$

Since $T_h^k f(x) = T_{kh} f(x) = f(x+kh)$, it is clear that

$$\triangle_h^r f(x) = (T_h - I)^r f(x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f(x+kh) \,.$$

Let the space of bounded continuous functions on $\mathbb{R}$ be denoted by $C = C(\mathbb{R})$. It is a Banach space under the supremum norm $f \to \|f\|_\infty = \sup\{|f(x)| : x \in \mathbb{R}\}$. The *r-th order modulus of continuity* of a function f in $L^p(\mathbb{R}),\ 1 \leq p < \infty$, is defined by

$$w_r(f,t)_p = \sup_{|h| \leq t} \|\triangle_h^r f\|_p \quad (f \in L^p,\ t > 0) \,,$$

when $p = \infty$, the L^p-norm is replaced by the norm in C. Each modulus $w_r(f,t)_p$, $(1 \le p \le \infty)$, is a nonnegative increasing function of $t > 0$; furthermore, for each fixed t, $w_r(\cdot,t)_p$ is a seminorm on L^p or C. It is following from the definition of higher order differences that

$$w_r(f,t)_p \le 2^r \|f\|_p \, .$$

Since $\triangle_{2h} = T_h^2 - I = (T_h + I)\triangle_h$, one sees also that

$$w_r(f,2t)_p \le 2^r w_r(f,t)_p \, .$$

Definition 8. *Suppose $\alpha > 0$ and $1 \le p,q \le \infty$. Let r be a positive integer with $r > \alpha$. The Besov space $B_p^{\alpha,q}$ consist of those f in L^p (if $p < \infty$) or C (if $p = \infty$) for which the norm*

$$\|f\|_{B_p^{\alpha,q}} = \begin{cases} \|f\|_p + \left\{ \displaystyle\int_0^\infty [t^{-\alpha} w_r(f,t)_p]^q \frac{dt}{t} \right\}^{1/q}, & (q < \infty), \\ \|f\|_p + \displaystyle\sup_{0<t<\infty} \left\{ t^{-\alpha} w_r(f,t)_p \right\}, & (q = \infty), \end{cases}$$

is finite.

Besov spaces are family of function spaces characterized in terms of moduli of smoothness and sharing many of the fundamental properties of Sobolev spaces, including embedding, restriction and extension, and interpolation properties. Peetre [P] contains an excellent historical account of these developments and a definition of Besov norms in terms of more general Littlewood-Paley decomposition. Working in $\mathbb{R}$, one fixes ϕ and ψ_j, $j = 1, 2, \ldots$ such that $\widehat{\phi}$ is supported in $(-2,2)$ and $\widehat{\psi}_j$ is supported in $(-2^{j+1}, 2^{j+1}) \backslash (-2^{j-1}, 2^{j-1})$, while $\widehat{\phi} + \sum \widehat{\psi}_j \equiv 1$ and $2^{-j\beta} |d^\beta \psi_j / dx^\beta| \le c_\beta$ for $\beta \in \mathbb{N}$. For $p \in [1,\infty]$, $q > 0$ and $\alpha \in \mathbb{R}$ one define the space

$$B_p^{\alpha,q}(\mathbb{R}) = \left\{ f \in \mathcal{S}'(\mathbb{R}) : \|\phi * f\|_{L^p} + \left(\sum_{j=1}^\infty (2^{\alpha j} \|\phi_j * f\|_{L^p})^q \right)^{1/q} < \infty \right\}.$$

In 1985, Frazier and Jawerth [FJ] provided a discrete description, essentially in terms of wavelet frames, followed shortly thereafter by Lemarié and Meyer's [LM] characterization of $B_p^{\alpha,q}$ by wavelet coefficient norms

$$\|f\|_{B_p^{\alpha,q}(\mathbb{R}^n)} = \|\{\langle f, \phi_k \rangle\}\|_{l^p} + \left(\sum_{j=1}^\infty (2^{j\alpha} \|\{\langle f, \psi_{jk,p'} \rangle\}_k\|_{l^p})^q \right)^{1/q}.$$

Here, ϕ is the scaling function of an MRA and ψ is the corresponding mother wavelet, while $\psi_{jk,p} = 2^{j/p} \psi(2^j x - k)$, also written $\psi_{I,p} = |I|^{1/2-1/p} \psi_I$. To define $B_p^{\alpha,\infty}$ one replaces the sum over j by a corresponding supremum. Originally the typical bandlimited wavelets were used but, as in Theorem 5, any sufficiently regular orthogonal or biorthogonal wavelet will do.

In this note, we discussed two families of function spaces. One is the family of Sobolev spaces and the other is the family of Besov spaces. Why we are interested in these spaces? Sobolev spaces are named after the Russian mathematician Sergei L. Sobolev. There are many criteria for smoothness of mathematical functions. The most basic criterion may be that of continuity. A considerably stronger notion of smoothness is that of differentiability (because functions that are differentiable are also continuous) and a yet stronger notion of smoothness is that the derivative also be continuous so called C^1. Differentiable functions are important in many areas, and in particular for differential equations. However, in the twentieth century, it was observed that the space C^1 (or C^2, etc.) was not exactly the right space to study solutions of differential equations. The Sobolev spaces are the modern replacement for these spaces in which to look for solutions of partial differential equations. Thus their importance lies in the fact that solutions of partial differential equations are naturally in Sobolev spaces rather than in the classical spaces of continuous functions.

In image processing, one of most important class of functions is a Bounded variation which is defined by

$$f \in BV([a,b]) \Leftrightarrow V_b^a(f) = \sup\left\{ \sum_{j=1}^{M} |f(x_j) - f(x_{j-1})| : a = x_0 < x_1 < \cdots < x_M = b \right\} < \infty \,,$$

Most images can be thought to belong the space of functions with bounded variation. Although, images are 2-dimensional functions, we just deal with one dimensional case. Let $B_p^{\alpha,q}(C) = \{f : \|f\|_{B_p^{\alpha,q}} \leq C\}$ and $BV(C) = \{f : V_b^a \leq C\}$. Then, for the periodic Besov spaces $B_p^{\alpha,q}([a,b])$ with corresponding periodic wavelet bases, there are constants $C_1 \leq C_2$ such that (see [D])

$$\frac{1}{C_1} \sum_{j=1}^{\infty} 2^{j/2} \|\{\langle f, \psi_{jk}\rangle\}_k\|_{l^1} \geq V_b^a(f) \geq \frac{1}{C_2} \sup_j 2^{j/2} \|\{\langle f, \psi_{jk}\rangle\}_k\|_{l^1} \,.$$

That is,

$$B_1^{1,1}(C_1) \subset BV(1) \subset B_1^{1,\infty}(C_2) \,.$$

We can see Bounded Variation actually sit in between two Besov spaces, thus understanding Besov spaces suffices to understand Bounded Variation.

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