DETERMINANTS

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1. Determinants: a Row Operation By-Product

The determinant is best understood in terms of row operations, in my opinion. Most books start by defining the determinant via formulas that are nearly impossible to use except on very small matrices. Since what is taught first is often the best learned, this is dangerous.

We will start with the idea that a determinant of a square matrix is a single number that can be calculated as a side product of Gaussian elimination performed on a square matrix A. You already know 95% of what it takes to calculate a determinant. The extra 5% is keeping track of some "magic numbers" that you multiply at together to create another "magic number" called the determinant of A.

(How mathematicians came to discover these magic numbers is another topic.)

Definition 1.1. We define the *factor* of every row operation as follows:

Type	assmuption	Row Operation	Factor
I	$j \neq k$	$\mathrm{Rj} \leftrightarrow \mathrm{Rk}$	-1
II	$\alpha \neq 0$	$\alpha\mathrm{Rj} \to \mathrm{Rj}$	$\frac{1}{\alpha}$
III	$j \neq k$	$\mathrm{Rj} + \beta\mathrm{Rk} \to \mathrm{Rj}$	1

Definition 1.2. We define the determinant $\det(A)$ of a square matrix as follows:

(a) The *determinant* of an n by n singular matrix is 0.
(b) The *determinant* of the identity matrix is 1.
(c) If A is non-singular, then the *determinant of* A is the product of the factors of the row operations in a sequence of row operations that reduces A to the identity.

The notation we use is $\det(A)$ or $|A|$. Generally, one drops the braces on a matrix if using the $|A|$ notation, so

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right).$$

The outer parantheses are often dropped, so

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

are the notations most used.

Notice this means you can calculate the determinant using any series of row operations you like that ends in I. What we are skipping (since this is not a theoretical class) is the reasoning that shows that the product of determinant factors comes out the same no matter what series of row operations you use.

Example 1.3. Find

$$\begin{vmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}.$$

Since

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$
$$\downarrow \tfrac{1}{2}\mathrm{R1} \to \mathrm{R1} \quad \text{factor: } 2$$
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$
$$\downarrow \mathrm{R3} + \mathrm{R1} \to \mathrm{R3} \quad \text{factor: } 1$$
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$
$$\downarrow \tfrac{1}{2}\mathrm{R2} \to \mathrm{R2} \quad \text{factor: } 2$$
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\downarrow \mathrm{R1} - \mathrm{R3} \to \mathrm{R1} \quad \text{factor: } 1$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

we have

$$\begin{vmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 2 \cdot 1 \cdot 2 \cdot 1 = 4.$$

Example 1.4. Find

$$\begin{vmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{vmatrix}.$$

Since

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$
$$\downarrow \tfrac{1}{2}\mathrm{R1} \to \mathrm{R1} \quad \text{factor: } 2$$
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$
$$\downarrow \mathrm{R3} + \mathrm{R1} \to \mathrm{R3} \quad \text{factor: } 1$$
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we can conclude that the original matrix is not invertible, so

$$\begin{vmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{vmatrix} = 0.$$

Notice that we do not need to know in advance if A is invertible. To find $\det(A)$ you can always use Gaussian elimination.

> If row operations lead to less than n pivots, the determinant is 0.

and

> If row operations lead to I, the determinant is the product of the row op factors.

Example 1.5. Find

$$\det \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$$
$$\Big\downarrow \tfrac{1}{2}\mathrm{R1} \to \mathrm{R1} \quad \text{factor: } 2$$
$$\begin{bmatrix} 1 & 2 \\ 1 & 6 \end{bmatrix}$$
$$\Big\downarrow \mathrm{R2} - \mathrm{R1} \to \mathrm{R2} \quad \text{factor: } 1$$
$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$$
$$\Big\downarrow \tfrac{1}{4}\mathrm{R1} \to \mathrm{R1} \quad \text{factor: } 4$$
$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$
$$\Big\downarrow \mathrm{R1} - 2\mathrm{R2} \to \mathrm{R1} \quad \text{factor: } 1$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

we have

$$\det \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix} = 2 \cdot 1 \cdot 4 \cdot 1.$$

2. Two-by-Two: an Easy Case

Two-by-two is the only size of matrix where there is a formula for the determinant that is faster to use than row operation method. If you have not seen this formula, here is how we can discover it.

Suppose we want

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and we are lucky to have $a \neq 0$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\Big\downarrow \tfrac{1}{a}\mathrm{R1} \to \mathrm{R1} \quad \text{factor: } a$$
$$\begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix}$$
$$\Big\downarrow \mathrm{R2} - c\mathrm{R1} \to \mathrm{R2} \quad \text{factor: } 1$$
$$\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & d - c\frac{b}{a} \end{bmatrix}$$
$$\|$$
$$\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix}$$
$$\Big\downarrow \tfrac{a}{ad-bc}\mathrm{R1} \to \mathrm{R1} \quad \text{factor: } \tfrac{ad-bc}{a}$$
$$\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix}$$
$$\mathrm{R1} - \tfrac{a}{b}\mathrm{R2} \to \mathrm{R2} \quad \text{factor: } 1$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

where we cheated a little. We need $ad - bc \neq 0$. If this is so, and still with $a \neq 0$, we have computed

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

This formula always holds, but let's skip the other cases (they are easier) and just state this result.

Lemma 2.1. *For any real numbers a, b, c, d,*

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

There is a formula that is a bit trickier than this that works for three-by-three. Starting with four-by-four there is no shortcut. You must either use row operations or the longer "row expansion" methods we'll get to shortly.

3. Elementary Matrices are Easy

Since elementary matrices are barely different from I, they are easy to deal with. As with their inverses, I recommend that you memorize their determinants.

Lemma 3.1.

(a) *An elementary matrix of type I has determinant -1.*

(b) *An elementary matrix of type II that has non-unit diagonal element* α *has determinant* α.
(c) *An elementary matrix of type III determinant* 1.

Rather than prove this, I offer some examples.

Example 3.2. Find

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Since

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\downarrow \mathrm{R1} - 2\mathrm{R3} \to \mathrm{R1} \quad \text{factor: } 1$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

we have

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Example 3.3. Find

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}.$$

Since

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
$$\downarrow \mathrm{R1} \leftrightarrow \mathrm{R2} \quad \text{factor: } -1$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

we have

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1.$$

Example 3.4. Find

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

where $\alpha \neq 0$.

Since

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\Big\downarrow \tfrac{1}{\alpha}\text{R2} \to \text{R2} \quad \text{factor: } \alpha$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

we have

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{vmatrix} = \alpha.$$

4. Upper Triangular is Easy: a Faster Algorithm

Let's consider a more substantial example than we've done so far.

Example 4.1. What is the determinant of

$$A = \begin{bmatrix} 0 & -2 & -4 \\ 1 & 4 & 8 \\ 1 & 6 & 15 \end{bmatrix}?$$

A solution is as follows:

$$\begin{bmatrix} 0 & -2 & -4 \\ 1 & 4 & 8 \\ 1 & 6 & 15 \end{bmatrix}$$
$$\downarrow \mathrm{R2} \leftrightarrow \mathrm{R1} \qquad \text{factor: } -1$$
$$\begin{bmatrix} 1 & 4 & 8 \\ 0 & -2 & -4 \\ 1 & 6 & 15 \end{bmatrix}$$
$$\downarrow \mathrm{R3} - \mathrm{R1} \to \mathrm{R3} \quad \text{factor: } 1$$
$$\begin{bmatrix} 1 & 4 & 8 \\ 0 & -2 & -4 \\ 0 & 2 & 7 \end{bmatrix}$$
$$\downarrow -\tfrac{1}{2}\mathrm{R2} \to \mathrm{R2} \quad \text{factor: } -2$$
$$\begin{bmatrix} 1 & 4 & 8 \\ 0 & 1 & 2 \\ 0 & 2 & 7 \end{bmatrix}$$
$$\downarrow \mathrm{R3} - 2\mathrm{R2} \to \mathrm{R3} \quad \text{factor: } 1$$
$$\begin{bmatrix} 1 & 4 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$
$$\downarrow \tfrac{1}{3}\mathrm{R3} \to \mathrm{R3} \quad \text{factor: } 3$$
$$\begin{bmatrix} 1 & 4 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\downarrow \mathrm{R1} + 4\mathrm{R2} \to \mathrm{R1} \quad \text{factor: } 1$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\downarrow \mathrm{R2} - 2\mathrm{R3} \to \mathrm{R2} \quad \text{factor: } 1$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so

$$\begin{aligned} \det(A) &= (-1) \cdot 1 \cdot (-2) \cdot 1 \cdot 3 \cdot 1 \cdot 1 \\ &= 6. \end{aligned}$$

Notice that after the matrix was in row echelon form, the remaining steps were type III operations that have factor 1. Thus we could have skipped these steps. In fact, it is very easy to calculate the determinant of upper triangular matrix.

Lemma 4.2. *The determinant of an upper triangular matrix is the product of its diagonal elements.*

Proof. If

$$A = \begin{bmatrix} d_1 & * & \cdots & * & * \\ 0 & d_2 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{n-1} & * \\ 0 & 0 & \cdots & 0 & d_n \end{bmatrix}$$

then it might be the some of the d_j are zero. If this is true, then A is singular, has determinant zero, and the product of the diagonal elements is zero as well.

Otherwise, we can reduce A to a diagoanal matrix using first a series of type III operations, which have factor 1. The diagonal matrix will be

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & d_n \end{bmatrix}$$

and from here we can reduce to the identity matrix using the operations

$$\begin{aligned} \frac{1}{d_1}\text{R1} &\to \text{R1} \\ \frac{1}{d_2}\text{R2} &\to \text{R2} \\ &\vdots \\ \frac{1}{d_n}\text{Rn} &\to \text{Rn} \end{aligned}$$

The product of all the determinant factors is

$$1 \times 1 \times \cdots \times 1 \times d_1 \times d_2 \times \cdots \times d_n = d_1 \times d_2 \times \cdots \times d_n.$$

□

So

The determinant of an upper triangular matrix is the product of the diagonal.

To properly say we can "stop when we reach upper diagaonal" we shall use the following. More generally, you can stop the row operations when you arrive at a matrix whose determinant you already know.

Lemma 4.3. *If a sequence of elementary row operations reduces A to B, and if the factors of these row operations are $r_1, \ldots, r_n$, then*

$$\det(A) = r_1 r_2 \ldots r_n \det(B).$$

Proof. We know that elementary row operations turn singular matrices into singular matrices. If A is singular then B is singular and

$$\det(A) = 0 = \det(B)$$

and the formula holds.

Suppose A (and B) are invertible, and that the operations we've found that take us from A to B are

$$\mathrm{Op}_1, \mathrm{Op}_2, \ldots, \mathrm{Op}_n.$$

We can find some additional elementary row operations to transform B to I. Let's call these

$$\tilde{\mathrm{Op}}_1, \tilde{\mathrm{Op}}_2, \ldots, \tilde{\mathrm{Op}}_m,$$

and call there factors

$$\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_m.$$

Then the longer sequence

$$\mathrm{Op}_1, \mathrm{Op}_2, \ldots, \mathrm{Op}_n, \tilde{\mathrm{Op}}_1, \tilde{\mathrm{Op}}_2, \ldots, \tilde{\mathrm{Op}}_m$$

will transform A to I. Therefore

$$\det(B) = \tilde{r}_1 \tilde{r}_2 \cdots \tilde{r}_m$$

and

$$\det(A) = r_1 r_2 \cdots r_n \tilde{r}_1 \tilde{r}_2 \cdots \tilde{r}_m$$

and we have our formula. □

Example 4.4. Calculate the the determinant of

$$A = \begin{bmatrix} 2 & 4 & 0 & 4 \\ 0 & 5 & 1 & 0 \\ 2 & 9 & 0 & 4 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

A solution is

$$\begin{bmatrix} 2 & 4 & 0 & 4 \\ 0 & 5 & 1 & 0 \\ 2 & 9 & 0 & 4 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$
$$\Big\downarrow \tfrac{1}{2}\mathrm{R1} \to \mathrm{R1} \quad \text{factor: } 2$$
$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 5 & 1 & 0 \\ 2 & 9 & 0 & 4 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$
$$\Big\downarrow \begin{array}{ll} \mathrm{R3} - 2\mathrm{R1} \to \mathrm{R3} & \text{factor: } 1 \\ \mathrm{R4} - \mathrm{R1} \to \mathrm{R4} & \text{factor: } 1 \end{array}$$
$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 5 & 1 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$
$$\Big\downarrow \mathrm{R3} - \mathrm{R2} \to \mathrm{R3} \quad \text{factor: } 1$$
$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and so the answer is

$$\begin{aligned} \det(A) &= 2 \cdot 1 \cdot 1 \cdot 1 \cdot (1 \cdot 5 \cdot (-1) \cdot (-1)) \\ &= 10. \end{aligned}$$

5. Products and Determinants

Theorem 5.1. *If A and B are n-by-n matrices, then*

$$\det(AB) = \det(A)\det(B).$$

Proof. If either A or B is not invertible, the product will also be noninvertible, and so the equation will be satisfied by being zero on both sides.

Suppose

$$\mathrm{Op}_1, \mathrm{Op}_2, \ldots, \mathrm{Op}_n$$

is a sequences of elementary row operations that transforms A to I. If the factors associated to these operations are $r_1, r_2, \ldots, r_n$ then

$$\det(A) = r_1 r_2 \cdots r_n.$$

We can apply these row operations to the left matrix in the product AB and transform this to IB. By Lemma 4.3,

$$\begin{aligned}\det(AB) &= r_1 r_2 \cdots r_n \det(IB) \\ &= r_1 r_2 \cdots r_n \det(B) \\ &= \det(A)\det(B).\end{aligned}$$

□

This theorem is imporant for all sorts of reasons. It can be paraphrased as

products pull apart from inside the determinant.

A nice thing this tells us that if A is invertible then

$$\det(A)\det(A^{-1}) = \det(I) = 1$$

and so

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Thus

The determinant of the inverse is the inverse of the determinant.

Transpose and Determinants

Example 5.2. What are $\det(A)$ and $\det(A^T)$ if

$$A = \begin{bmatrix} 1 & 0 & \pi & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}?$$

Since

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \pi & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

both A and $A^T$are type III elementary, and so

$$\det(A) = \det(A^T) = 1.$$

Example 5.3. What are $\det(A)$ and $\det(A^T)$ if

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In this case $A^T = A$ and A is a type II elementary matrix, so

$$\det(A) = \det(A^T) = -1.$$

Example 5.4. What are $\det(A)$ and $\det(A^T)$ if

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In this case $A^T = A$ and A is a type I elementary matrix, so

$$\det(A) = \det(A^T) = \frac{2}{3}.$$

I hope you see the pattern here.

Theorem 5.5. *If I is a square matrix, then*

$$\det(A^T) = \det(A).$$

Proof. If A is singular, so is A^T and both matrices have determanent 0. Otherwise, we can write A as a product of elementary matrices

$$A = E_1 E_2 \cdots E_k$$

and so

$$A^T = E_k^T E_{k-1}^T \cdots E_1^T.$$

This means

$$\det(A) = \det(E_1)\det(E_2)\cdots\det(E_k)$$

and

$$\det(A^T) = \det(E_k^T)\det(E_{k-1}^T)\cdots\det(E_1^T).$$

Recall that while matrix multiplication is not commutative, the product here is a product of real numbers. So using commutativity, we have

$$\det(A^T) = \det(E_1^T)\det(E_2^T)\cdots\det(E_k^T)$$

and so all we need to do is verify that

$$\det(E^T) = \det(E)$$

for every elementary matrix. I'll leave this to your imagination. □

In blue, we have

The transpose leave the determinant alone.

The transpose of a lower triangular matrix is upper triangular, with the same diagonal.

Lemma 5.6. *If A is upper or lower triangular, the determinant of A is the product of the diagonal elements.*

Since the transpose mixes left with right, we can sort out the effect of column operations. They act on the determinant just like row operations.

Lemma 5.7. *Suppose A is a square matrix. If we obtain A_1 as the result of a column operation, then*

$$\det(A) = r\det(A_1)$$

where

(a) $r = \alpha$ if the column operation was αCk $\to$ Ck with $\alpha \neq 0$,
(b) $r = -1$ if the column operation was Cj $\leftrightarrow$ Ck, with j $\neq k$, and
(c) $r = 1$ if the column operation was Cj + βCk $\to$ Cj,with $j \neq k$.

So you can

use column operations when computing determinants,

but

don't use column operations for computing inverses

and

don't use column operations when solving linear systems.

I hesitate to mention what doesn't work, in general, but using column operations is very tempting to a lot of students. No, no, no. You can mix row and column operations when computing determinants, and that's about it.

6. Partitioned Matrices

If we have a partioned matrix

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

can we calulate $\det(A)$ from a formula involving the smaller matrices? Yes, sometimes, but the formula is nastier than you would think. However, there is a nice special case.

Example 6.1. Compute, where possible,the determinants of

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix},$$

$$B = \begin{bmatrix} 8 & 8 & 4 \\ 0 & 1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and

$$X = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}.$$

Here 0 is necessarily meaning

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and so we are looking at a matrix X that looks nice if seen in terms of blocks:

$$\begin{aligned} X &= \begin{bmatrix} 2 & 4 & 8 & 8 & 4 \\ 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \\ &= \left[\begin{array}{cc|ccc} 2 & 4 & 8 & 8 & 4 \\ 1 & 3 & 0 & 1 & 0 \\ \hline 0 & 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]. \end{aligned}$$

First of all, B does not have a determinant because it is not square.

Now consider A. We could use the two by two rule, but let's work this out with row operations, as this will guide us when working with the big matrix X. So, working out $\det(A)$:

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$
$$\downarrow \tfrac{1}{2}\mathrm{R1} \to \mathrm{R1} \quad \text{factor: } 2$$
$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$
$$\downarrow \mathrm{R2} - \mathrm{R1} \to \mathrm{R2} \quad \text{factor: } 1$$
$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

which has only ones on the diagonal, so

$$\det(A) = 2 \cdot 1 = 2.$$

Now we do the same for D :

$$\begin{bmatrix} 3 & 0 & -3 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$
$$\downarrow \tfrac{1}{3}\mathrm{R1} \to \mathrm{R1} \quad \text{factor: } 3$$
$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$
$$\downarrow \begin{array}{ll} \mathrm{R2} - \mathrm{R1} \to \mathrm{R2} & \text{factor: } 1 \\ \mathrm{R3} - \mathrm{R1} \to \mathrm{R3} & \text{factor: } 1 \end{array}$$
$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$
$$\downarrow \tfrac{1}{2}\mathrm{R3} \to \mathrm{R3} \quad \text{factor: } 2$$
$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

which means

$$\det(D) = 3 \cdot 1 \cdot 1 \cdot 2 = 6.$$

Now we can take care of X. In fact, we start by using the operations we used on A. Then we use the operations we used on D, but shifted down two rows. This will reduce X to the

point of being upper triangular with ones on the diagonal.

$$\begin{bmatrix} 2 & 4 & 8 & 8 & 4 \\ 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$\downarrow \frac{1}{2}\text{R1} \to \text{R1}$ factor: 2

$$\begin{bmatrix} 1 & 2 & 4 & 4 & 2 \\ 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$\downarrow \text{R2} - \text{R1} \to \text{R2}$ factor: 1

$$\begin{bmatrix} 1 & 2 & 4 & 4 & 2 \\ 0 & 1 & -4 & -3 & -2 \\ 0 & 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$\downarrow \frac{1}{3}\text{R3} \to \text{R3}$ factor: 3

$$\begin{bmatrix} 1 & 2 & 4 & 4 & 2 \\ 0 & 1 & -4 & -3 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$\downarrow \text{R4} - \text{R3} \to \text{R4}$ factor: 1

$\downarrow \text{R5} - \text{R3} \to \text{R5}$ factor: 1

$$\begin{bmatrix} 1 & 0 & 4 & 4 & 2 \\ 0 & 1 & -4 & -3 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$\downarrow \frac{1}{2}\text{R3} \to \text{R3}$ factor: 2

$$\begin{bmatrix} 1 & 0 & 4 & 4 & 2 \\ 0 & 1 & -4 & -3 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and since this has all diagonal elements equal to 1, we have

$$\det(X) = 2 \cdot 1 \cdot 3 \cdot 1 \cdot 1 \cdot 2 = 12.$$

Thus we see that

$$\det(X) = \det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det(A) \det(D).$$

This equality happened because of how the row operations transfered from A and D over to X.

Theorem 6.2. *Suppose A is an n-by-n matrix, D is an m-by-m matrix, and B is n-by-m. The partitioned matrix*

$$X = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

has determinant equal to

$$\det(A) \det(D),$$

I.e,

$$\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det(A) \det(D).$$

With proper assumptions on the sizes of the matrices, we also have the formula

$$\det \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} = \det(A) \det(D).$$

Proof. Here we have a case where a formal proof clouds the mind. Here is an informal argument.

The row operations that convert A to the smaller I work on the big matix to give

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \to \cdots \to \begin{bmatrix} I & B_1 \\ 0 & D \end{bmatrix}$$

where we don't really care what B_1 looks like. Then you use a sequence of row operations that convert D to the correct sized identity, but applied to the lower rows of the partitioned matrix and you get

$$\begin{bmatrix} I & B_1 \\ 0 & D \end{bmatrix} \to \cdots \to \begin{bmatrix} I & B_1 \\ 0 & I \end{bmatrix}.$$

This is upper triangular with all ones on the diagona. The final product of factors is just the product of the factors you computed for A times the product of the factors you computed for D. This gives the first result.

The second formula follows from the first using the transpose. □

7. Expanding (slowly) on Rows and Columns

In the book you will see how a determinent can be computed by "expanding along a row" or "expanding along a column." With that method, you compute an n-by-n determinant as a weighted sum of n different $(n-1)$-by-$(n-1)$ determinants. This involves ugly notation, so I will ask that you read about this from the book.

Example 7.1. Calculate the determinent of

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix}$$

using expansion along the top row at each stage.

$$\begin{aligned}
& \begin{vmatrix} 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{vmatrix} \\
= \; & 2\begin{vmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{vmatrix} \\
= \; & 2\begin{vmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{vmatrix} \\
= \; & 2\left(\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}\right) \\
& \quad - \left(\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}\right) \\
& \quad - \left(\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}\right) \\
= \; & 2\left((-3) - (1) + 2(5)\right) \\
& \quad - \left((-3) - (0) + 2(3)\right) \\
& \quad - \left((5) - (3) + (-1)\right) \\
= \; & 2(6) - (3) + (1) \\
= \; & 8
\end{aligned}$$

You have seen that there is an alternating sign pattern used in the expansion. It may seem that this arrises out of nowhere, but it comes down to counting type II row operations and how many -1 factors there are.

Let's ponder the case of four by four matrices. A very special case of block matrix computations, as in the last section, tells us that

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{11} \det \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

Using type II elementary row opertations, we also find that

$$\begin{aligned}\det\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix} &= -\det\begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix} \\ &= -a_{21}\det\begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\det\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix} &= -\det\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix} \\ &= \det\begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix} \\ &= a_{31}\det\begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}\end{aligned}$$

and using three type II moves,

$$\det\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = -a_{41}\det\begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

This is where the alternating signs come from, at least when one is expanding along the first column.

To explain the rest of the formula, we need to know the following:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

This is a special case of the following result that is very imporant to the theory of determinants. Again, it is possible to explain in terms of row operations and transposes, but I will skip that. So, without proof, here is this result about adding columns withing determinants.

Theorem 7.2. *If* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ *and* $\mathbf{b}_1$ *are size* n *column vectors, then*

$$\begin{vmatrix} (\mathbf{a}_1 + \mathbf{b}_1) & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{vmatrix} = \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{vmatrix} + \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{vmatrix}.$$

A similar statement is true for any other column position, and for matrices described by rows.

For example, this theorem is telling us

$$\begin{vmatrix} 2 & 2 & 2 \\ 2 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 2 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 2 \\ 2 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix}$$

since

$$2 \quad 1 \quad 0$$

plus

$$0 \quad 1 \quad 2$$

is

$$2 \quad 2 \quad 2 \ .$$

The reason this expansion method is so bad for most calculations, is that while a 3-by-3 matrix can be written as 3 two-by-two determinants, a 4-by-4 becomes 12 two-by-two determinants, and so forth, until a 10-by-10 becomes

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 1,814,400$$

two-by-two determinants.

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