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We start out by recalling Cotlar's Lemma and its simple proof

Lemma: Suppose $\{T_j\}$ is a finite collection of bounded operators on L^2. Assume that we have $\gamma(j)$, $j \in \mathbb{Z}$, with

$$A = \sum_{j=-\infty}^{+\infty} \gamma(j) < +\infty$$

and

$$\|T_i^* T_j\| \leq \gamma(i-j)^2$$

$$\|T_i T_j^*\| \leq \gamma(i-j)^2$$

Then $T = \sum T_j$ verifies

$$\|T\| \leq A$$

I briefly recall the proof: Recall that $\|T\|^{2m} = \|(T^*T)^m\|$. We write

$$(T^*T)^m = \sum_{i_1, \ldots, i_{2m}} T_{i_1}^* T_{i_2} \cdots T_{i_{2m}}$$

We first associate the factors in each summand

as $(T^*_{i_1} T_{i_2})$ $(T^*_{i_{2m-1}} T_{i_{2m}})$ and

get $\|T^*_{i_1} \cdots T_{i_{2m}}\| \le \gamma^2(i_1 - i_2) \cdots \gamma^2(i_{2m-1} - i_{2m})$

We can also associate the factors as

$T^*_{i_1}(T_{i_2} T^*_{i_3}) \cdots (T_{i_{2m-2}} T^*_{i_{2m-1}}) T_{i_{2m}}$,

so that, since $\|T_{i_j}\| \le \gamma(0) \le A$, we get

$\|T^*_{i_1} T_{i_2} \cdots T_{i_{2m}}\| \le A^2 \gamma^2(i_2 - i_3) \cdots \gamma^2(i_{2m-2} - i_{2m-1})$

Taking the geometric means gives

$$\|(T^*T)^m\| \le \sum_{i_1 \cdots i_{2m}} A \gamma(i_1 - i_2)\gamma(i_2 - i_3) \cdots \gamma(i_{2m-1} - i_{2m})$$

We then sum in i_1, using $\sum_{i_1} \gamma(i_1 - i_2) \le A$, etc, to obtain $\|(T^*T)^m\| \le A^{2m} \sum_{i_{2m}} 1$.

Suppose we have N summands. We then obtain

$\|T\| \le A N^{1/2m}$, and letting $m \to \infty$

proves the Lemma.

(We can replace $\mathbb{Z}$ by $\mathbb{Z}^\nu$ with no proof).

Remarks. The original version (1953) of the Lemma was proven by Cotlar for T_j self-adjoint and mutually commuting. It was first used to obtain the L^2 boundedness of the "ergodic Hilbert transform", a situation where the Fourier transform was not available. The general version given was proved by Cotlar (67) and independently by E.M. Stein. It appears in a paper by Knapp-Stein from 1971, where they apply it to obtain the L^2 boundedness of intertwining operators given by singular integrals on nilpotent groups, in the context of representation theory for semi-simple groups.

I will now turn to pseudo-differential operators and some of the applications to pde. Thus, let $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and consider the operator

$$T_a f(x) = \int e^{ix\cdot\xi} a(x,\xi) \hat{f}(\xi) d\xi$$

This class of operators is a direct descendant of the variable coefficient singular integrals of Calderón-Zygmund and was first introduced by Kohn-Nirenberg (1965) and Unterberger-Bokobza (1964). The most common formulation we use today was introduced by Hörmander (1965, 1967). The function $a(x,\xi)$ is the symbol and the "standard" class of symbols $S^m_{1,0}$ is given by $\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \leq A_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$

This class of operators form a graded algebra, closed under adjoints. Moreover, for instance, the symbol of the composition is the product of the symbols, plus "lower order". Also, it is not difficult to show that T_a is L^2 bounded if $a \in S^0_{1,0}$. In connection with hypoelliptic equations, Hörmander introduced the classes $S^m_{\rho,\delta}$ which are given by

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq A_{\alpha,\beta} (1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}$$

When $0 \leq \delta < \rho \leq 1$, everything goes through as before. The case when $0 \leq \delta = \rho \leq 1$ was called the "exotic symbol class". When $\delta = \rho = 1$ it was shown by example that operators (without) in $S^0_{1,1}$ need not be L^2 bounded (the "forbidden symbol class"). The question of what happened for $S^0_{\rho,\delta}$,

$0 \le \rho < 1$ remained open until it was answered by Calderón-Vaillancourt (1971-72)

Theorem: (Calderón-Vaillancourt) If $a \in S^0_{\rho,\rho}$ $0 \le \rho < 1$, T_a is L^2 bounded.

The importance of the issue was suggested by Mizohata, Kumano-Go and other japanese mathematicians. The proof was by application of Cotlar's lemma. I will now give a sketch of the proof in the case $\rho = 0$. Note that the assption on a in this case is that its derivatives are bounded. The assptio is clearly symmetric in x and ξ and we can thus consider (in view of Plancherel's theorem)

$$Sf(x) = \int a(x,\xi)\, e^{ix\cdot\xi} f(\xi)\, d\xi$$

If $\sum_{i \in \mathbb{Z}^m} \phi(x-i) = 1$, where $\operatorname{supp} \phi \subset \{|x_j| \leq 1\}$

and $\alpha = (i, i') \in \mathbb{Z}^{2m}$, $\beta = (j, j') \in \mathbb{Z}^{2m}$, then

one sets $a_\alpha(x, \xi) = \phi(x-i)\, a(x, \xi)\, \phi(\xi - i')$.

One then verifies that

$$\| S_\alpha^* S_\beta \| \leq A \left(1 + |\alpha - \beta|\right)^{-2N}$$

$$\| S_\alpha S_\beta^* \| \leq A \left(1 + |\alpha - \beta|\right)^{-2N}, \text{ N large}$$

and Cotlar's lemma finishes the job. In order to check this, one uses that if

$Kf(x) = \int k(x, y) f(y)\, dy$ and

$$\sup_x \int |k(x,y)|\, dy \leq 1, \quad \sup_y \int |k(x,y)|\, dx \leq 1$$

then $\|K\| \leq 1$. One then observes that

$$\left(S_\alpha^* S_\beta\right)(f)(\xi) = \int s_{\alpha,\beta}(\xi, \eta) f(\eta)\, d\eta, \text{ where}$$

$$s_{\alpha,\beta}(\xi, \eta) = \int \overline{a}_\alpha(x, \xi)\, a_\beta(x, \eta)\, e^{i x \cdot (\eta - \xi)}\, dx$$

In this integral one integrates by parts, using

$$(I - \Delta_x)^N e^{ix\cdot(\eta-\xi)} = (1+|\eta-\xi|^2)^N e^{ix\cdot(\eta-\xi)}$$

and the observation that $\bar{a}_\alpha(x,\xi)$ and $a_\beta(x,\eta)$ have disjoint x support unless $i-j \in Q_1$.

Thus, $|s_{\alpha,\beta}(\xi,\eta)| \leq A_N \dfrac{\phi(\xi-i')\,\phi(\eta-j')}{(1+|\xi-\eta|^2)^{2N}}$ if $i-j \in Q_1$, 0 elsewhere, which gives our bound.

The Calderón-Vaillancourt theorem and its proof found a remarkable application to pde in the work of R. Beals and C. Fefferman (1974) on the local solvability of linear partial differential operators of principal type, which settled a long-standing conjecture of Nirenberg – Trèves.

I will now turn to more recent applications to non-linear Schrödinger

equations. This has been a long-standing project, joint with G. Ponce and L. Vega (90-06)

The issue is the study of the local (in time) well-posedness of the initial value problem (quasi-linear Schrödinger equations)

$$\left\{\begin{aligned} \partial_t u &= -i\, a_{jk}(x,t,u,\bar{u},\nabla u,\nabla\bar{u})\,\partial_{x_j x_k} u + \\ &+ \vec{b}_1(x,t,u,\bar{u},\nabla u,\nabla\bar{u})\cdot\nabla u + \\ &+ \vec{b}_2(x,t,u,\bar{u},\nabla u,\nabla\bar{u})\cdot\nabla\bar{u} + \\ &+ c_1(x,t,u,\bar{u})\,u + c_2(x,t,u,\bar{u})\,\bar{u} \\ u|_{t=0} &= u_0 \end{aligned}\right.$$

In the nineties, our attention was focused in the so-called semi-linear case, when the second order operator is constant coefficient, i.e. $\sum_{j=1}^{k}\partial_{x_j}^2 - \sum_{j=k+1}^{n}\partial_{x_j}^2 = \mathcal{L}$, $1\le k\le n$ (98).

In studying this problem, we were forced to understand the linear Cauchy problem (in simplified form)

$$(LCP)\quad \begin{cases} \partial_t = i\mathcal{L}u + \vec{b}_1(x)\nabla u + c_1 \cdot u \\ u|_{t=0} = u_0 \end{cases}$$

and understand the issue of its well-posedness in (say) L^2. It turns out that there was previous work in this problem, which goes back to Mizohata and his former student Takeuchi (80, 81). Let $\tilde{\xi} = (\xi_1, \ldots, \xi_k, -\xi_{k+1}, \ldots, -\xi_n)$, which is a bi-characteristic for $\mathcal{L}$. Then, the Mizohata result is that a necessary condition for the L^2 well-posedness of (LCP) is:

$$\left| \mathrm{Im} \int_0^t \vec{b}_1(x+s\tilde{\omega}).\omega \, ds \right| \leq C$$

for all $\omega \in S^{n-1}$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. Thus, for instance $\vec{b}_1(x) = i.e_1$ does not work!

To illustrate our approach to (LCP), let's assume that $\vec{b}_1(x) \in C_0^\infty(B_1)$, $c_1 \in C_b^\infty(\mathbb{R}^n)$.

Our idea is to apply the "energy method" to the equation verified by $v = Cu$, where C is an invertible operator, with symbol in $S^0_{0,0}$. Thus, let us calculate

$$\frac{d}{dt}\langle Cu, Cu\rangle = \frac{d}{dt}\int Cu\,\overline{Cu}\,dx =$$

$$= \langle Ci\Delta u, Cu\rangle + \langle C\vec{b}_1.\nabla u, Cu\rangle + \langle Cc_1 u, Cu\rangle +$$

$$+ \langle u, Ci\Delta u\rangle + \langle Cu, C\vec{b}_1.\nabla u\rangle + \langle Cu, Cc_1 u\rangle$$

Integration by parts gives

$$\langle i\mathcal{L}Cu, Cu\rangle + \langle Cu, i\mathcal{L}Cu\rangle = 0$$

Thus, we can rewrite this as

$$\frac{d}{dt}\langle Cu, Cu\rangle = 2\mathrm{Re}\langle i[C\mathcal{L}-\mathcal{L}C]u, Cu\rangle +$$

$$+2\mathrm{Re}\langle C\vec{b}\cdot\nabla u, Cu\rangle +$$

$$+ O(\|u(t)\|_{L^2}^2) \quad , \text{ since}$$

C is L^2 bounded (we hope). The idea is to construct C so that the first 2 terms "almost" cancel, C is L^2 bounded and invertible, which then gives, for $v = Cu$,

$\frac{d}{dt}\langle v, v\rangle = O(\|v\|_{L^2}^2)$, which yields L^2 control of v for short time, and hence for u.

In order to achieve the above cancellation one writes things down at the symbol level. If $C(x,\xi)$ is the symbol of C, a calculation shows that the symbol of $i[CQ-QC] = -2\tilde{\xi}\cdot\nabla_x C(x,\xi)$ while the symbol of $C\vec{b}\cdot\nabla$ is $iC(x,\xi)\vec{b}_1(x)\cdot\xi$. One then has to make "small"

$$-2\tilde{\xi}\cdot\nabla_x C(x,\xi) + iC(x,\xi)\vec{b}_1(x)\cdot\xi$$

Since we need C to be invertible, c has to be non-zero, so we write $C(x,\xi) = \exp(\gamma(x,\xi))$. One then has, in the expression above

$$C(x,\xi)\{-2\tilde{\xi}\nabla_x\gamma(x,\xi) + i\vec{b}_1(x)\cdot\xi\}$$

, which needs to be small (preferably 0).

But, the equation can be integrated exactly, and the choice

$$\tilde{\gamma}(X,\xi) = -\frac{i}{2}\int_0^\infty \vec{b}_1(X+s\frac{\xi}{|\xi|})\cdot\frac{\xi}{|\xi|}\,ds$$

makes the bracket 0. (This explains the Mizohata condition mentioned before).
We seem to be done, but what symbol bounds does $\tilde{\gamma}$ verify? they are

$$\left|\partial_X^\alpha \partial_\xi^\beta \tilde{\gamma}(X,\xi)\right| \leq A_{\alpha,\beta}\left(\frac{1+|X|}{|\xi|}\right)^{|\beta|}$$

and this is optimal. Unfortunately, this symbol class is too large and the operators $C(X,\xi) = \exp(\tilde{\gamma})$, with $\tilde{\gamma}$ verifying these bounds need not be L^2 bounded!
What to do?: We now define, for R large

$$\gamma(X,\xi) = \theta(|\xi|/R)\cdot\Psi\left(\frac{R\langle X\rangle}{\langle\xi\rangle}\right)\tilde{\gamma}(X,\xi),$$

where $\theta \equiv 1$ for $|\xi|$ large, $\equiv 0$ for $|\xi|$ small and $\psi = 1 - \theta$. We can now check that γ and $c \in S^0_{0,0}$ and for R large T_R will be invertible and the above procedure can be carried out thanks to Calderón-Vaillancourt and Gårding's lemma.

In the general quasi-linear case, which we treated in 2004-06 (also joint with my former Phd student C. Rolvung), it turns out that this trick does not suffice and we are forced to face the issue of whether $\exp(\tilde{\gamma}(x,\xi))$, with $\tilde{\gamma}$ defined above gives rise to a bounded L^2 operator. The answer turns out to be yes. The proof is a

delicate application of (once more!) Cotlar's lemma. One also needs to develop an "almost calculus" for such operators, again by the use of Cotlar's lemma. In the end, one is able to obtain the local well-posedness of "general" quasi-linear Schrödinger equations.