Paraproducts and the bilinear Calderón-Zygmund theory

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Outline of the talk

- The paraproduct paradigm
- Variations and applications
- Bilinear Calderón-Zygmund singular integral operators

All the paraproducts, the paraproduct

"The name *paraproduct* denotes an idea rather than a unique definition; several definitions exist and can be used for the same purposes."

S. Janson and J. Peetre, TAMS 1988.

Leibnitz rule

For $f, g \in C_0^1(\mathbb{R})$ we have $(fg)' = f'g + fg'$.

Hence

$$f(x)g(x) = \int_{-\infty}^{x} f'(t)g(t)\,dt + \int_{-\infty}^{x} f(t)g'(t)\,dt$$

Define

$$\Pi_1(f,g)(x) = \int_{-\infty}^{x} f'(t)g(t)\,dt$$

Properties

Π_1 *satisfies*

1. $fg = \Pi_1(f,g) + \Pi_1(g,f)$ *(Product reconstruction)*
2. $\Pi_1(f,g)' = f'g$ *(Leib rule = 'half' of Leibnitz rule)*
3. *If* $h(0) = 0$, *then* $\Pi_1(f, h'(f)) = h(f)$ *(Linearization)*

$\Pi_1 : X \times Y \to Z$ not well-suited for L^p-spaces (no Hölder inequality).

A. Calderón[1] (1965)

For F, G analytic on the upper-half plane $\{s + it : s \in \mathbb{R}, t > 0\}$, Calderón defined

$$\Pi_2(F, G)(s) = -i \int_0^{\infty} F'(s + it) G(s + it)\, dt.$$

Properties

Π_2 *verifies*

1. $FG = \Pi_2(F, G) + \Pi_2(G, F)$ *(Product reconstruction)*
2. $\Pi_2(F, G)' = \Pi_2(F', G) + \Pi_2(F, G')$ *(Leibnitz rule)*
3. *If* $H(0) = 0$, *then* $\Pi_2(F, H'(F)) = H(F)$ *(Linearization)*
4. *For* $1 < p, q, r < \infty$ *with* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$

$$\|\Pi_2(F, G)\|_{L^r} \le c\, \|F\|_{H^p}\, \|G\|_{H^q} \quad \text{(Hölder's inequality)}$$

[1] Proc. Nat. Acad. Sci., 53, 1092–1099.

Simple complex manipulations

$$F(s+it) = F_1(s,t) + iF_2(s,t)$$
$$G(s+it) = G_1(s,t) + iG_2(s,t)$$

Write

$$F_1(s,t) = (f_1 * P_t)(s),$$

where

$$P(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R},$$

and

$$P_t(x) = \frac{1}{t} P\left(\frac{x}{t}\right), \quad t > 0.$$

Set $Q(x) = P'(x)$ so that $\int Q(x)\,dx = P(\infty) - P(-\infty) = 0$ and

$$(P_t)'(x) = \frac{1}{t^2} P'\left(\frac{x}{t}\right) = \frac{1}{t} Q_t(x).$$

Back to reality

Rewrite

$$\begin{aligned} F'(s+it) &= \frac{\partial F_1}{\partial s}(s,t) + i\frac{\partial F_2}{\partial s}(s,t) \\ &= \frac{\partial (f_1 * P_t)}{\partial s}(s) + i\frac{\partial (f_2 * P_t)}{\partial s}(s) \\ &= f_1 * (P_t)'(s) + if_2 * (P_t)'(s) \\ &= \frac{1}{t}(f_1 * Q_t)(s) + i\,\frac{1}{t}(f_2 * Q_t)(s) \end{aligned}$$

and since $G(s+it) = (g_1 * P_t)(s) + i(g_2 * P_t)(s)$, we only care for

$$\Pi_3(f,g)(s) := \int_0^\infty (Q_t * f)(s)(P_t * g)(s)\,\frac{dt}{t}$$

No more complex variables!

Prehistoric paraproducts

$$\Pi_3(f,g)(s) = \int_0^\infty (Q_t * f)(s)(P_t * g)(s)\,\frac{dt}{t}$$

has a discrete version (think $t = 2^{-j}$)

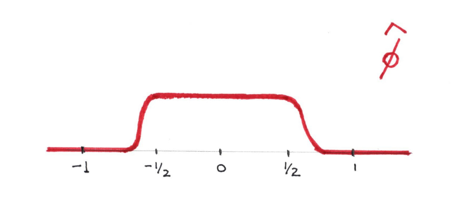
$$\Pi_4(f,g)(s) = \sum_{j\in\mathbb{Z}} (Q_j * f)(s)(P_j * g)(s)$$

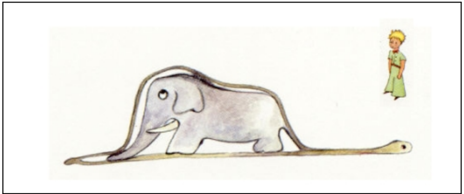
Question

What properties do Π_3 and Π_4 have?

$\phi \in \mathcal{S}(\mathbb{R}^n)$, real,
radially symmetric, and

$$\chi_{\{|\xi|\leq 1/2\}} \leq \hat{\phi} \leq \chi_{\{|\xi|\leq 1\}}$$





Why should any one be frightened by a bump function?

Elements of the Littlewood-Paley theory

Define $\psi \in \mathcal{S}(\mathbb{R}^n)$ by

$$\hat{\psi}(\xi) := \hat{\phi}(\xi/2) - \hat{\phi}(\xi), \quad \xi \in \mathbb{R}^n.$$

For $j \in \mathbb{Z}$, set

$$\widehat{\psi_j}(\xi) := \hat{\psi}(\xi/2^j) \quad \text{and} \quad \widehat{\phi_j}(\xi) := \hat{\phi}(\xi/2^j)$$

Hence, $\mathrm{supp}(\widehat{\psi_j}) \subset \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, for $j \in \mathbb{Z}$, and

$$\sum_{j \in \mathbb{Z}} \widehat{\psi_j}(\xi) = 1, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

Low-pass and band-pass filters

Define the operators S_j and Δ_j by

$$S_j(f) = \phi_j * f,$$

and

$$\Delta_j(f) = \psi_j * f.$$

In particular,

$$S_j(f) = \sum_{k \le j} \Delta_k(f) \quad \text{and} \quad f = \sum_{j \in \mathbb{Z}} \Delta_j(f),$$

Product vs. Paraproduct

Given $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$fg = \left(\sum_{j\in\mathbb{Z}} \Delta_j(f)\right)\left(\sum_{k\in\mathbb{Z}} \Delta_k(g)\right) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \Delta_j(f)\Delta_k(g)$$

Introduce the (idea of a) paraproduct between f and g as

$$\Pi_5(f,g) := \sum_{j\in\mathbb{Z}}\sum_{k\leq j} \Delta_j(f)\Delta_k(g) = \sum_{j\in\mathbb{Z}} \Delta_j(f)S_j(g)$$

That is,

$$\Pi_5(f,g) = \sum_{j\in\mathbb{Z}} (\psi_j * f)(\phi_j * g)$$

Compare with

$$\Pi_4(f,g) = \sum_{j\in\mathbb{Z}} (Q_j * f)(P_j * g)$$

Allowing for errors

Although Π_5 gives perfect product reconstruction, we consider

$$\Pi_6(f,g) = \sum_{j\in\mathbb{Z}} (\psi_j * f)(\phi_{j-2} * g)$$

so that

$$fg = \Pi_6(f,g) + \Pi_6(g,f) + R(f,g),$$

where

$$R(f,g) = \sum_{\substack{j,k\in\mathbb{Z}\\ |j-k|<2}} (\psi_j * f)(\psi_k * g).$$

Why the 2? Why allowing error at all?

To make room for bigger elephants.

The paraproduct paradigm

Π_6 verifies

Product reconstruction

$$fg = \Pi_6(f,g) + \Pi_6(g,f) + R(f,g)$$

Hölder's inequality (or other bilinear estimates)

For $1 < p, q, r < \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$

$$\|\Pi_6(f,g)\|_{L^r} \leq c\,\|f\|_{L^p}\,\|g\|_{L^q}$$

Para-Leib rule (or other "commutativity" properties)

$$\partial^\alpha \Pi_6(f,g) = \tilde{\Pi}_6(\partial^\alpha f, g)$$

Here $\tilde{\Pi}_6$ is another paraproduct(!).

Linearization formula (modulo smooth errors)

Proving the Kato-Ponce inequality

In 1988, T. Kato and G. Ponce[2] proved that, if $\alpha > 0$ and $1 < p_1, p_2, q_1, q_2, r < \infty$, with

$$1/r = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2,$$

and $\widehat{D^\alpha h}(\xi) = |\xi|^\alpha \hat{h}(\xi)$, then

$$\|D^\alpha(fg)\|_{L^r} \leq C\left(\|D^\alpha f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|D^\alpha g\|_{L^{q_2}}\right)$$

Short proof due to C. Muscalu, J. Pipher, T. Tao, and C. Thiele[3]

$$\begin{aligned}\|D^\alpha(fg)\|_{L^r} &= \|D^\alpha(\Pi_5(f,g)) + D^\alpha(\Pi_5(g,f))\|_{L^r} \\ &= \left\|\tilde{\Pi}_5(D^\alpha f, g) + \tilde{\Pi}_5(D^\alpha g, f)\right\|_{L^r} \\ &\leq C\left(\|D^\alpha f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|D^\alpha g\|_{L^{q_2}}\right)\end{aligned}$$

[2] Comm. Pure Appl. Math. 41, no. 7, 891–907.

[3] Acta Math. 193 (2004), no. 2, 269–296.

J.-M. Bony[4] (1981)

Paradifferential operators - Paraproducts - Paramultiplication

Suppose that $f \in C^\alpha$, $g \in C^\beta$ (both bounded), with $0 < \alpha < \beta$. Then

$$\overbrace{fg}^{C^\alpha} = \overbrace{\Pi_6(f,g)}^{C^\beta} + \overbrace{\Pi_6(g,f)}^{C^\alpha} + \overbrace{R(f,g)}^{C^{\alpha+\beta}}$$

Suppose $F \in C^\infty(\mathbb{R})$ with $F(0) = 0$. Then

$$F(u) = \Pi_6(u, F'(u)) + e_F(u),$$

where $e_F(u)$ is smoother than u.

[4] Ann. Sci. École Norm. Sup. 14, 209–246.

R. R. Coifman and Y. Meyer (1978)

Au-delà des Opérateurs Pseudo-différentiels

au-delà $=$ beyond

beyond $=$ para

Paradifferential operators

J. Peetre[5] (1967-1974), H. Triebel[6] (1977)

$$fg = \sum_{j\in\mathbb{Z}} (\psi_j * f)(\phi_j * g) + \sum_{j\in\mathbb{Z}} (\psi_j * g)(\phi_j * f)$$

In the context of pointwise multiplication of function spaces.

[5] New thoughts on Besov spaces. Chapter 7

[6] Ann. Mat. Pura Appl. (4) 114 (1977) 87–102.

Some famous applications. Apologies for any omissions

1. Bony's linearization formula
2. Boundedness of Calderón commutators (Coifman-Meyer)
3. $T(1)$ and $T(b)$ theorems (David-Journé-Semmes)
4. Pointwise multipliers of function spaces (Peetre, Sickel, Triebel,...)
5. Fluid dynamics (Cannone, Chemin, Lemarié-Rieusset, Meyer,...)
6. Decomposition of operators (Coifman-Meyer, Gilbert-Nahmod, Grafakos-Kalton, Lacey-Thiele...)
7. Bilinear pseudo-differential operators, 'reduced symbols' (Coifman-Meyer, Grafakos-Torres, Bényi-Torres)
8. Paracommutators (Janson-Peetre)
9. Compensated compactness (Coifman, P.L. Lions, Meyer, Semmes, Peng, Wong,...)
10. Dyadic versions (P. Auscher, O. Beznosova, O. Dragicevic, L. Grafakos, N. Katz, M. Lacey, X. Li, T. Mei, C. Muscalu, F. Nazarov, M. C. Pereyra, S. Petermichl, T. Tao, C. Thiele, S. Treil, A. Volberg, L. Ward, B. Wick,...)

As bilinear multipliers

We have

$$\Pi_6(f,g) = \sum_{j\in\mathbb{Z}} (\psi_j * f)(\phi_{j-2} * g)$$

Using the Fourier transform

$$\Pi_6(f,g)(x) = \sum_{j\in\mathbb{Z}} \int\int \widehat{\psi}_j(\xi)\widehat{\phi_{j-2}}(\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix(\xi+\eta)}\, d\xi d\eta$$

A closer look at the bilinear symbols

$$\widehat{\psi}_j(\xi)\widehat{\phi_{j-2}}(\eta)$$

For ξ and η such that $2^{j-1} \le |\xi| \le 2^{j+1}$ and $|\eta| \le 2^{j-2}$ it holds

$$2^{j-2} \le |\xi + \eta| \le 2^{j+2}$$

Bigger elephants

Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$, radial, such that

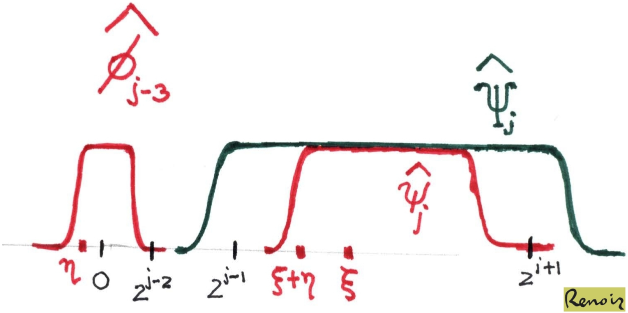
$$\widehat{\Psi}(\omega) = 1, \quad |\omega| \in [2^{-2}, 2^2]$$

and

$$\text{supp}(\widehat{\Psi}) \subset \{\omega \in \mathbb{R}^n : |\omega| \in [2^{-2} - 1/8, 2^2 + 1/8]\}.$$

Therefore,

$$\widehat{\Psi_j}(\xi + \eta)\widehat{\psi_j}(\xi)\widehat{\phi_{j-2}}(\eta) = \widehat{\psi_j}(\xi)\widehat{\phi_{j-2}}(\eta).$$



One way to use Ψ_j

$$\Pi_6(f,g)(x) = \sum_{j\in\mathbb{Z}} \int\int \widehat{\psi}_j(\xi)\widehat{\phi_{j-2}}(\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix(\xi+\eta)}\, d\xi d\eta$$

$$= \sum_{j\in\mathbb{Z}} \int\int \widehat{\Psi_j}(\xi+\eta)\widehat{\psi}_j(\xi)\widehat{\phi_{j-2}}(\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix(\xi+\eta)}\, d\xi d\eta$$

$$= \sum_{j\in\mathbb{Z}} \Psi_j * ((\psi_j * f)(\phi_{j-2} * g))(x).$$

By duality,

$$\langle \Pi_6(f,g), h\rangle = \sum_{j\in\mathbb{Z}} \langle (\psi_j * f)(\phi_{j-2} * g), \Psi_j * h\rangle.$$

Advantage: immediate access to the Littlewood-Paley pieces of h.

Typical example

Let $f \in BMO$ and $g, h \in L^2$

$$
\begin{aligned}
&|\langle \Pi_6(f,g), h\rangle| = |\sum_{j\in\mathbb{Z}} \langle (\psi_j * f)(\phi_{j-2} * g), \Psi_j * h\rangle| \\
&\leq \sum_{j\in\mathbb{Z}} \|(\psi_j * f)(\phi_{j-2} * g)\|_{L^2} \|\Psi_j * h\|_{L^2} \\
&\leq \left(\sum_{j\in\mathbb{Z}} \int |\psi_j * f|^2 |\phi_{j-2} * g|^2\right)^{1/2} \left(\sum_{j\in\mathbb{Z}} \int |\Psi_j * h|^2\right)^{1/2} \\
&\leq C \|f\|_{BMO} \|\mathcal{M} g\|_{L^2} \|\mathcal{S} h\|_{L^2} \leq C \|f\|_{BMO} \|g\|_{L^2} \|h\|_{L^2}
\end{aligned}
$$

Freezing a variable: linear singular integrals

That is,

$$\Pi_6 : BMO \times L^2 \to L^2.$$

Moreover, for **fixed** $f \in BMO$, the mapping

$$g \mapsto \Pi_6(f, g)$$

has a Calderón-Zygmund kernel. Thus,

$$g \mapsto \Pi_6(f, g)$$

is a **linear** Calderón-Zygmund singular integral operator. This explains

$$\Pi_6 : L^\infty \times C^\alpha \to C^\alpha,$$

and many other mapping properties of the form

$$\Pi_6 : L^\infty \times X \to X.$$

A convenient third wheel

The bilinear operator

$$\Pi_6(f,g) = \sum_{j\in\mathbb{Z}} (\psi_j * f)(\phi_{j-2} * g) = \sum_{j\in\mathbb{Z}} \Psi_j * ((\psi_j * f)(\phi_{j-2} * g))$$

leads to considering bilinear operators of the form

$$\Pi_7(f,g)(x) := \sum_{j\in\mathbb{Z}} \phi_j^1 * ((\phi_j^2 * f)(\phi_j^3 * g))(x)$$

$$= \int\int \left(\sum_{j\in\mathbb{Z}} \int \phi_j^1(x-w)\phi_j^2(w-y)\phi_j^3(w-z)\, dw \right) f(y)g(z)\, dydz$$

$$=: \int\int K_{\Pi_7}(x,y,z) f(y)g(z)\, dydz,$$

where ϕ^1, ϕ^2, and ϕ^3 are smooth functions, some of them with appropriate cancelation.

Dyadic cubes

We write $Q \in \mathcal{D}$ if Q is a dyadic cube.

That is, for some $k \in \mathbb{Z}^n$, $\nu \in \mathbb{Z}$

$$Q = \{x \in \mathbb{R}^n : k_i \leq 2^\nu x_i \leq k_i + 1; i = 1, \ldots, n\}$$

Set $x_Q = 2^{-\nu} k$ and also write $Q = Q_{\nu k}$.

Notice that $|Q| = 2^{-\nu n}$.

Towards a molecular representation I

Think $j = \nu$, so that

$$K_{\Pi_7}(x,y,z) = \sum_{\nu \in \mathbb{Z}} \int \phi_\nu^1(x-w)\phi_\nu^2(w-y)\phi_\nu^3(w-z)\,dw$$

$$= \sum_{\substack{\nu\in\mathbb{Z}\\ k\in\mathbb{Z}^n}} \int_{Q_{\nu k}} 2^{\nu n}\phi^1(2^\nu(x-w))2^{\nu n}\phi^2(2^\nu(w-y))2^{\nu n}\phi^3(2^\nu(w-z))\,dw$$

$$= \sum_{\substack{\nu\in\mathbb{Z}\\ k\in\mathbb{Z}^n}} \frac{|Q_{\nu k}|^{-\frac{1}{2}}}{|Q_{\nu k}|} \int_{Q_{\nu k}} 2^{\frac{3\nu n}{2}}\phi^1(2^\nu(x-w))\phi^2(2^\nu(w-y))\phi^3(2^\nu(w-z))\,dw$$

$$= \sum_{\nu\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |Q_{\nu k}|^{-\frac{1}{2}} \phi_Q^1(x)\phi_Q^2(y)\phi_Q^3(z) + E(x,y,z).$$

Here $\phi_Q^j(x) = 2^{\frac{\nu n}{2}}\phi^j(2^\nu(x - 2^{-\nu}k))$, for $Q = Q_{\nu k}$ and $j = 1,2,3$, and $E(x,y,z)$ is the kernel of a smoothing operator (see M.-Naibo, 2009)

Towards a molecular representation II

Define the bilinear kernel

$$K_{\Pi_8}(x,y,z) = \sum_{\nu\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |Q_{\nu k}|^{-\frac{1}{2}} \phi_Q^1(x)\phi_Q^2(y)\phi_Q^3(z)$$

and its associated bilinear operator

$$\Pi_8(f,g)(x) = \sum_{\nu\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |Q_{\nu k}|^{-\frac{1}{2}} \langle \phi_Q^2, f\rangle \langle \phi_Q^3, g\rangle \phi_Q^1(x).$$

Recall that

$$\phi_Q^j(x) = 2^{\frac{\nu n}{2}} \phi^j(2^\nu(x - 2^{-\nu}k)), \quad Q = Q_{\nu k}, \quad j = 1,2,3.$$

Smooth molecules

Let $Q \in \mathcal{D}$ and $M, N \in \mathbb{N}$. A smooth molecule of *regularity* M and *decay* N associated to Q is a function denoted by

$$\phi_Q = \phi_{Q_{\nu k}} = \phi_{\nu k}$$

that satisfies

$$|\partial^\gamma \phi_{\nu k}(x)| \leq \frac{C_{\gamma,N} 2^{\nu n/2} 2^{|\gamma|\nu}}{(1 + 2^\nu |x - 2^{-\nu} k|)^N}, \quad x \in \mathbb{R}^n,$$

for all $|\gamma| \leq M$.

Yes, just like $\phi_Q(x) = 2^{\frac{\nu n}{2}} \phi(2^\nu (x - 2^{-\nu} k))$ for a smooth ϕ.

M and N are specified during the applications.

Molecular paraproducts

Let $\{\phi_Q^1\}_{Q\in\mathcal{D}}$, $\{\phi_Q^2\}_{Q\in\mathcal{D}}$, $\{\phi_Q^3\}_{Q\in\mathcal{D}}$ be three families of molecules.

The associated molecular paraproduct is given by

$$\Pi_9(f,g)(x) = \sum_{Q\in\mathcal{D}} |Q|^{-1/2}\langle f,\phi_Q^1\rangle\langle g,\phi_Q^2\rangle\phi_Q^3(x)$$

Heuristics for the para-Leib rule.

Duality yields

$$\langle\Pi_9(f,g),h\rangle = \sum_{Q\in\mathcal{D}} |Q|^{-1/2}\langle f,\phi_Q^1\rangle\langle g,\phi_Q^2\rangle\langle\phi_Q^3,h\rangle.$$

Excellent set up for molecular representations of functions.

Π_9 as a bilinear CZ SIO. Bényi-M.-Nahmod-Torres (2008)

Motto: Don't freeze, bilinearize!

1. Bilinear almost diagonal estimates
2. Molecular paraproducts have **bilinear** Calderón-Zygmund kernels
3. Molecular paraproducts with cancelation are **bilinear** Calderón-Zygmund operators
4. Mapping properties of the form $X \times Y \to Z$ (as opposed to $X \times Y \to Y$)

Bilinear almost orthogonality estimates

For every $N > n+1$ there is a constant C, depending only on N and n, such that for any $\gamma, \nu, \mu, \lambda \in \mathbb{Z}$ and any $x, y, z \in \mathbb{R}^n$ the following inequality holds

$$\sum_{k \in \mathbb{Z}^n} \frac{2^{-\gamma n} 2^{\nu n/2} 2^{\mu n/2} 2^{\lambda n/2}}{[(1+2^{\nu}|x-2^{-\gamma}k|)(1+2^{\mu}|y-2^{-\gamma}k|)(1+2^{\lambda}|z-2^{-\gamma}k|)]^{5N}}$$

$$\leq \frac{C\, 2^{-\max(\mu,\nu,\lambda)n/2}\, 2^{\mathrm{med}(\mu,\nu,\lambda)n/2}\, 2^{\min(\mu,\nu,\lambda)n/2}}{((1+2^{\min(\nu,\mu)}|x-y|)(1+2^{\min(\mu,\lambda)}|y-z|)(1+2^{\min(\nu,\lambda)}|x-z|))^N}$$

Paraproducts have bilinear C-Z kernels

Estimating the size of $K_{\Pi_9}(x,y,z)$. Assume $N > 10n + 10$.

$$|K_{\Pi_9}(x,y,z)| \le \sum_{\nu\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} 2^{n\nu/2} |\phi^1_{\nu k}(y)||\phi^2_{\nu k}(z)||\phi^3_{\nu k}(x)|$$

$$\le \sum_{\nu\in\mathbb{Z}} \frac{C 2^{2n\nu}}{[(1+2^\nu|x-y|)(1+2^\nu|z-y|)(1+2^\nu|x-z|)]^N}$$

$$\le \frac{C}{(|x-y|+|z-y|+|x-z|)^{2n}}.$$

To estimate $K_{\Pi_9}(x,y,z) - K_{\Pi_9}(x',y,z)$, ask for $M = 1$.

No cancelation needed.

The $L^2 \times L^2 \to L^1$ bound (cancelation required)

Suppose $N > 10n + 10$, $M = 1$, and $\{\phi^1_Q\}_{Q\in\mathcal{D}}$, $\{\phi^2_Q\}_{Q\in\mathcal{D}}$ with cancelation.

$$|\langle \Pi_9(f,g),h\rangle| \leq \sum_{Q\in\mathcal{D}} |\langle f,\phi^1_Q\rangle||\langle g,\phi^2_Q\rangle|2^{n\nu/2}|\langle h,\phi^3_Q\rangle|$$

$$\leq \left(\sum_{Q\in\mathcal{D}} |\langle f,\phi^1_Q\rangle|^2\right)^{1/2} \left(\sum_{Q\in\mathcal{D}} |\langle g,\phi^2_Q\rangle|^2\right)^{1/2} \sup_{Q\in\mathcal{D}} 2^{n\nu/2}|\langle h,\phi^3_Q\rangle|$$

$$\leq C\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^\infty}.$$

Further extensions

1. Paraproducts with Dini continuous molecules, M.-Naibo, 2008.
2. Paraproducts and bilinear C-Z operators in spaces of homogeneous type, M. (PhD thesis, KU 2005) and Grafakos-Liu-M.-Yang (work in progress)

Thank you!