Two weight estimates, Clark measures and rank one perturbations of unitary operators

Sergei Treil[1]
(joint work with C. Liaw[2])

[1]Department of Mathematics
Brown University
[2]Department of Mathematics
Baylor University

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Rank one perturbations

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$$U_\gamma := U + (\gamma - 1)bb_1^*, \qquad \|b\| = 1, \quad b_1 := U^* b, \quad \gamma \in \mathbb{C}.$$

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- WLOG: b is cyclic, so $U = M_\xi$ in $L^2(\mu)$, $\mu(\mathbb{T}) = 1$; $b \equiv \mathbf{1}$, therefore $b_1(\xi) = \overline{\xi}$.

Spectral theorem

- If U is cyclic, i.e. if for some $b \in \mathcal{H}$ we have $\mathrm{span}\{U^n b : n \in \mathbb{Z}\} = \mathcal{H}$, then $U = M_\xi$ in $L^2(\mu)$. Vector b is called *cyclic* vector for U.
- Measure μ is not unique, but we can pick a measure $\mu = \mu_b$ associated to b,

$$(U^n b, b) = \int_{\mathbb{T}} \xi^n d\mu_b(\xi) \qquad \forall n \in \mathbb{Z},$$

  the measure μ_b is uniquely defined.
- If $\Phi : \mathcal{H} \to L^2(\mu)$, $\mu = \mu_b$ is the unitary operator such that $U = \Phi^{-1} M_\xi \Phi$, then $\Phi b = \mathbf{1}$.

Back to rank 1 perturbations

- $U_\gamma = U + (\gamma - 1)bb_1^*$, $\gamma \in \mathbb{T}$, b cyclic, $b_1 = U^*b$.
- Let b is a cyclic vector for U. It is not hard to show that then b is cyclic for all U_γ.
- We consider spectral measures $\mu_\gamma = \mu_{\gamma,b}$ for operators U_γ,
  In this case $\Phi_\gamma b \equiv \mathbf{1}$ for all $\gamma \in \mathbb{T}$.
  Since $\|b\| = 1$ all measures μ_γ are probability measures.
- The measures μ_γ are called Clark measures.
- So let $U = U_1$ be M_ξ in $L^2(\mu)$, $\mu = \mu_1$.
  Then
  $$b \equiv 1, \qquad b_1 = U^*b \equiv \overline{\xi}.$$

Goal: Want to describe unitary operators intertwining U_γ and its model: unitary $\Phi_\gamma : L^2(\mu) \to L^2(\mu_\gamma)$,

$$U_\gamma \Phi_\gamma = M_z \Phi, \qquad U_\gamma b \equiv \mathbf{1}.$$

Relations for Clark measures

- The measure μ_γ is defined

$$((U_\gamma - zI)^{-1}b, b) = \int_{\mathbb{T}} \frac{d\mu_\gamma(\xi)}{z-\xi} \qquad \forall z \notin \mathbb{T}.$$

- U_γ is a rank 1 perturbation of $U_1 = M_\xi$ in $L^2(\mu)$. Using the formula

$$(I - ac^*)^{-1} = I + \frac{1}{d}ac^*, \qquad d = 1 - (c, a),$$

one can compute the resolvent and find relations between μ_γ.

Relations for Clark measures

- Namely, let

$$T\mu(z) = \int_{\mathbb{T}} \frac{d\mu(\xi)}{1 - \overline{\xi} z}.$$

- Define $\theta(z)$ by

$$1 - \theta(z) = \frac{1}{T\mu(z)}$$

- Then

$$1 - \overline{\gamma}\theta(z) = \frac{1}{T\mu_\gamma(z)}$$

- Can rewrite

$$T\mu_\gamma(z) = \frac{T\mu(z)}{1 - (1 - \gamma)T\mu(z)}.$$

Relations for Clark measures

Function θ was introduced for a reason.

- One can show that $\theta \in H^\infty$, $\|\theta\|_\infty \leq 1$.
- Therefore the function $F_\gamma = \dfrac{1+\overline{\gamma}\theta}{1-\overline{\gamma}\theta}$ has positive real part.
- The measures μ_γ are the measures whose Poisson extension give $\operatorname{Re} F_\gamma$.

Pretend to be physicists

Let Φ be an integral operator with kernel $K(z,\xi)$

$$\Phi f(z) = \int_{\mathbb{T}} K(z,\xi) f(\xi) d\mu(\xi).$$

Want:

$$\Phi_\gamma (M_\xi + (\gamma - 1) b b_1^*) = M_z \Phi_\gamma .$$

Recall that $b \equiv \mathbf{1}$, $\Phi_\gamma b \equiv \mathbf{1}$ (in $L^2(\mu_\gamma)$), $b_1 = U^* b \equiv \overline{\xi}$ (in $L^2(\mu)$), so $\Phi_\gamma b b_1^*$ is an integral operator with kernel $1 \cdot \xi$.

$$K(z,\xi)\xi + (\gamma - 1)\xi = z K(z,\xi).$$

Solving for K we get

$$K(z,\xi) = (1-\gamma)\frac{\xi}{\xi - z} = (1-\gamma)\frac{1}{1 - \overline{\xi} z}$$

Acting as boring mathematicians

Theorem (C. Liaw, S. Treil)

Under the above assumptions

$$\Phi_\gamma f(z) = f(z) + (1-\gamma)\int_{\mathbb{T}} \frac{f(\xi)-f(z)}{1-\overline{\xi}z}\,d\mu(\xi).$$

for all $f \in C^1(\mathbb{T})$.

Is it a SIO with kernel $1/(1-\overline{\xi}z)$?

- Will show next lecture that the operators T_r with kernels $1/(1-r\overline{\xi}z)$, $\xi, z \in \mathbb{T}$ are uniformly (in $r \neq 1$) bounded (it also follows from results of V. Kapustin).
- Follows from results of A. Poltoratskii that the boundary values $T_\pm f$ (from inside and outside) of the Cauchy integral

$$Tf(z) = \int_{\mathbb{T}} \frac{f(\xi)}{1-\overline{\xi}z}\,d\mu(\xi)$$

  exist μ_γ-a.e.

$$\Phi_\gamma f(z) = f(z) + (1-\gamma)\int_{\mathbb{T}} \frac{f(\xi) - f(z)}{1-\bar{\xi}z} d\mu(\xi).$$

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  exist μ_γ-a.e.
- That implies the operators $T_+ : L^2(\mu) \to L^2(\mu_\gamma)$ are bounded and

$$\Phi_\gamma f = (\mathbf{1} - (1-\gamma)T_\pm \mathbf{1})f + (1-\gamma)T_\pm f$$

Alternative representation

From the previous slide:

$$\Phi_\gamma f = (\mathbf{1} - (1-\gamma)T_\pm \mathbf{1})f + (1-\gamma)T_\pm f$$

It is known (Aronszajn–Donoghue) that $\mu_\mathrm{s} \perp (\mu_\gamma)_\mathrm{s}$, so

$$T_\pm \mathbf{1} = \frac{1}{1-\gamma} \qquad (\mu_\gamma)_\mathrm{s}\text{-a.e.},$$

which agrees with known results.

Proof of the representation formula

$$\Phi_\gamma f(z) = f(z) + (1-\gamma)\int_{\mathbb{T}} \frac{f(\xi)-f(z)}{1-\bar{\xi}z} d\mu(\xi).$$

Idea of the proof:

$$U_\gamma = M_\xi + (\gamma-1)bb_1^*, \qquad b = \mathbf{1},\ b_1 \equiv \xi.$$

and

$$\Phi_\gamma(M_\xi + (\gamma-1)bb_1^*) = M_z\Phi_\gamma.$$

can be rewritten as

$$\Phi_\gamma M_\xi = M_z\Phi_\gamma + (1-\gamma)(\Phi_\gamma b)b_1^*.$$

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Right multiplying (*) by M_ξ and using (*) we get

$$\begin{aligned}\Phi_\gamma M_\xi^2 &= M_z \Phi_\gamma M_\xi + (1-\gamma)(\Phi_\gamma b) b_1^* M_\xi \\ &= M_z^2 \Phi_\gamma + (1-\gamma)\left(M_z(\Phi_\gamma b) b_1^* + (\Phi_\gamma b) b_1^* M_\xi\right).\end{aligned}$$

Iterating and using $\Phi_\gamma b = \mathbf{1}$ we get

$$\Phi_\gamma M_\xi^n = M_z^n \Phi_\gamma + (1-\gamma)\sum_{k=0}^{n-1}(M_z^k \mathbf{1}) b_1^* M_\xi^{n-k-1}.$$

Applying this identity to $b = \mathbf{1}$ we get for $f(\xi) \equiv \xi^n$, $n \geq 0$

$$\begin{aligned}\Phi_\gamma f(z) &= z^n + (1-\gamma)\sum_{k=0}^{n-1} z^k \int_{\mathbb{T}} \xi^{n-k} d\mu(\xi) \\ &= z^n + (1-\gamma)\int_{\mathbb{T}} \frac{\xi^n - z^n}{1-\bar{\xi} z} d\mu(\xi).\end{aligned}$$

Action of Φ_γ on $\overline{\xi}^n$

Applying the identity

$$\Phi_\gamma M_\xi^n = M_z^n \Phi_\gamma + (1-\gamma)\sum_{k=0}^{n-1}(M_z^k \mathbf{1}) b_1^* M_\xi^{n-k-1}.$$

to $f(\xi) \equiv \overline{\xi}^n$ and multiplying the result by $\overline{z}^n$ we get

$$\begin{aligned}\overline{z}^n &= (\Phi_\gamma f)(z) + (1-\gamma)\overline{z}^n \sum_{k=0}^{n-1} z^k \int_{\mathbb{T}} \xi^{n-k}\overline{\xi}^n d\mu(\xi) \\ &= (\Phi_\gamma f)(z) + \int_{\mathbb{T}} \overline{z}^n \overline{\xi}^n \frac{\xi^n - z^n}{1-\overline{\xi} z} d\mu(\xi)\end{aligned}$$

Rigidity theorem

Theorem (Rigidity Theorem)

Let a probability measure μ *on* $\mathbb{T}$ *be supported on at least two distinct points. Let* $\gamma \in \mathbb{T} \setminus \{1\}$, *and let* $\mathcal{V} f$ *be defined for* C^1 *functions* f

$$\mathcal{V} f(z) = f(z) + (1 - \gamma) \int_{\mathbb{T}} \frac{f(\xi) - f(z)}{1 - \bar{\xi} z} d\mu(\xi)$$

Assume $\mathcal{V}$ *extends to a bounded operator from* $L^2(\mu)$ *to* $L^2(\nu)$ *and assume* $\ker \mathcal{V} = \{0\}$.
Then $\exists h > 0$ *such that* $1/h \in L^\infty(\nu)$, *and* $M_h \mathcal{V}$ *is a unitary operator from* $L^2(\mu) \to L^2(\nu)$ *(equivalently, that* $\mathcal{V} : L^2(d\mu) \to L^2(|h|^2\, d\nu)$ *is unitary).*
Moreover, the measure $|h|^2 \nu$ *is exactly the Clark measure* μ_γ, *and* $\mathcal{V}$ *treated as the operator* $L^2(\mu) \to L^2(\mu_\gamma)$ *is exactly the operator* Φ_γ.

The main idea of the proof: similar unitary operators are unitarily equivalent