Regularizations of Singular Integral Operators
(joint work with C. Liaw)

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Hilbert and Cauchy Transforms:

- **Hilbert transform:**
  \[
  Tf(x) = (\pi)^{-1} \int_{\mathbb{R}} \frac{f(y)}{x - y} dy
  \]
  (integral is defined as principal value)
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- Sometimes the integration is with respect to an arbitrary measure \( \mu \) on \( \mathbb{R} \) or on \( \mathbb{C} \).
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- Sometimes the integration is with respect to an arbitrary measure \( \mu \) on \( \mathbb{R} \) or on \( \mathbb{C} \).
- The case with \( \mu \) on \( \mathbb{C} \) (Cauchy Transform) appears in problems related to analytic capacity. The measure \( \mu \) is “one-dimensional”
  \[ \mu(D_{x,r}) \leq Cr \]
  and one is interested in boundedness of such operators in \( L^2(\mu) \).
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  and one is interested in boundedness of such operators in \( L^2(\mu) \).

- But sometimes we are interested in boundedness \( L^2(\mu) \rightarrow L^2(\nu) \), and there is no apriori restrictions on \( \mu \) and \( \nu \).
An example:

- Let $A := M_t$ in $L^2(\mu)$, and let $A_\alpha = A + \alpha(\cdot, b)b$, where $b$ is cyclic for $A$.
  WLOG can assume that $b \equiv 1$.
- By the spectral theorem $A_\alpha$ unitarily equivalent to $M_s$ in $L^2(\mu_\alpha)$.

$$M_s U_\alpha = U_\alpha A_\alpha, \quad U_\alpha : L^2(\mu) \rightarrow L^2(\mu_\alpha) \text{ unitary.}$$

$U_\alpha b$ is cyclic, so WLOG can assume $U_\alpha b = U_\alpha 1 = 1$.

Informal reasoning gives us:

if $U$ is an integral operator with Kernel $K(s, t)$, then $K(s, t) = \alpha/(s - t)$. 
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Indeed,

$$sK(s, t) = U_\alpha M_t + \alpha(\cdot, b)U_\alpha b$$

$$= K(s, t)t + \alpha 1$$

so $K(s, t) = \alpha/(s - t)$.
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**Informal reasoning gives us:**

If $U$ is an integral operator with Kernel $K(s, t)$, then $K(s, t) = \alpha/(s - t)$.

**Theorem (C. Liaw, S.Treil, 2009)**

$$U_\alpha f(s) = f(s) - \alpha \int_{\mathbb{R}} \frac{f(t) - f(s)}{t - s} d\mu(t), \quad \forall f \in C^1_c.$$
Rigidity theorem

Theorem (C. Liaw–S. Treil, 2009)

Let an operator $V$ defined on $C^1_c$ by

$$Vf(s) = f(s) - \alpha \int_{\mathbb{R}} \frac{f(t) - f(s)}{t - s} d\mu(s)$$

extends to a bounded operator $L^2(\mu) \to L^2(\nu)$, and let $\ker V = \{0\}$. Then there exists $h \geq 0$ such that $1/h \in L^\infty(\nu)$, and $M_h V$ is a unitary operator from $L^2(\mu) \to L^2(\nu)$.

Moreover, the unitary operator $U := M_h V$ gives the spectral representation of the operator $A_\alpha := M_t + \alpha(\cdot, b)b$, $b \equiv 1$, in $L^2(\mu)$, namely $UA_\alpha = M_s U$, where $M_s$ is the multiplication by the independent variable $s$ in $L^2(\nu)$.

In other words, all operators of such type appear from rank one perturbations.
Calderón–Zygmund operators

A Calderón–Zygmund operator of dimension $d$ in $\mathbb{R}^N$, $d \leq N$, is an integral operator, bounded in $L^2$ and with kernel $K$ satisfying the following growth and smoothness conditions

1. $|K(x, y)| \leq \frac{C_{cz}}{|x - y|^d}$ for all $x, y \in \mathbb{R}^N, x \neq y$.

2. There exists $\alpha > 0$ such that

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C_{cz} \frac{|x - x'|^\alpha}{|x - y|^{d+\alpha}}$$

for all $x, x', y \in \mathbb{R}^N$ such that $|x - x'| < |x - y|/2$.

Has singularity at the diagonal $x = y$. The measure is at best $d$-dimensional, $\mu(B_{x, r}) \leq Cr^d$, so integral is not well defined.

How to interpret?
Let $m$ be a function in $\mathbb{R}^N$, $m \equiv 0$ in a neighborhood of 0, and $m(x) \equiv 1$ in a neighborhood of $\infty$.

Kernels $K_\varepsilon(x, y) = K(x - y)m((x - y)/\varepsilon)$ do not have singularity near the diagonal $x = y$, so the integral

$$T_\varepsilon f(x) = \int K_\varepsilon(x, y)f(y)d\mu(y)$$

is well defined for $f \in L^\infty_c$.

We say that $T$ is bounded if $T_\varepsilon$ are uniformly (in $\varepsilon$ bounded).

If $m(x) = 1_{[1, \infty)}(|x|)$ we get the classical truncation. Smooth truncations are also used (and are more natural from my point of view).

Does boundedness depend on regularization?
Interpreting CZO: axiomatic approach

- We assume that bilinear form $\langle Tf, g \rangle$ is defined on a dense set; in classical situations when one works with Lebesgue measure in $\mathbb{R}^N$ or on a smooth manifold it is often $C^\infty_c \times C^\infty_c$.

- The fact that $T$ is an integral operators means simply that

$$\langle Tf, g \rangle = \int \int K(x, y)f(y)g(x)d\mu(y)d\nu(x)$$

for compactly supported $f$ and $g$ with separated supports.

- A multiplication operator $M_\varphi$, $M_\varphi f = \varphi f$ is an operator with kernel $K(x, y) \equiv 0$.

- If $\mu = \nu$, $\mu(B_{x,r}) \leq Cr^d$, the generalization of Cotlar inequality implies that the truncations $T_\varepsilon$ are uniformly bounded for all reasonable truncating functions $m$.

Nothing was known in the two-weight case!
Radon measures $\mu$ and $\nu$ in $\mathbb{R}^n$ are fixed.

$\mu$ and $\nu$ do not have common atoms; no other restriction is assumed.

A singular kernel $K(x, y)$ is a function in $L^2_{\text{loc}}(\mu \times \nu)$ off the diagonal $x = y$ (can blow-up at the diagonal).

**Definition (Restricted boundedness)**

A kernel $K$ is called $L^p$ restrictedly bounded if

$$|\langle Tf, g \rangle| = \left| \int \int K(x, y)f(y)g(x) d\mu(y) d\nu(x) \right| \leq C \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\nu)}$$

for all $f, g \in L^\infty_c$, supp $f \cap$ supp $g = \emptyset$.

Here $1/p + 1/p' = 1$.

The best constant $C$ is called the restricted norm.
Main result

Recall:

- $m$ is a cut-off function, $m \equiv 0$ in a neighborhood of 0 and $m \equiv 1$ in a neighborhood of $\infty$.
- $T_\varepsilon$ are “truncated operators” with kernels $K_\varepsilon$,
  \[ K_\varepsilon(x, y) = K(x, y)m\left(\frac{x - y}{\varepsilon}\right); \]
- Measures $\mu$ and $\nu$ do not have common atoms.
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**Theorem (C. Liaw, S. Treil)**

Let \( K \) be an \( L^p \) restrictedly bounded singular kernel, and let \( m \in C^\infty \). Then the regularized operators \( T_\varepsilon \) are uniformly (in \( \varepsilon \)) bounded.
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In fact it is sufficient to assume that \( 1 - m \) is in the Sobolev space \( H^s \), \( s > N/2 \).

Recall: \( f \in H^s \iff \int_{\mathbb{R}^N} (1 + |x|)^{2s} |\hat{f}(x)|^2 dx < \infty. \)
Classical truncations

For interesting operators the result holds with classical truncations 
\( m(x) = 1_{[1,\infty)}(|x|) \) as well!

Interesting operator: convolution kernel, \( K(x, y) = \tilde{K}(y - x) \),

\[
\tilde{K}(x) = A(|x|)B(x/|x|),
\]

where \( A(r) \geq 0 \) for all \( r > 0 \) and \( B \) is a function (with values in some \( \mathbb{R}^m \)) in the Sobolev space \( H^k \), \( k > N/2 \) on the unit sphere \( S_{N-1} \) in \( \mathbb{R}^N \).

Examples

- \( \tilde{K}(z) = 1/z \), \( z \in \mathbb{C} \) — Cauchy Transform.
- \( \tilde{K}(z) = 1/z^2 \), \( z \in \mathbb{C} \) — Beurling–Ahlfors Transform;
- \( \tilde{K}(x) = x/|x|^{d+1}, x \in \mathbb{R}^N \) — Riesz Transform of order \( d \) in \( \mathbb{R}^N \).
Recall: Riesz Transform of order $d$ is convolution integral operator with kernel $K(x, y) = (x - y)/|x - y|^{d+1}$.

**Theorem (C. Liaw, S. Treil)**

For Riesz Transform of order $d$ the restricted $L^p$ boundedness, $1 < p < \infty$, implies the following generalized two-weight Muckenhoupt $A^d_p$ condition of order $d$;

$$\sup_B (\text{diam } B)^{-d} \mu(B)^{1/p'} \nu(B)^{1/p} < \infty;$$

here the supremum is taken over all balls in $\mathbb{R}^N$.

In particular, if $\mu = \nu$ then $\mu$ is $d$-dimensional, $\mu(B_{x,r}) \leq Cr^d$. It was known before, but the proofs were ugly.

If $d = N$ then $\mu_s \perp \nu_s$. 
A simple idea: Schur multipliers

- Replacing $K(x, y)$ by $K(x, y)e^{-iax}e^{iay}$, $a \in \mathbb{R}$ does not change its restricted norm.
- So if $\rho \in L^1(\mathbb{R}^N)$, then for the kernel

$$\int_{\mathbb{R}} K(x, y)e^{-iax}e^{iay} \rho(a)da = K(x, y)\hat{\rho}(x - y)$$

its restricted norm increases at most $\|\rho\|_1$ times.
- Functions $m$ in the Wiener Algebra $W = \mathcal{F}(L^1) + c$ are Schur multipliers (multipliers with respect to restricted norm).
- If $m = \hat{\rho}$, $\rho \in L^1$, then $m(s/\varepsilon) = \mathcal{F}\{x \mapsto \varepsilon^N \rho(\varepsilon x)\}(s)$, and

$$\int_{\mathbb{R}^N} |\varepsilon^N \rho(\varepsilon x)|dx = \int_{\mathbb{R}^N} |\rho(x)|dx = \|\rho\|_1.$$

so the Schur multipliers $m((x - y)/\varepsilon)$ are uniformly (in $\varepsilon$) bounded.
If \( \varphi \in H^k, \ k > N/2 \), then \( \varphi \in \mathcal{F}(L^1) \) (easy exercise in Cauchy–Schwartz).

Note that the rescaling \( \varphi \mapsto \varphi_\varepsilon, \ m_\varepsilon(x) = m(x/\varepsilon) \) does change Sobolev norm, but the norm in the Wiener Algebra \( W \) is preserved.

So, if \( 1 - m \in C_C^\infty \subset W^k, \ k > N/2 \), then the kernels \( K_\varepsilon \),

\[
K_\varepsilon(x, y) m((x - y)/\varepsilon)
\]

are uniformly (in \( \varepsilon \)) restrictedly bounded.
Schur multipliers: an example

Let \( \rho(x) = e^{-x}1_{[0,\infty)} \), and let

\[
m(s) := 1 - \hat{\rho}(s) = \frac{s}{s - i}.
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Schur multipliers: an example

Let $\rho(x) = e^{-x}1_{[0,\infty)}$, and let

$$m(s) := 1 - \hat{\rho}(s) = \frac{s}{s - i}.$$

Then $m(s/\varepsilon) = \frac{s}{s - i\varepsilon}$. 

We used Schur multipliers before in complex analysis, probably without noticing it!
Let $\rho(x) = e^{-x}1_{[0,\infty)}$, and let

$$m(s) := 1 - \hat{\rho}(s) = \frac{s}{s - i}.$$

Then $m(s/\varepsilon) = \frac{s}{s - i\varepsilon}$.

Then for $K(x, y) = 1/(x - y)$ we have

$$K_\varepsilon(x, y) = \frac{1}{x - y} \times \frac{x - y}{x - y - i\varepsilon} = \frac{1}{x + i\varepsilon - y}$$

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Proposition

Let $K \in L^2_{\text{loc}}(\mu \times \nu)$ (everywhere, not just off the diagonal $x = y$), and let $K$ be restrictedly bounded with restricted norm $C$. Then the integral operator $T$ with kernel $K$ is bounded, $\|T\| \leq 2C$.

Note that $T$ is well defined on $f \in L^\infty_c$.

Idea of the proof:

- Restricting everything to a compact set assume WLOG that $T$ is compact $L^2(\mu) \to L^2(\nu)$. Take $f$ and $g$.
- Considering $f$ only on “almost half” of the support and $g$ on “almost the other half”, we get the sequences $f_n, g_n$ with separated supports such that $f_n \to \frac{1}{2} f, g_n \to \frac{1}{2} g$ weakly.
- $\langle Tf_n, g_n \rangle$ can be estimated because of restricted boundedness, and $\langle Tf_n, g_n \rangle \to \frac{1}{4} \langle Tf, g \rangle$.
Lemma (One weight version)

Let $\sigma$ be a Radon measure without atoms in $\mathbb{R}^N$. There exist Borel sets $E_n^1, E_n^2, n \in \mathbb{N}$ such that $\text{dist}(E_n^1, E_n^2) > 0 \ \forall n$.

1. The operators $P_n^k, P_n^k f := 1_{E_n^k} f, k = 1, 2$ converge to $\frac{1}{2} I$ in weak operator topology in $L^2(\sigma)$.

2. For any $p \in [1, \infty)$ and for $k = 1, 2$

$$\lim_{n \to \infty} \| 1_{E_n^k} f \|_{L^p(\sigma)} = 2^{-1/p} \| f \|_{L^p(\sigma)}, \quad \forall f \in L^p(\sigma).$$

- Trivially to do for Lebesgue measure
- More work needed for general case.

One can get a two weight version from this Lemma.
Lemma (Two weight version)

Let $\mu$ and $\nu$ be Radon measures in $\mathbb{R}^N$ without common atoms. Here exist Borel sets $E^1_n, E^2_n, n \in \mathbb{N}$ such that $\text{dist}(E^1_n, E^2_n) > 0 \ \forall n$.

1. The operators $P^1_n$ and $P^2_n$, given by
   
   \[ P^1_n f := 1_{E^1_n} (f_{\mu c} + \frac{1}{2} f_{\mu a}) \]
   \[ P^2_n g := 1_{E^2_n} (g_{\nu c} + \frac{1}{2} g_{\nu a}) \]
   converge to $\frac{1}{2} I$ in weak operator topology of $L^2(\mu)$ and $L^2(\nu)$ respectively.

2. For any $p \in [1, \infty)$ and for any $f \in L^p(\mu)$, $g \in L^p(\nu)$

   \[ \lim_{n \to \infty} \| 1_{E^1_n} f \|_{L^p(\mu)} \leq 2^{-1/p} \| f \|_{L^p(\mu)}, \]
   \[ \lim_{n \to \infty} \| 1_{E^2_n} g \|_{L^p(\nu)} \leq 2^{-1/p} \| g \|_{L^p(\nu)}. \]

- If $\mu$ and $\nu$ do not have atoms, apply one weight lemma to $\sigma = \mu + \nu$: the sets $E^1_n, E^2_n$ from this lemma would do a trick.
- Atoms: add one atom to each set at each step, and remove small neighborhoods of added atoms from the other set.
Proof of the one-weight lemma

**Lemma (One weight version)**

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1. The operators $P_{n}^k, P_{n}^kJ := 1_{E_{n}^k}J$, $k = 1, 2$ converge to $\frac{1}{2}I$ in weak operator topology in $L^2(\sigma)$.

2. For any $p \in [1, \infty)$ and for $k = 1, 2$

   $$\lim_{n \to \infty} \|1_{E_{n}^k}J\|_{L^p(\sigma)} = 2^{-1/p}\|J\|_{L^p(\sigma)}, \quad \forall J \in L^p(\sigma).$$
Proof of the one-weight lemma

- Since $P_n$ are always contractions, sufficiently to check everything on dense sets.
- Sufficient to check everything only on $1_Q$ for standard dyadic cubes in $\mathbb{R}^N$, i.e. on cubes of form $Q = 2^k([0, 1)^N + j)$, $k \in \mathbb{Z}$, $j \in \mathbb{Z}^N$.
- Sufficient to show that given $\varepsilon > 0$ and the size $2^k$ we can split all the cubes of this size almost in half with relative error $\varepsilon$:

$$\left| \sigma(Q \cap E^{1,2}) - \frac{1}{2} \sigma(Q) \right| \leq \varepsilon \sigma(Q).$$

Note that the estimate for size $2^k$ implies the same estimate for bigger dyadic cubes.
Proof of the one-weight lemma: splitting a cube

Want to split cubes with relative error $\varepsilon$,

$$\left| \sigma(Q \cap E^{1,2}) - \frac{1}{2} \sigma(Q) \right| \leq \varepsilon \sigma(Q).$$

Let us construct the disjoint sets $E^{1,2}$, we then shrink them to make separated.

- Pick a size $\delta = 2^{-m}$ such that $\sigma(R) \leq \frac{1}{2} \varepsilon \sigma(Q)$ for every dyadic cube $R \subset Q$ of this size. Follows from continuity of the measure.
- Split cubes $Q$ into dyadic cubes $R$ of this size, and order these cubes.
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- Split cubes $Q$ into dyadic cubes $R$ of this size, and order these cubes.
- Put $R_1$ to $E^1$ and $R_2$ to $E^2$. 
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- Then always put the next cube to the set of smaller measure $\sigma$. 
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- Pick a size $\delta = 2^{-m}$ such that $\sigma(R) \leq \frac{1}{2} \varepsilon \sigma(Q)$ for every dyadic cube $R \subset Q$ of this size. Follows from continuity of the measure.
- Split cubes $Q$ into dyadic cubes $R$ of this size, and order these cubes.
- Put $R_1$ to $E^1$ and $R_2$ to $E^2$.
- Then always put the next cube to the set of smaller measure $\sigma$.
- We end up with error $\leq \frac{1}{2} \varepsilon \sigma(Q)$. 

Proof of the one-weight lemma: shrinking the cubes

We constructed disjoint sets $E^{1,2}$, now we want to make them separated.

- Replace each small cube $R$ by $rR$ where $r < 1$ and close to 1;
- $rR$ means dilation with respect to the corner, not the center: $r[0, 1)^N = [0, r)^N$, i.e. we dilate this cube with respect to the origin.
- With such dilation, $R = \bigcup_{r \in (0,1)} rR$, and thus it follows from countable additivity that
  \[
  \sigma(R) = \lim_{r \to 1^-} \sigma(rR).
  \]
Assumption about no common atoms is not really restriction: Interaction between $\mu_c$ and $\nu$ and the interaction between $\mu$ and $\nu_c$ is given by the main result. Interaction between $\mu_a$ and $\nu_a$ is described by a matrix.

The result about two weight Muckenhoupt condition is obtained by multiplying $K$ by an appropriate Schur multiplier to get the kernel satisfying

$$K(x, y) \geq r^{-d} \quad \text{for } |x - y| \leq r.$$ 

The modified kernel is positive in some direction.

The result about classical truncations is obtained similarly: we use an appropriate Schur multiplier to get the positive kernel dominating the difference between the classical and smooth truncation.