Cora Sadosky: Her Mathematics and Mentorship

Rodolfo H. Torres

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An Afternoon in Honor of Cora Sadosky

New Mexico Analysis Seminar

Albuquerque, New Mexico

April 4, 2014

I want to thank the organizers for asking me to present this talk about Cora Sadosky.

In particular I want to thank:

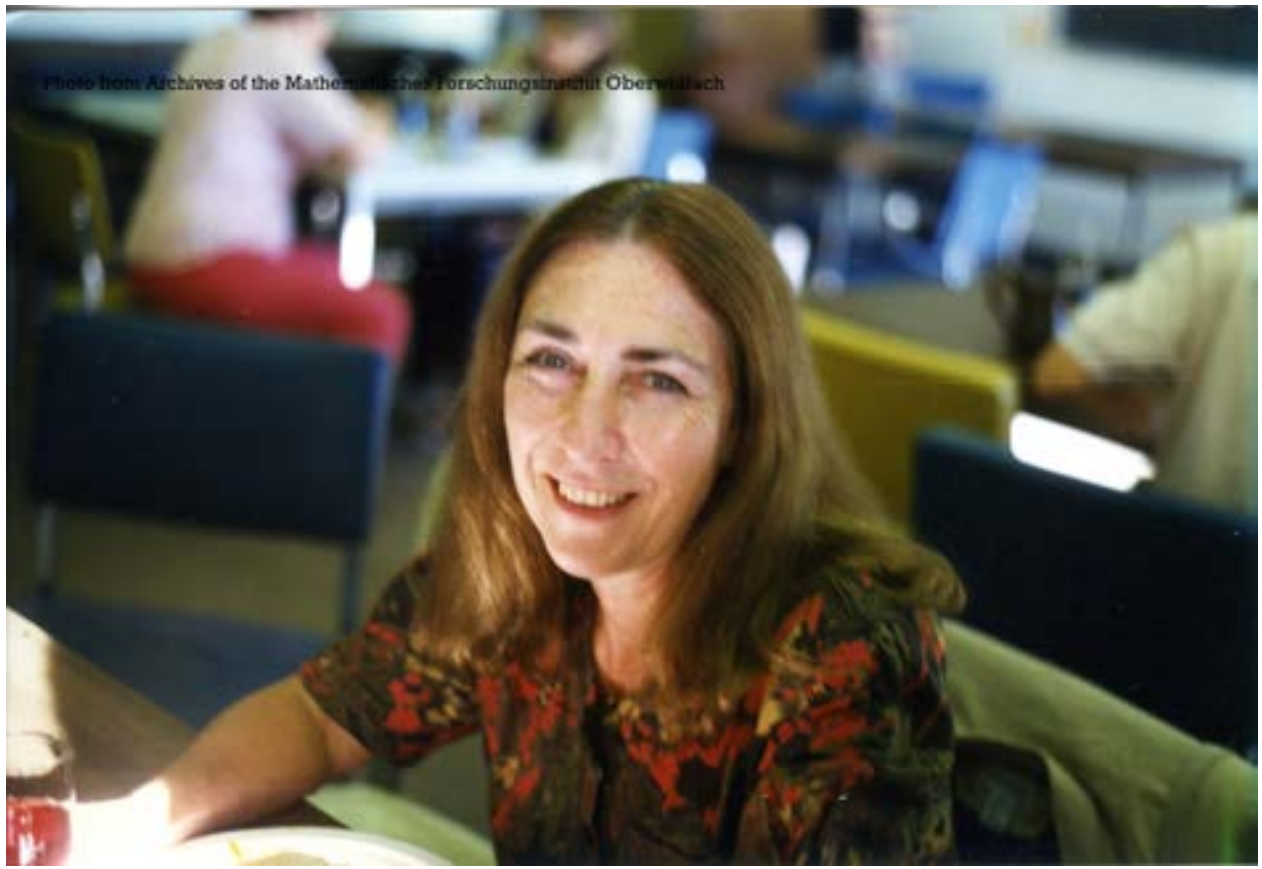
Cristina Pereyra for the idea to have this “Afternoon in Honor of Cora Sadosky”

Andrea Nahmod for encouraging me to give this talk despite my initial resistance

and very specially

Estela Gavosto for her enormous help preparing this talk.

Give me your tired, your poor,
Your huddled masses yearning to breathe free...[1]



Cora Sadosky
May 23, 1940 - December 3, 2010

[1] From *The New Colossus* by Emma Lazarus; the poem engraved on the Statue of Liberty

"Cora was a vibrant, strong minded and outspoken woman, who fought all her life for human rights, who helped uncountable young mathematicians without expecting anything in return.

[2] *Remembering Cora Sadosky*, E.A. Gavosto, A. Nahmod, C. Pereyra, G. Ponce, R.T., W. Urbina, AWM Newsletter, Volumen: 41 (2) 2011.

"Cora was a vibrant, strong minded and outspoken woman, who fought all her life for human rights, who helped uncountable young mathematicians without expecting anything in return.

Cora was a phenomenal mathematician, in a time when it was not easy for women, and she championed this cause all her life, becoming president of the AWM in the early 90s." [2]

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Cora use to tell us that

"Antoni Zygmund taught her to judge mathematicians by their theorems; that when someone spoke highly of a mathematician, Zygmund would always like to see the theorems such mathematician had proved. She did the same. If you said to her that you like mathematician X she would ask you to explain his/her mathematics." [3]

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Unfortunately I will only cover a small part of her mathematical accomplishments, but I will also try to provide you with glimpses of her personality through quotes of her own words

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"But she [Cora] never let disagreements she may have with a mathematics colleague on other issues affect her appreciation of his/her mathematics." [4]

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Brief Biographical Information

Cora was the daughter of two mathematicians

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[6] Picture borrowed from www.sobretiza.com.ar/2013/05/17/clementina-cumplio-52-anos/#axzz2xO23TGBJ

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Cora Ratto de Sadosky (1912-1981)[5]

Manuel Sadosky (1914-2005)[6]

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She studied in Paris with M. Frechet but actually got her PhD in Mathematics in her 40's, back in Buenos Aires working with M. Cotlar. With him, she authored one of the first modern books in Linear Algebra in Spanish and also edited a prestigious publication series at the UBA (with authors like L. Schwartz and A.P. Calderón).

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She created the Albert Einstein Foundation to provide financial support to talented students. A prize in her name was established in Vietnam in 1996. It is awarded to young women participating in the math olympics.[7]

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He and Cora Ratto were instrumental in creating a modern School of Sciences with a tremendous impact in the scientific landscape of Argentina. But in 1966 the university was taken over by the military (losing its autonomy), causing the resignation of 400 scientists to their positions and eventually a life in exile for Cora's family in Uruguay, Venezuela, and Spain.

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He returned to Argentina in the 80's with a new democratic government and became Secretary of Science and Technology. He was elected "Illustrious citizen" of the city of Buenos Aires in 2003.[8]

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Like her mother, she resigned in protest from the UBA in 1966 after the military intervention. After several temporary positions, including one at Johns Hopkins Univ., and periods where she had to take on other type of jobs to make a living, Cora and family had to leave Argentina in 1974 because of political persecution. They all moved to Venezuela, where Cora continued her lifelong collaboration with Cotlar. In 1980 she moved to Howard University where she became a full professor in 1985.[9]

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Cora returned for a year to Argentina in 1984. She conducted the rest of her mathematical career in the US but continued with her interest in Argentina and Venezuela where she helped many young mathematicians, some of whom continued with Cora's assistance their mathematical careers in the US.
After she retired from Howard, Cora moved to California to live close to her daughter Cora Sol, son in law Tom and granddaughter Sasha.

Other Academic Positions/Honors

- Member in residence of the Institute of Advanced Studies, Princeton, 1978-1979.
- Visiting Professorship for Women, NSF, spent a year at the Institute of Advanced Studies, Princeton, 1983-1984.
- Visiting Professor University of Buenos Aires, 1984-1985.
- Career Advancement Award, NSF, spent a year at the Mathematical Science Research Institute, Berkeley, 1987-1988.
- Visiting Professorship for Women, NSF, spent a year at the University of California Berkeley, 1995-1996.
- Fellow of the American Association for the Advancement of Science.

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Research Areas and Collaborators

Cora Sadosky wrote about 60 articles in several areas of Harmonic Analysis and Operator Theory.

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Publications (by number in area)

Fourier analysis Functional analysis Functions of a complex variable History and biography Integral equations Integral transforms, operational calculus Operator theory Probability theory and stochastic processes Real functions Systems theory; control

Co-authors (by number of collaborations)

Alpay, Daniel Arocena, Rodrigo Ball, Joseph A. Bolotnikov, Vladimir Chang, Der-Chen E. Christ, Michael Cotlar, Mischa Dafni, Galia Dijksma, Aad Fabes, Eugene Barry Ferguson, Sarah H. Kenig, Carlos E. Pott, Sandra Rovnyak, James Vinnikov, Victor Weiss, Guido L. Wheeden, Richard L.

A better idea about her specific research topics is given by a Wordle cloud of a collection of titles of her articles

Some samples of Cora Sadosky's Mathematics

a) *Parabolic Singular Integral Operators*

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Cora did her thesis in the subject under the direction of Calderón and Zygmund. In her own words:

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"I was obsessed with parabolic singular integrals, which seemed the natural object to study after Calderón's success with elliptic and hyperbolic PDEs. Calderón encouraged me in that interest, and, as the problem was in the air, very soon afterwards a first paper on the subject appeared by B. Frank Jones.

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and she added

[10] *Alberto Pedro Calderón (1920-1998)*, M. Christ, C.E. Kenig, C. Sadosky, and G. Weiss, Notices Amer. Math. Soc. 45 (1998), no. 9, 1148-1153.

and she added

"...through C-Z correspondence, we found out that Zygmund had assigned one of his students, Eugene Fabes, a problem close to mine and that we had both proved the pointwise convergence of parabolic singular integrals (by different methods)! Panic struck;

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"...through C-Z correspondence, we found out that Zygmund had assigned one of his students, Eugene Fabes, a problem close to mine and that we had both proved the pointwise convergence of parabolic singular integrals (by different methods)! Panic struck; Calderón defended my priority on the problem, but all was solved amicably, and upon my return Gene and I wrote our first result as a joint paper." [10]

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B.F. Jones (1964) introduced singular integrals of the type

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where $K(x,t) = 0$ for $t < 0$, $K(\lambda x, \lambda^m t) = \lambda^{-d-m} K(x,t)$ for some $m > 0$, $\int_{\mathbb{R}^d} K(x,1)dx = 0$ and K has some appropriate smoothness.

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These singular integrals arise in parabolic differential equations

$$\frac{\partial u}{\partial t}(x,t) = (-1)^m P(D)u(x,t), \quad (x,t) \in \mathbb{R}^d \times (0,\infty)$$

for P and appropriate homogeneous polynomial of degree m.

If Γ is the fundamental solution, then

$$u(x,t) = \int_0^t \int_{\mathbb{R}^d} \Gamma(x-y, t-s) f(y,s)\, dyds$$

solves $\dfrac{\partial u}{\partial t} - (-1)^m P(D)u = f$.

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For example, if $d = 1$ and we consider the heat equation (m=2), taking two derivatives of the fundamental solution lead to integrals with kernels of the form

$$K(x,t) \approx \frac{1}{t^{3/2}} \left(\frac{x^2}{8t} - \frac{1}{4} \right) e^{\frac{-x^2}{4t}}$$

John proved the convergence in L^p-sense and posed the question about the almost everywhere convergence in $\mathbb{R}^n \times (0, \infty)$.

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Theorem (Fabes-Sadosky)
If $f \in L^p$ for $1 < p < \infty$ and

$$T^* f(x,t) = \sup_{\epsilon > 0} |T_\epsilon f(x,t)|,$$

then

$$\|T^* f\|_p \lesssim \|f\|_p.$$

More generally Sadosky (1966) studied operators given by kernels with homogeneities of the form

$$K(\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_n} x_n) = \lambda^{-(\alpha_1+\cdots+\alpha_n)} K(x_1, \ldots, x_n)$$

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as principal values singular integrals with respect to a proper pseudo distance which matches the homogeneity of the problem. For example, for appropriate m,

$$\rho(x, y) = \left(\sum_{j=1}^{n} |x_j - y_j|^{m/\alpha_j} \right)^{1/m}$$

Sadosky also considered fractional singular integral versions of the above operators, and led to work with Cotlar (1967) on *"quasi-homogeneous" Bessel potential spaces*, extending to the parabolic setting work at the time of N. Aronszajn, K.T. Smith and A.P. Calderón.

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(For the young ones in the audience: yes there was a time when we did not have internet.)

b) *The Helson-Szegö theorem*

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Mischa Cotlar (1912-2007) [11]

An incredibly talented mathematician whose only formal education was his PhD from Chicago, Cotlar was also a devoted pacifists and, not surprisingly, a close friend of Cora's family.

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The 2007 New Mexico Seminar website in his honor reads:

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[13] *On the life and work of Mischa Cotlar*, C. Sadosky, Rev. Un. Mat. Argentina, 49-2 (2010), i-iv.

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"Mischa Cotlar was an exceptional mathematician and human being. Generations of mathematicians in Venezuela, Argentina, and other Latin American countries grew under his guidance. He was one of the world experts in harmonic analysis and operator theory."[12]

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Speaking of their joint work, Cora wrote:

"*In Caracas, Mischa and I began to collaborate in earnest and together we established an ambitious research program. Mathematically, our Caracas exile was extraordinarily productive. Although Mischa was part of the Zygmund school, he had an astonishing intellectual affinity with the Ukrainian school of Mathematics lead by Professors Krein and Gohberg, the leaders of the extraordinarily original and fertile school of operator theory. In spite of my analytic upbringing, I could not resist Mischa's daring approaches to operator theory."* [13]

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Indeed, the ambitious Cotlar-Sadosky program brought to light beautiful connections between different areas in the study of moments theory and lifting theorems for measures, Toeplitz forms, Hankel operators, and scattering systems. All this was done both in a very general setting, but also on concrete applications in harmonic analysis involving weighted norm inequalities and the ubiquitous role of BMO in the intersection of many topics in analysis.

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I will briefly discuss some of their achievements related to the Helson-Szegö Theorem, highlighting only a couple of remarks about different characterizations as described in their own works.

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Indeed, the ambitious Cotlar-Sadosky program brought to light beautiful connections between different areas in the study of moments theory and lifting theorems for measures, Toeplitz forms, Hankel operators, and scattering systems. All this was done both in a very general setting, but also on concrete applications in harmonic analysis involving weighted norm inequalities and the ubiquitous role of BMO in the intersection of many topics in analysis.

I will briefly discuss some of their achievements related to the Helson-Szegö Theorem, highlighting only a couple of remarks about different characterizations as described in their own works.

I believe Sergei will talk in more details about work of Arozena, Cotlar and Sadosky and the connection between he Helson-Szegö Theorem and Toeplitz forms.

Helson and Szegö (1960) proved that for $\omega \geq 0$ and $d\mu = \omega dx$,

$$\int_{\mathbb{T}} |Hf|^2 \, d\mu \lesssim \int_{\mathbb{T}} |f|^2 \, d\mu$$

if and only if

$$\omega = e^{u+Hv}, \qquad u, v \in L^\infty, \qquad \|v\|_\infty < \pi/2 \tag{1}$$

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Let R_2 be the collection of all such measures.

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"When A. Zygmund read the manuscript of the paper [by Helson-Szegö] he remarked to the authors that they had given the necessary and sufficient condition for the boundedness of the Hilbert transform in L^2 , solving an important problem. The consequences of this result continue to be of interest in a variety of analysis settings."

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Of course there is also the Hunt-Muckenhoupt-Wheeden Theorem (1973) and we also have that H is bounded if and only if $\omega \in A_2$, i.e.

$$\sup_I \left(\frac{1}{|I|}\int_I \omega\, dx\right)\left(\frac{1}{|I|}\int_I \omega^{-1}\, dx\right) < \infty \tag{2}$$

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The condition in (1) is good to construct weights but it is almost useless to verify that something is a weight; while for the condition in (2) the situation is the opposite.

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If $f \in BMO$, then by the John-Nirenberg inequality $e^{cf} \in A_2$ for some appropriate c. It follows from the H-M-W Theorem that H is bounded on $L^2(e^{cf})$ and by the H-S Theorem

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However, as Cora observed:

"The trick of the proof sketched above is to apply the necessary condition of one theorem and the sufficient condition of the other. Those happen to be the hard part of both theorems..." [14]

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A *moment problem* related to the H-S Theorem is that of characterizing the Fourier coefficients of the weights ω. It is interesting how Cotlar and Sadosky saw the moment problem and generalized Toeplitz operators as natural substitutes for *Cotlar's Lemma* on almost orthogonality:

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"Cotlar's Lemma showed that the Hilbert transform basic property on boundedness on L^2 is related to a more general property of operators admitting certain type of decomposition. Since the Lemma was not suited to characterize the positive measures μ such that H is bounded in $L^2(\mu)$ it was natural to think that such characterization could also be related to a general property of certain operators."

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"...we have the following moment problem: to characterize the Forurier transform $\hat{\mu}$ of such μ, i.e. those sequences $s : \mathbb{Z} \to \mathbb{C}$ such that there exist a measure $\mu \in R_2$ with $\hat{\mu} = s(n)$."[15]

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Such measures give rise to *generalized Toeplitz kernels* (GTK): $K : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ such that for some $M > 1$,

$$K(m,n) = \begin{cases} (M-1)s(m-n) & \text{if } sign(m) = sign(n) \\ (M+1)s(m-n) & \text{if } sign(m) \neq sign(n) \end{cases}$$

is positive definite. Note that the restriction of K to each quadrant of $\mathbb{Z} \times \mathbb{Z}$ is translation invariant or Toeplitz.

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"Since, by the Bochner theorem, every positive definite Toeplitz kernel is the Fourier transform of a positive measure, Cotlar and Sadosky set to extend this result to GTKs, ..."[16] and use the results then in applications to the Hilbert transform.

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The power of these ideas is that they work also for the *two-weight problem* for the Hilbert transform by relating matrices of bilinear forms with certain invariances to "positive" measures.

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The building blocks of these matrices are the invariant (under shifts in $\mathbb{T}$) transformations of the form

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writing as before $f = f_1 + f_2$ and $Hf = -if_1 + if_2$, this is equivalent to

$$\int_{\mathbb{T}} f_1\bar{f}_1\, d\mu_{11} + \int_{\mathbb{T}} f_1\bar{f}_2\, d\mu_{12} + \int_{\mathbb{T}} f_2\bar{f}_1\, d\mu_{21} + \int_{\mathbb{T}} f_2\bar{f}_2\, d\mu_{22} \geq 0$$

where

$$\mu_{11} = \mu_{22} = M^2\nu - \mu \qquad \mu_{12} = \mu_{21} = M^2\nu + \mu$$

One also has

$$\left| \int_{\mathbb{T}} f_1 \bar{f}_2 \, d\mu_{12} \right| \leq \left(\int |f_1|^2 \, d\mu_{11} \right)^{1/2} \left(\int |f_2|^2 \, d\mu_{22} \right)^{1/2} \qquad (3)$$

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Cotlar-Sadosky showed that this can be *lifted* to a (complex) measure ρ differing form μ_{12} by a an analytic function $h \in H^1(\mathbb{T})$ so that (3) holds for all $(f_1, f_2) \in V \times V$ and hence

$$|\rho(D)| \leq \mu_{11}(D)^{1/2} \mu_{22}(D)^{1/2}$$

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They obtain the following result:

Theorem (Cotlar-Sadosky)

A pair of positive measures μ and ν on $\mathbb{T}$ satisfy

$$\int_{\mathbb{T}} |Hf|^2 \, d\mu \leq M^2 \int_{\mathbb{T}} |f|^2 \, d\nu$$

if and only if

$$|(M^2\nu + \mu - h dx)(D)| \leq (M^2\nu - \mu)(D)$$

for some $h \in H^1(\mathbb{T})$ and all Borel sets $D \subset \mathbb{T}$.

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Cotlar-Sadosky gave other characterizations using the notion of u-bounded operators on Banach lattices and also obtained, among other results, versions of the Helson-Szegö Theorem for the product Hilbert transform.

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Theorem (Cotlar-Sadosky)
A weight ω on on $\mathbb{T}^2$ satisfy

$$\int_{\mathbb{T}^2} |Hf|^2 \omega \leq M^2 \int_{\mathbb{T}^2} |f|^2 \omega$$

where $H = H_1 H_2$, H_j the Hilbert transform in the variable j, if and only if $\log \omega \in bmo(\mathbb{T}^2)$ with

$$\log \omega = u_1 + H_1 v_1 = u_2 + H_2 v_2$$

for u_j, v_j real valued in $\mathbb{T}^2$, $u_j \in L^\infty$ and $\|v_j\|_\infty \leq \pi/2 - \epsilon_M$.

The *bmo* space is precisely the space of all functions Φ so that for some $f_j, g_j \in L^\infty(\mathbb{T}^2)$

$$\Phi = f_1 + H_1 g_1 = f_2 + H_2 g_2$$

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$$\|\Phi\|_{bmo} = \inf\{\max_{j=1,2}\{\|f_j\|, \|g_j\|\} \text{ over all decompositions}\}$$

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This space *bmo* is also characterized by bounded mean oscillations on rectangles,

$$\|\Phi\|_{bmo} = \sup_{I,J\subset\mathbb{T}} \frac{1}{|I||J|} \int_{I\times J} |\Phi(x_1, x_2) - \Phi_{IJ}| dx_1 dx_2 \qquad (4)$$

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Other *BMO* spaces surfaced in Cotlar-Sadosky work on the product Hilbert transform and Hankel operators and on further collaborations of Cora with S. Ferguson and with S. Pott. All such spaces coincide in 1-d but they become different substitutes for *BMO* in product dimensions.

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Another natural BMO_{rec} space is defined to be the space of functions Φ such that

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One has $BMO_{prod} \subset BMO_{rec}$

Ferguson-Sadosky characterized the spaces *bmo* and BMO_{rec} in terms of (big) Hankel and little Hankel operators. We shall only state the characterization of *bmo*.

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Let $P : L^2 \to \mathcal{H}^2$ be the projection operator and $P^{\perp} = I - P$. Define the Hankel operator Γ_{Φ} with symbol Φ, to be $\Gamma_{\Phi}(f) = P^{\perp}(\Phi f)$ and let $M_{\Phi} f = \Phi f$

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Theorem (Ferguson-Sadosky)

The following are equivalent

- *The function* $\Phi \in$ *bmo*
- Γ_Φ *and* $\Gamma_{\bar{\Phi}}$ *are both bounded on* $\mathcal{H}^2$
- The commutators $[M_\Phi, H_j]$ are bounded on L^2
- The commutator $[M_\Phi, H_1 H_2]$ is bounded on L^2

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They also showed that if $\Phi \in BMO_{prod}$ then

$$\|[[M_\Phi, H_1], H_2]\|_{L^2 \to L^2} \lesssim \|\Phi\|_{prod}$$

the converse was proved by Ferguson-Lacey (2002). This provided a dual formulation of the so-called weak factorization problem in the context biholomorphic Hardy spaces in two variable. That is, the problem of being able to write $h \in \mathcal{H}^1$

$$h = \sum_j f_j g_j$$

with $f_j, g_j \in \mathcal{H}^2$ and

$$\sum_j \|f_j\|_2 \|g_j\|_2 \leq c\|h\|_1$$

Cora's Impact on the Profession

According to the Mathematics Genealogy Project, Cora had officially one PhD student: Sandra Farrier, Howard University, 2005. However she played a tremendously influential role in the life of many young mathematicians (and some no so young too).

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According to the Mathematics Genealogy Project, Cora had officially one PhD student: Sandra Farrier, Howard University, 2005. However she played a tremendously influential role in the life of many young mathematicians (and some no so young too).

"In particular she interacted with many students in Venezuela and in Argentina. In her years in Caracas, Cora was very influential on a group of Venezuelan mathematicians, María Dolores Morán, Ramón Bruzual, Marisela Domínguez and Stefania Marcantognini, among others, and a Uruguayan mathematician, Rodrigo Arocena, that got their PhD degrees at the Universidad Central de Venezuela (UCV). She also aided others pursue their doctorates in the US, including Gustavo Ponce who went to Courant Institute in 1978 and later Cristina Pereyra who went to Yale University in 1987.

Likewise, during her sabbatical year in Buenos Aires in 1984-1985 she helped many Argentinian mathematicians come to the US for their doctoral degrees; among them, José Zero who went to University of Pennsylvania, Estela Gavosto and Rodolfo Torres who went to Washington University, and Andrea Nahmod and Lucas Monzón who went to Yale University." [17]

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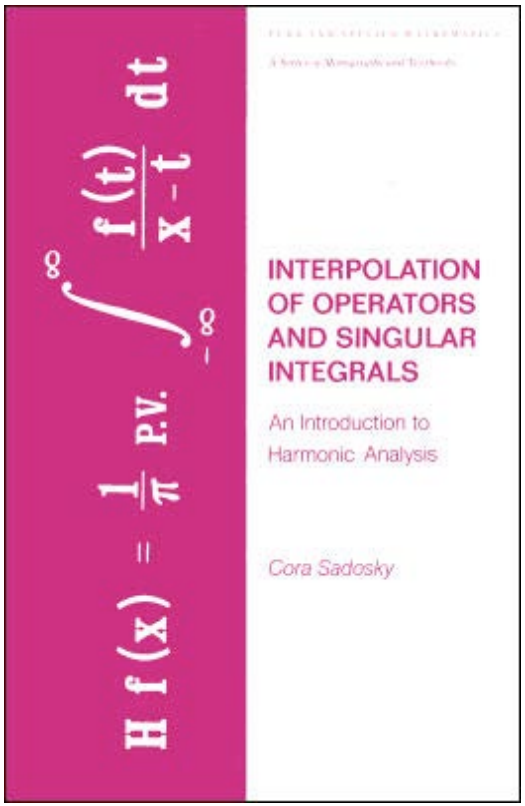
Many other students benefited from courses she taught in several countries. These lectures notes gave rise to her monograph *Interpolation of Operators and Singular Integrals: An Introduction to Harmonic Analysis*, Marcel Dekker, Inc., 1979.

The book was conceived as a very accessible introduction for students to materials that were only covered in textbooks at the time by the famous treatises of E. Stein, *Singular Integrals and Differentiability Properties of Functions*, and E. Stein and G. Weiss *Introduction to Fourier Analysis on Euclidean Spaces*, which trained generations of harmonic analysts.

[17] *Remembering Cora Sadosky*, AWM Newsletter, Volumen: 41 (2) 2011.

In the preface of the book Cora wrote:



"I hope that the book will be accessible to a wide audience that includes graduate students first approaching the subject... The initial inspiration on the treatment of this subject comes from magnificent courses given by E. M. Stein, G. Weiss, and A. P. Calderón, which I attended as a graduate student a the University of Chicago and the University of Buenos Aires. The overall influence is that of Professor A. Zygmund who taught me how beautiful singular integrals are and induced the will to try to share with others the pleasure of their beauty."

Many of us were helped by Cora in countless ways. In particular Estela and I benefited, among many other things, from the way she understood the "two-body problem" and brought awareness about this issue in our profession when few were paying attention.

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Many of us were helped by Cora in countless ways. In particular Estela and I benefited, among many other things, from the way she understood the "two-body problem" and brought awareness about this issue in our profession when few were paying attention.

"Cora was also concerned with many other aspects of the academic life and she often told us to learn about them and encouraged us to be proactive. She taught us that for any change to take place people really need to get involved." [18]

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- Member of the Human Resources Advisory Committee, Mathematical Sciences Research Institute, Berkeley, 2002(?)-05(?)
- Member of the Nominating Committee, AMS, 2001-03
- Member of the Council of the AMS, 1987-88 and 1995-98
- Member of the Committee on Science Policy, AMS, 1996-98
- Member of the Committee on the Profession, ASM, 1995-96
- Member of the Committee on Human Rights of Mathematicians, AMS, 1990-96
- President of the Association for Women in Mathematics from 1993-95
- Member of the Committee on Cooperation with Latin American Mathematicians, AMS, 1990-92

"Cora Sadosky fought many battles and in many fronts. She had tremendous convictions against many injustices, gender inequality, and discrimination in our society. She often took the flag of the underrepresented, underserved, and underestimated. She chose to fight many of her battles from within the mathematical community, sometimes even risking her own mathematical career. She showed a lot of courage in this sense and never worried about the consequences for her. She was never afraid to express her views." [19]

[19] *Remembering Cora Sadosky*, AWM Newsletter, Volumen: 41 (2) 2011.

...and expressing her views indeed she did...

[20] *On Issues of Immigration and Employment For Mathematicians*, C. Sadosky, Notices of the AMS 44 (1997), 1474-1477

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She wrote several articles and was often quoted in others about issues on women and young mathematicians, immigration, and discrimination. She had a vision of inclusiveness. For example speaking of immigration policies and the profession, she ones wrote in the *Notices*

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"In the mathematics departments of many U.S. universities a substantial percentage of professors-in particular, of mathematicians-are foreign born. This would be unthinkable in Europe! It is a sign of how more socially open this country is with respect to the rest of the world. We should be collectively proud of the U.S. openness, which underlines an extraordinary social dynamism. The enlightened acceptance foreigners receive at our universities ought to be promoted as an example to follow." [20]

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"Many young mathematicians are trying to develop research careers in difficult circumstances. Research mathematicians in nonresearch environments, including those at nondoctoral institutions, need support. For ages many women have faced these difficulties without help, and some have survived as mathematicians. Their experiences could help others, and their losses should not be repeated. Now some see promising young mathematicians take positions at non-research institutions, and they cry foul. Instead, we should help devise support systems to make small-college positions compatible with research."

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"Finding ways to support an active research population in the U.S. will become increasingly difficult in an era of globalization of the world economy, where the competition is fierce and international."

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A believer in equal opportunity and affirmative action properly done, Cora organized in 1994 an AWM panel on "Are Women Getting All the Jobs". She spoke often to dispel the notion that women were taking a disproportional percentage of the jobs in the 90's (which AMS data at the time also showed it was not the case) with spirited statements:

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"We strongly believe that this is false and dangerous, that pitting one group of under/unemployed mathematicians agains another is just the old tactic of dividing people with similar interests in order to exploit them all." [21]

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Other times she preferred to rely on irony and humor to convey a message:

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"Don't hold your breath, this year there was no appointment of a new woman mathematician to the National Academy of Sciences..." [23]

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Some of Cora's accomplishments as president of the AWM included

"...the move of AWM headquarters to the University of Maryland and the concurrent staff changes. She increased AWMs international connections and involvement in science policy, in particular initiating (in coordination with other organizations) the first Emmy Noether Lecture at an ICM in 1994 and representing AWM at the International Congress of Mathematics Education in 1993." [24]

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Reflecting on the AWM, Cora wrote:

"Our Association really makes an impact on the situation of women in mathematics. ... Still, women continue to face formidable problems in their development as mathematicians ... To successfully confront these problems, we need the ideas and the work, the enthusiasm and the commitment of all–students and teachers and researchers and industrial mathematicians–of every woman and every man who stands for "women's right to mathematics"." [24]

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Gustavo Ponce, Estela Gavosto, Wilfredo Urbina, Andrea Nahmod and Carlos Perez, Berkeley,1988

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Conference in Honor of Mischa Cotlar's 80th Birthday, Caracas ,1994[25]

[25] Photo courtesy of Wilfredo Urbina

Show video courtesy of Mario Milman

Cora Sadosky, Virginia Vera Pomalaza, Roxana Lopez

Cora with student reporters

Kovalevskaia Fund in Perú, May 1994 [26]

[26] Captured from the AWM Newsletter, 24 (1994).

Professor A.P. Calderón and some of his students, Chicago, 1996.[27]

[27] Photo courtesy of Eduardo Gatto via Wilfredo Urbina

Two presidents

From left, Mary Gray, Professor of Mathematics and Statistics at American University and the first president of the Association for Women in Mathematics(AWM), and Cora Sadosky, Professor of Mathematics at Howard University and eleventh president of the AWS, taking a break between sessions.

National Forum on Promoting Diversity at the Graduate Level in Mathematics, MSRI, Berkeley, 2008.[28]

[28] Photo captured from the Proceedings of the Forum.

Cora Sadosky and Estela Gavosto in a panel at the same MSRI Forum

Kabe Moen, Virginia Naibo, Diego Maldonado, Cora Sadosky, RT, and Arpad Benyi, Albuquerque, 2009.[29]

[29] Photo courtesy of Erika Ward.

...and at the end of the day it was (is) all about Mathematics:

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"Strange as it may seem at the end of this long report, I started out with the impression I had little to say. That impression stemmed from my doing something it may be unwise for a current AWM president to do: I plunged into mathematics head on for three full weeks. I came out of that stint dazed, vaguely guilty and deeply happy. It is very clear that I enjoy like crazy doing mathematics! "

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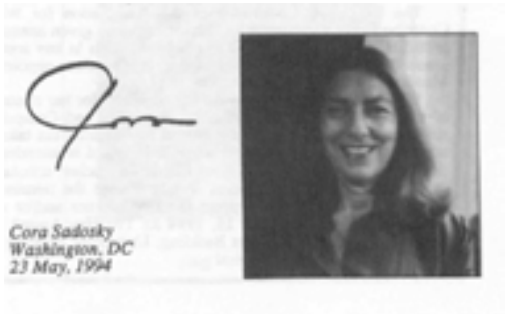
"But what remains with me is the sense of elation. I did not prove the Riemann Conjecture. My work was modest, but it gave me so much pleasure to do it! Thus I close this conversation with a wish to each of you for this summer: do some of the mathematics you want to do and do it with great pleasure!" [30]



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"A mighty woman with a torch, ..."

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Thank you for your attention

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