Harmonic Analysis and Non-Linear Dynamics

Dedicated to memory of Cora Sadosky

Alex Stokolos (Georgia Southern University)
and
Dmitriy Dmitrishin, Anna Khamitova, Anatolii Korenovskyi, Alexei Solyanik (Odessa, Ukraine)

13th New Mexico Analysis Seminar

April 5, 2014

Settings

- Consider the system

$$x_{n+1} = f(x_n), \quad x_n \in \mathbb{R}, \quad n \in \mathbb{Z}_+$$

- Assume the system has an unstable equilibrium point x^*, $f(x^*) = x^*$.
- Assume that chaos is observed for a certain range of parameter(s).
- We want to stabilize the system using a control that would work for all parameters from the specified range: $x_{n+1} = f(x_n) + u$, $u = u(x_n, x_{n-1}, \ldots, x_{n-N+1})$
- **Question A**: *Can stabilization be achieved by control with a bounded depth of used prehistory?*
- **Question B**: *If so, what is the minimal depth of prehistory needed?*

Examples of linear control

- Extended Time Delay Auto-Synchronization (ETDAS)

$$x_{n+1} = f(x_n) + u_n, \qquad u_n = \epsilon(x_n - x_{n-1}) + Ru_{n-1}, \quad |R| < 1.$$

- N-Time Delay Auto-Synchronization (NTDAS)

$$x_{n+1} = f(x_n) + \gamma \left(x_n - \frac{1}{N} \sum_{k=1}^{N} x_{n-k} \right).$$

- Predictive

$$x_{n+1} = f(x_n) + \epsilon(f_{(s+1)p+1}(x_n) - f_{sp+1}(x_n)).$$

- Generalized linear DFC

$$x_{n+1} = f(x_n) - \sum_{j=1}^{N-1} \varepsilon_j (x_{n-j} - x_{n-j+1}).$$

Upper and lower bounds

- Linearized system ($x_n = x^* + \delta_n$, $\mu := f'(x^*)$)

$$\delta_{n+1} = \mu\delta_n - \sum_{j=1}^{N} a_j \delta_{n-j+1}, \qquad \sum_{j=1}^{N} a_j = 0.$$

- Characteristic equation

$$\chi(\lambda) = \lambda^N + (-\mu + a_1)\lambda^{N-1} + a_2\lambda^{N-2} + \cdots + a_N =: \lambda^N - \mu\lambda^{N-1} + p(\lambda)$$

- $$\chi(1) = 1 - \mu > 0 \Longrightarrow \mu < 1.$$

- $$x_{n+1} = \mu x_n \Longrightarrow x_n = C\mu^n, \quad \text{blows up if } \mu > 1.$$

- $$\text{Vieta's Theorem implies} \quad 1 - \mu + \sum_{j=1}^{N} a_j < 2^N \Longrightarrow \mu > 1 - 2^N.$$

- If $N = 2$, we get $1 - 2^N = 1 - 4 = -3 < \mu < 1$.

Domain of stability for fixed $a_1, \dots, a_N$

- It is not difficult to show that for any given $\mu \in (1 - 2^N, 1)$ there are coefficients $a_1, \dots, a_N$, $a_1 + \cdots + a_N = 0$ such that the polynomial

$$\lambda^N + (-\mu + a_1)\lambda^{N-1} + a_2\lambda^{N-2} + \cdots + a_N$$

  is stable.
- **Question C**: What is the maximal length of a connected component?

Theorem A

The maximal length of a connected component is 4

Generalized non-linear DFC

- Consider the following system with non-linear control

$$x_{n+1} = f(x_n) - u,$$

$$u = -\sum_{j=1}^{N-1} \varepsilon_j \left(f(x_{n-j+1}) - f(x_{n-j}) \right), \; |\varepsilon_j| < 1, \; j = 1, \ldots .N-1.$$

- It can be written as

$$x_{n+1} = \sum_{j=1}^{N} a_j f(x_{n-j+1}),$$

- where $a_1 = 1 - \varepsilon_1, a_j = \varepsilon_{j-1} - \varepsilon_j, j = 2, \ldots, N-1, a_N = \varepsilon_{N-1}$. Note that

$$\sum_{j=1}^{N} a_j = 1.$$

- Characteristic equation:

$$\lambda^n - f'\left(x^*\right) \sum_{j=1}^{N} a_j \lambda^{n-j} = 0$$

- or

$$\lambda^N - \mu(a_1\lambda^{N-1} + a_2\lambda^{N-2} + \cdots + a_N) = 0.$$

- The stability condition: $|\lambda| < 1$.
- We need to maximize over all $a_1 + ... + a_N = 1$ the length of the interval for μ containing zero, provided that the polynomial $\lambda^N - \mu\left(a_1\lambda^{N-1} + \cdots + a_N\right)$ is Schur stable.
- Similarly to the linear case, $1 - 2^N < \mu < 1$.

- $$\mathcal{P}_\mu = \left\{ \lambda^N - \mu\left(a_1\lambda^{N-1} + \cdots + a_N\right),\ \sum_{j=1}^N a_j = 1 \right\}$$
- Since on the boundary of the stability region we have
$$\frac{1}{\mu} - \sum_{j=1}^N a_j e^{-ijt} = 0 \quad \Longleftrightarrow \quad \mu \sum_{j=1}^N a_j e^{-ijt} = 1$$
- we would like to estimate the quantity
$$J_N = \inf_{a_1+\cdots+a_N=1} \left[\max_{t\in[0,\pi]} \left\{ \left| \sum_{j=1}^N a_j e^{-ijt} \right| \ : \quad \arg\left(\sum_{j=1}^N a_j e^{-ijt} \right) = \pi \right\} \right]$$
- or, equivalently,
$$J_N = \inf_{a_1+\cdots+a_N=1} \left[\max_{t\in[0,\pi]} \left\{ \left| \sum_{j=1}^N a_j \cos jt \right| : \sum_{j=1}^N a_j \cos jt < 0,\ \sum_{j=1}^N a_j \sin jt = 0 \right\} \right]$$
- The condition $|J_N| \cdot |\mu| < 1$ guarantees the existence of $a_1, \ldots, a_N$ that make $\mathcal{P}_\mu$ stable.

Fejér-Jackson-Gronwall inequality

- Let's choose $a_j = 1/(j\ell(N))$, where $\ell(N) = 1 + 1/2 + ... + 1/N$.
- $$J_N \leq \frac{1}{\ell(N)} \max_{t\in[0,\pi]} \left\{ \left| \sum_{j=1}^{N} \frac{\cos jt}{j} \right| : \sum_{j=1}^{N} \frac{\cos jt}{j} \leq 0;\ \sum_{j=1}^{N} \frac{\sin jt}{j} = 0 \right\}.$$
- The Fejér-Jackson-Gronwall inequality says that for any $N \geq 1$ and $0 < t < \pi$,
$$0 < \sum_{j=1}^{N} \frac{\sin jt}{j} < 2.$$
- $$J_N \leq \frac{1}{\ell(N)} \left| \sum_{j=1}^{N} \frac{(-1)^j}{j} \right| \leq \frac{C}{\ln N}.$$

Uniform choice of a_j - Dirichlet kernel

- $$\frac{1}{N}\max_{t\in[0,\pi]}\left\{\left|\frac{\sin(Nt/2)}{\sin t/2}\right| : \arg\left(\frac{e^{-it}e^{-iNt/2}}{e^{-it/2}}\right)=\pi\right\}.$$
- $$\max_k \frac{1}{N}\left|\frac{\sin(Nt_k/2)}{\sin t_k/2}\right| : \ t_k/2=\frac{\pi(1+2k)}{N+1},\ Nt_k/2=\frac{\pi N(1+2k)}{N+1}.$$
- $$Nt_k/2=\frac{\pi N(1+2k)}{N+1}=\frac{\pi(N\pm 1)(1+2k)}{N+1}=\pi(1+2k)-\frac{\pi(1+2k)}{N+1}.$$
- Therefore, $J_N \le \frac{1}{N}$.
- What could be better?
- Only $\frac{1}{N^2}$.

Köbe Quarter Theorem and Suffridge polynomials

- Köbe Quarter Theorem: *Let* $F(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ *be univalent in* $\mathbb{D} = \{|z| < 1\}$. *Then* $F(\mathbb{D})$ *contains the disk with center* 0 *and radius* $1/4$.
- Köbe function

$$f(z) = \frac{z}{(1-z)^2} = \sum_{k=1}^{\infty} k z^k; \quad -\frac{1}{4} \notin f(\mathbb{D}).$$

- Suffridge polynomials

$$S_N(z) = \sum_{k=1}^{N} \csc \frac{\pi}{N+1} \left(1 - \frac{k-1}{N}\right) \sin \frac{\pi k}{N+1} z^k.$$

- They obey an "asymptotic Köbe Quarter Theorem": the k-th coefficient approaches k and

$$\inf_{z \in \partial\mathbb{D}} |S_N(z)| = |S_N(-1)| \to \frac{1}{4}, \; N \to \infty.$$

Theorem B

We have

$$J_N = \tan^2 \frac{\pi}{2(N+1)} \sim \frac{1}{N^2}$$

and the optimal control coefficients are uniquely defined by

$$\varepsilon_j = \sum_{k=j+1}^{N} a_k^0, \; j = 1, \ldots, N-1,$$

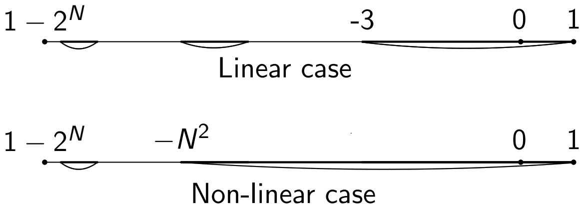
where

$$a_k^0 = 2 \cdot \tan \frac{\pi}{2(N+1)} \cdot (1 - \frac{k}{N+1}) \cdot \sin \frac{\pi k}{N+1}, \; k = 1, \; \ldots \; , N.$$

If x^* *is an equilibrium point and* $f'(x^*) \in (-m, -1]$, *then*

$$N = \left\lfloor \frac{\pi}{2 \cot^{-1} \sqrt{m}} \right\rfloor - 1 \sim \sqrt{m}.$$

Linear control versus non-linear one



Example

Let $f : [0,\ 1] \to [0,\ 1]$ be a one-parameter logistic map

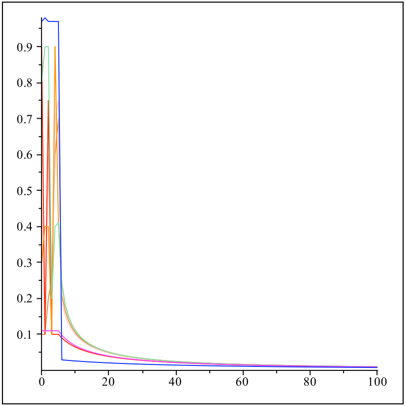
$$f(x) = h \cdot x \cdot (1 - x), \quad 0 \le h \le 4.$$

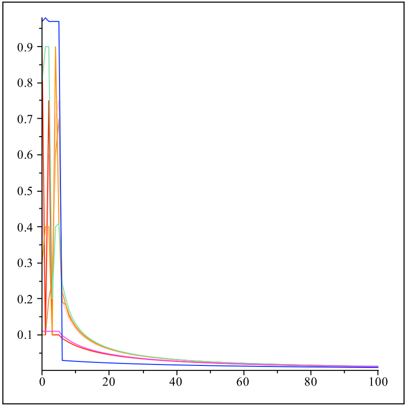
For $h \in (3,\ 4]$ the equilibrium point $x^* = 1 - \frac{1}{h}$ is unstable and the multiplier $\mu \in [-2,\ -1)$. Therefore $\frac{\pi}{2 \cdot \cot^{-1}\sqrt{2}} \approx 2.55$ and the minimal depth of prehistory is $N = 1$. The optimal strength coefficient $\varepsilon_1^0 = \frac{1}{3}$ and the optimal control is

$$u = -\frac{1}{3}\left(f(x_n) - f(x_{n-1})\right).$$

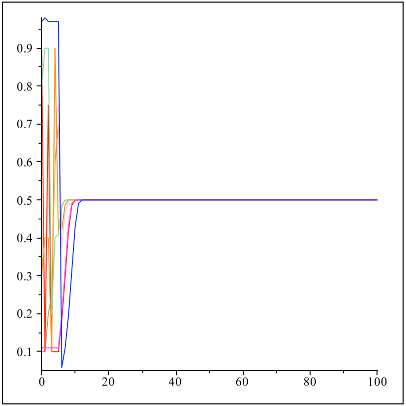
That is the closed-loop system $x_{n+1} = f(x_n) + u$ has locally stable equilibrium points for all $h \in (3,\ 4]$.

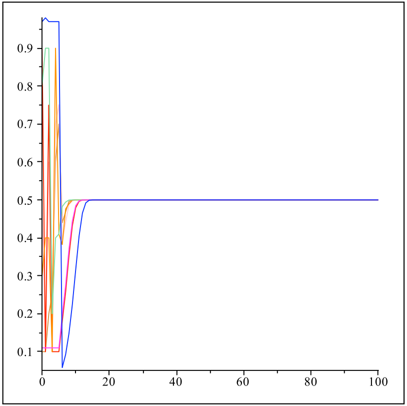
Logistic map, $h = 1$



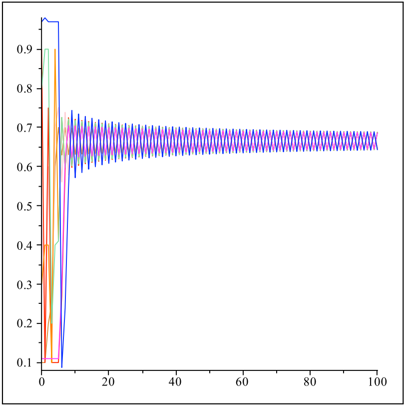


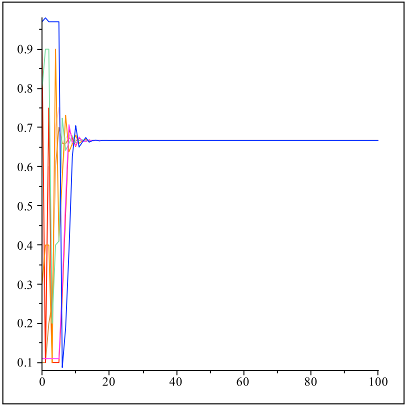
Logistic map, $h = 2$



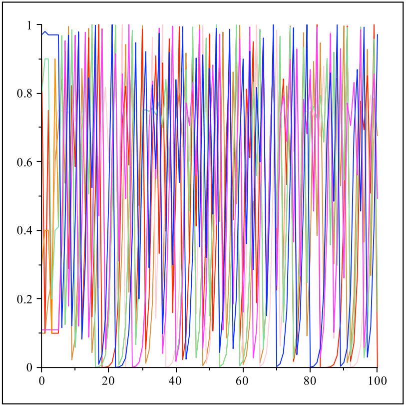


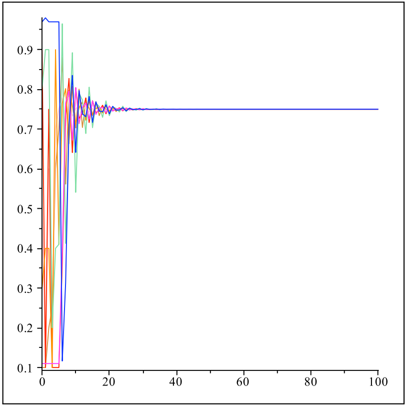
Logistic map, $h = 3$



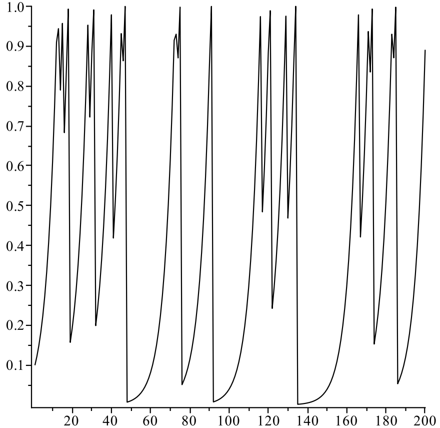


Logistic map, $h = 4$



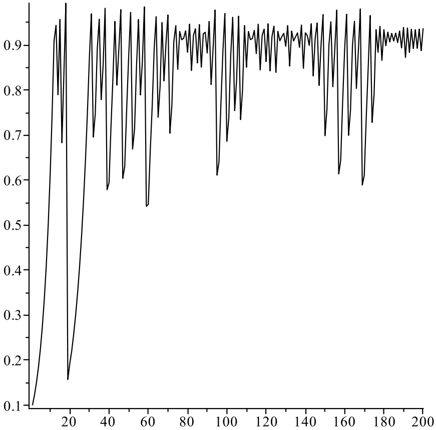


$f(x) = 1.22x(1 - x^{20})$



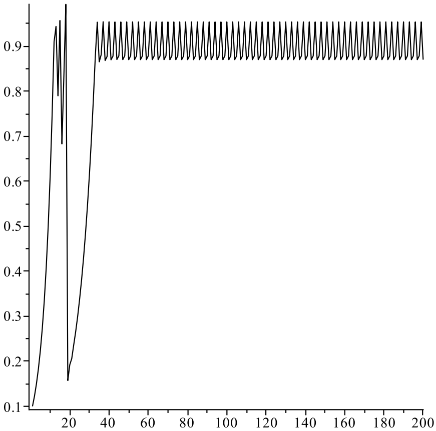
$$x_{n+1} = f(x_n)$$

$f(x) = 1.22x(1 - x^{20})$



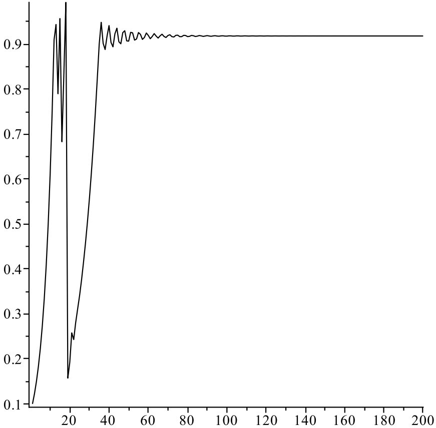
$$x_{n+1} = f(x_n) - \frac{1}{3}(f(x_n) - f(x_{n-1}))$$

$f(x) = 1.22x(1 - x^{20})$



$$x_{n+1} = f(x_n) - 0.6944(f(x_n) - f(x_{n-1})) - 0.3236(f(x_{n-1}) - f(x_{n-2}))$$

$f(x) = 1.22x(1 - x^{20})$



$$x_{n+1} = f(x_n) - 0.5606(f(x_n) - f(x_{n-1})) - 0.1464(f(x_{n-1}) - f(x_{n-2})) - 0.0763(f(x_{n-2}) - f(x_{n-3}))$$

A cycle

- $$x_{n+1} = f\left(x_n\right), \quad x_n \in \mathbb{R}^1, \quad n = 1, 2, \ldots$$
- unstable cycle (η_1, η_2) :

$$\eta_2 = f\left(\eta_1\right), \qquad \eta_1 = f\left(\eta_2\right)$$

- the cycle multiplier:

$$\mu = f'\left(\eta_1\right) \cdot f'(\eta_2) \in \left(-m, -1\right), \quad m > 1$$

A cycle

One can locate a cycle by stabilizing the iterative map $f(f(x))$.

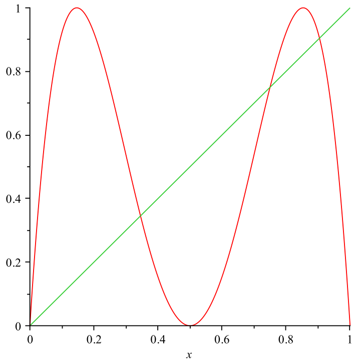


Figure: 2-Iterated logistic map

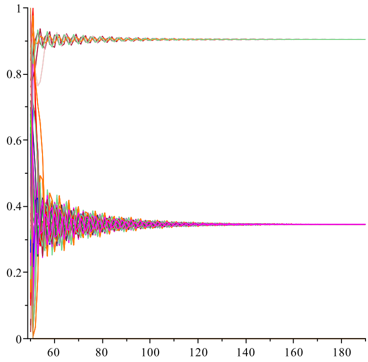


Figure: Equillibria

2-cycle control

- $$u = -\sum_{j=1}^{N-1} \varepsilon_j \left(f\left(x_{n-2j+2}\right) - f\left(x_{n-2j}\right)\right), |\varepsilon_j| < 1$$
- $$\lambda^{2(N-1)+1} + k\left(a_1\lambda^{2(N-1)} + a_2\lambda^{2(N-1)-2} \ldots + a_N\right)$$
- where $a_1 = 1 - \varepsilon_1$, $a_j = \varepsilon_{j-1} - \varepsilon_j$, $j = 2, \ldots, N-1$, $a_N = \varepsilon_{N-1}$, so $a_1 + \cdots + a_N = 1$ and $k = \pm i\sqrt{|\mu|}$.
- $$J_N = \inf_{\sum_{j=1}^{N} a_j = 1} \left[\max_{t \in [0, \frac{\pi}{2}]} \left\{ \left| \sum_{j=1}^{N} a_j e^{-i(2j-1)t} \right| : \arg\left(\sum_{j=1}^{N} a_j e^{-i(2j-1)t} \right) = \frac{\pi}{2} \right\} \right]$$
- $|\mu \cdot (J_N)^2| < 1$ implies Schur *robust* stability of characteristic polynomial.

Fejér-Jackson-Gronwall inequality

- Let $a_j = 1/(j\,\ell(N))$
- $$J_N = \inf_{\sum_{j=1}^N a_j = 1} \left[\max_{t\in\left[0,\,\frac{\pi}{2}\right]} \left\{ \left| \sum_{j=1}^{N} a_j e^{-i(2j-1)t} \right| :\ \arg\left(\sum_{j=1}^{N} a_j e^{-i(2j-1)t} \right) = \frac{\pi}{2} \right\} \right]$$
$$\le \frac{1}{\ell(N)} \max_{t\in[0,\pi]} \left\{ \left| \sum_{j=1}^{N} \frac{\sin(2j-1)t}{j} \right| :\ \sum_{j=1}^{N} \frac{\sin(2j-1)t}{j} \ge 0; \right.$$
$$\left. \sum_{j=1}^{N} \frac{\cos(2j-1)t}{j} = 0 \right\}.$$

Fejér-Jackson-Gronwall inequality

- $$\sum_{j=1}^{N}\frac{\sin(2j-1)t}{j} = 2\sum_{j=1}^{N}\frac{\sin(2j-1)t}{2j-1} + \sum_{j=1}^{N}\sin(2j-1)t\left(\frac{1}{j}-\frac{2}{2j-1}\right) \pm 2\sum_{j=1}^{N}\frac{\sin 2jt}{2j}$$
$$= 2\sum_{j=1}^{2N}\frac{\sin jt}{j} - \sum_{j=1}^{N}\frac{\sin(2j-1)t}{j(2j-1)} + \sum_{j=1}^{N}\frac{\sin(j(2t))}{j}.$$
- $$\left|\sum_{j=1}^{N}\frac{\sin(2j-1)t}{j}\right| \le C$$
- $$J_N \le \frac{C}{\ell(N)} \le \frac{C_1}{\log N} \to 0.$$

Theorem C

The following statements are valid:

(*i*)
$$J_N = \frac{1}{N}.$$

(*ii*) *The optimal coefficients are uniquely defined by*

$$\epsilon_j = \sum_{k=j+1}^{N} \frac{2(N-k)+1}{N^2}, \quad j = 1, \dots, N-1.$$

(*iii*) *If* η_1 *and* η_2 *are the points of an unstable cycle such that* $f'(\eta_1)f'(\eta_2) \in (-m, -1]$, *then* $N = 2(N_0 - 1)$, *where* N_0 *is the smallest integer satisfying* $m \cdot N_0^{-2} < 1$.

Example

For the one-parameter logistic map $f(x) = hx(1-x), \quad 1+\sqrt{6} < h \leq 4,$

$$\eta_1 = \frac{1+h-\sqrt{h^2-2h-3}}{2h},\ \eta_2 = \frac{1+h+\sqrt{h^2-2h-3}}{2h},\ \mu \in [-4,-1).$$

The strength coefficients are $\epsilon_1 = \frac{4}{9},\ \epsilon_2 = \frac{1}{9}$.

2-cycle stabilization

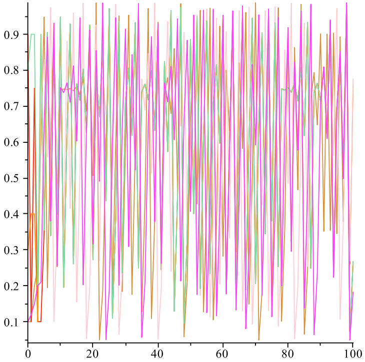


Figure: Chaotic solutions

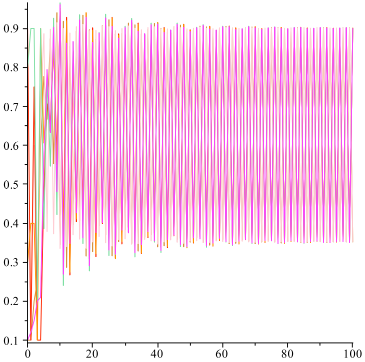


Figure: Stable solutions

Thank you!