Geometric measure theory as a tool in free boundary regularity problems

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Geometric measure theory as a tool in free boundary regularity problems

- Introduction - Motivation
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1. Introduction

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open connected domain such that $\operatorname{int}\Omega^c \neq \emptyset$.

Dirichlet problem in Ω: given a bounded continuous function f on $\partial\Omega$ ($f \in C_b(\partial\Omega)$) does there exist a solution $u_f \in C_b(\overline{\Omega}) \cap C^2(\Omega)$ to

$$(1) \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \quad ? \end{cases}$$

Definition. Ω is *regular* if $\forall\, f \in C_b(\partial\Omega)$ there exists $u_f \in C_b(\overline{\Omega}) \cap C^2(\Omega)$ satisfying (1).

Remark. If Ω is bounded and regular by the maximum principle

$$(2) \qquad |u_f(X)| \leq \max_{\partial\Omega}(f) \qquad \forall\, X \in \Omega.$$

Thus for $X \in \Omega$ the linear operator

$$L_X : C(\partial\Omega) \to \mathbb{R} \quad \text{where} \quad L_X f = u_f(X)$$

is bounded.

By the Riesz Representation theorem for $X \in \Omega$ there exists a Radon measure ω^X satisfying

$$(3) \qquad u_f(X) = \int_{\partial\Omega} f(Q) d\omega^X(Q)$$

$\forall\, f \in C_b(\partial\Omega)$. Since $u_1(X) = 1$, (2) implies that ω^X is a probability measure.

ω^X is the harmonic measure of Ω with pole X

Let

$$F(X)=\begin{cases} -\frac{1}{2\pi}\log|X| & \text{if } \ n \ = \ 1 \\ \frac{1}{(n-1)\sigma_n|X|^{n-1}} & \text{if } \ n \ \geq \ 2, \end{cases}$$

where $\sigma_n = |\mathbb{S}^n|$. Then

$$u(X)=\int_{\mathbb{R}^{n+1}} F(X-Y)\varphi(Y)dY$$

satisfies

$$-\Delta u=\varphi \text{ in } \ \mathbb{R}^{n+1},$$

i.e $\Delta F = -\delta_{X=0}$ where δ is the Dirac delta function.

Green's formula: let $u, v \in C^1(\overline{\Omega}) \cap C^2(\Omega)$, where Ω is a C^1 domain then

$$(4) \quad \int_\Omega u\Delta v - \int_\Omega v\Delta u = \int_{\partial\Omega} \Big(u\frac{\partial v}{\partial \nu} - -v\frac{\partial u}{\partial \nu}\Big) d\,\sigma.$$

Here σ denotes the surface measure to $\partial\Omega$, and ν the outward pointing unit normal to $\partial\Omega$.

If Ω is regular then for $X \in \Omega$ solve

$$\begin{cases} \Delta u_X = 0 & \text{in } \Omega \\ u_X(Q) = F(Q - X) & \text{for } Q \in \partial\Omega \end{cases}$$

Then $G(X,Y) = F(Y - X) - u_X(Y)$ satisfies

$$\begin{cases} \Delta G(X,\cdot) &= -\delta_X & \text{in } \Omega \\ G(X,Q) &= 0 & \text{for } Q \in \partial\Omega \end{cases}$$

$G(X,\cdot)$ is the Green function of Ω with pole X

Applying (4) in $\Omega \backslash B(X,\epsilon)$ to $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ and $v(Y) = G(X,Y)$, and letting $\epsilon \to 0$ we have

$$(5) \qquad u(X) = -\int_{\partial\Omega} u(Q) \frac{\partial G(X,Q)}{\partial \nu} d\,\sigma(Q) - \int_{\Omega} G(X,Y) \Delta u(Y) dY.$$

If u satisfies (1) then (5) becomes

$$(6) \qquad u(X) = -\int_{\partial\Omega} f(Q) \frac{\partial G(X,Q)}{\partial \nu} d\,\sigma(Q).$$

The maximum principle, (3) and (6) ensure that

$$(7) \qquad k_X(Q) = d\omega^X(Q) = -\frac{\partial G(X,Q)}{\partial \nu} d\,\sigma(Q).$$

k_X is the Poisson kernel of Ω with pole X.

Using (7) and applying (4) in $\Omega \cap B(Q,R)$ where $Q \in \partial\Omega$ and $2R < |X-Q|$ to $G(X,\cdot)$ and $\varphi \in C_c^\infty(B(Q,R))$ we have

(8)
$$\int_\Omega G(X,Y)\Delta\varphi(Y)\,dY = \int_{\partial\Omega} \varphi(Q)d\,\omega^X(Q).$$

Example: Let $\Omega = B(0,r)$ for $X \in B(0,r)$ and $Q \in \partial B(0,r)$,

$$k_X(Q) = \frac{r^2 - |X|^2}{\sigma_n r |X-Q|^{n+1}}.$$

In particular $k_0(Q) = \frac{1}{\sigma_n r^n}$.

Classical boundary regularity results

If Ω is C^∞, $\vec{n} \in C^\infty$ $\Longrightarrow$ $\log k_X \in C^\infty$

If Ω is $C^{k+1,\alpha}$, $\vec{n} \in C^{k,\alpha}$ $\Longrightarrow$ $\log k_X \in C^{k,\alpha}$

If Ω is $C^{1,\alpha}$, $\vec{n} \in C^{0,\alpha}$ $\Longrightarrow$ $\log k_X \in C^{0,\alpha}$

Kellogg

Question: What happens as $\alpha \to 0$?

If Ω is C^1, $\vec{n} \in C^0$ $\Longrightarrow$ $\log k_X \in VMO(\partial\Omega)$

Jerison-Kenig

The free boundary regularity problem for the Poisson kernel addresses the question of whether, under the appropriate hypothesis the previous implications are equivalences

Theorem [AC]. Assume that:

1. "Ω satisfies the divergence theorem, and that the surface measure of $\partial\Omega$ has Euclidean growth,"

2. "$\partial\Omega$ is flat enough,"

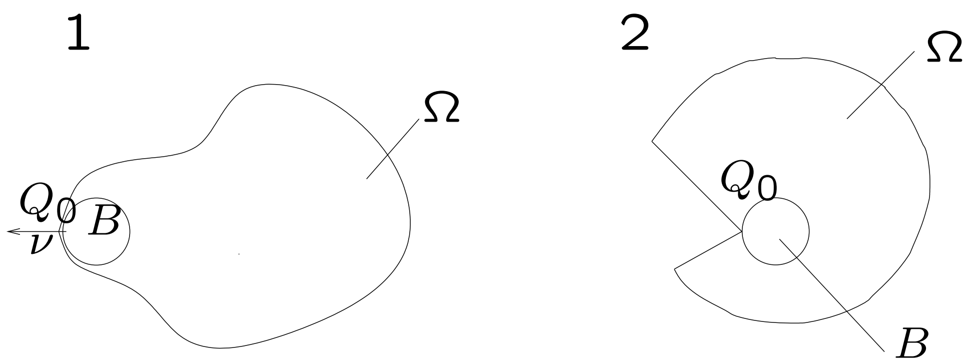
3. '$\log k_X \in C^{0,\beta}$ for some $\beta \in (0, 1)$,'

then Ω is a $C^{1,\alpha}$ domain for some $\alpha \in (0, 1)$ which depends on β.

2. Harmonic and subharmonic functions

Definition. Let $Q_0 \in \partial\Omega$, we say that Ω satisfies the interior sphere condition at Q_0 if there is an open ball $B \subset \Omega$ so that $\partial\Omega \cap \overline{B} = \{Q_0\}$.

Examples.



Theorem. (Hopf boundary point lemma) Assume that u is harmonic in Ω, $Q_0 \in \partial\Omega$ and

- u is continuous at Q_0.
- $u(Q_0) < u(X)$ for all $x \in \Omega$.
- Ω satisfies the interior sphere condition at Q_0.

If the outward unit normal to $\partial\Omega$ at Q_0 exists,

$$-\frac{\partial u}{\partial \nu}(Q_0) = -\nabla u \cdot \nu(Q_0) > 0.$$

Otherwise, if the outward unit normal does not exist then

$$\liminf_{\substack{X \to Q_0 \\ \text{(non-tangentially)}}} \frac{u(X) - u(Q_0)}{|X - Q_0|} > 0.$$

Let $D \subset \mathbb{R}^n$ be an open connected domain.

Definition. A function $f \in C(D)$ is said to be subharmonic if $\forall \phi \in C_c^\infty(D)$, with $\phi \geq 0$

$$\int_D f \Delta \phi \geq 0.$$

If $f \in C^2(D)$, f is subharmonic if and only if $\Delta f \geq 0$.

Remark. If f is subharmonic the operator $L : C_c^\infty(D) \to \mathbb{R}$ defined by

$$L(\phi) = \int_D \Delta\phi f$$

is a non-negative bounded linear operator. By the Riesz Representation Theorem there exists a non-negative Radon measure λ such that

$$L(\phi) = \int_D \phi d\lambda \qquad \forall \phi \in C_c^\infty(D).$$

For f is subharmonic and $\phi \in C_c^\infty(D)$ we have

$$\int \phi \Delta f = \int_D f \Delta \phi$$

where

$$d\lambda = \Delta f \geq 0 \text{ as a Radon measure.}$$

Representation formula for subharmonic functions

Let $f \in C(D) \cap W^{1,2}_{\text{loc}}(D)$ be subharmonic in D. Let $\overline{B(y,r)} \subset D$ and let $G_r(x,-)$ denote the Green's function of $B(y,r)$ with pole at $x \in B(y,r)$ then

$$f(x) = -\int_{\partial B(y,r)} f(q) \frac{\partial G_r(x,q)}{\partial \nu} d\sigma(q) - \int_{B(y,r)} G_r(x,z) \Delta f(z).$$

In particular

$$f(y) = \fint_{\partial B(y,r)} f(q) d\sigma(q) - \int_{B(y,r)} G_r(y,z) \Delta f(z)$$

Mean value inequality for subharmonic functions

Let $f \in C(D) \cap W^{1,2}_{\text{loc}}(D)$ be subharmonic in D. If $\overline{B(y,r)} \subset D$ then

$$f(y) \leq \fint_{\partial B(y,r)} f(q) d\sigma(q),$$

and for $x \in B(y,r)$

$$\begin{aligned} f(x) &\leq -\int_{\partial B(y,r)} f(q) \frac{\partial G_r(x,Q)}{\partial \nu} d\sigma(q) \\ &\leq \frac{r^2 - |x-y|^2}{\sigma_{n-1} r} \int_{\partial B(y,r)} \frac{f(z)}{|z-x|^n} d\sigma(z). \end{aligned}$$

Recall the following equivalent definition of subharmonicity.

Theorem. Let $f \in C(D)$, f is subharmonic in D if and only if for every ball B such that $\overline{B} \subset D$, and every harmonic function $h \in C(\overline{B})$ satisfying $f \leq h$ on ∂B then $f \leq h$ in B.

3. Non-tangentially accessible domains - NTA

Definition. A domain Ω is non-tangentially accessible (NTA) if there exists constants $M > 2$ and $R > 0$ ($R = \infty$ if Ω is unbounded) such that $\forall\, Q \in \partial\Omega$, $\forall\, r \in (0, R)$

1. Ω satisfies the corkscrew condition:

   there exists $A = A(Q, r) \in \Omega$ such that

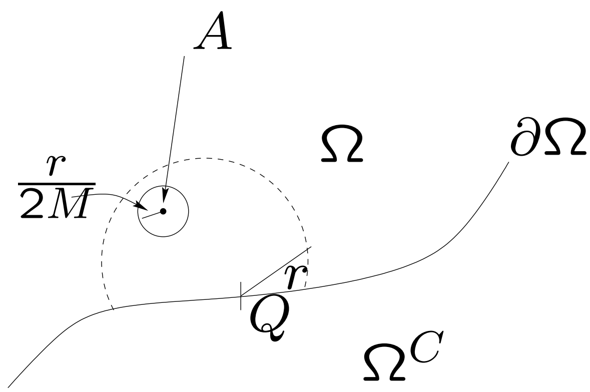
   $$\frac{r}{M} \leq |A - Q| \leq r \text{ and } d(A, \partial\Omega) \geq \frac{r}{M}$$

2. Ω^C satisfies the corkscrew condition.

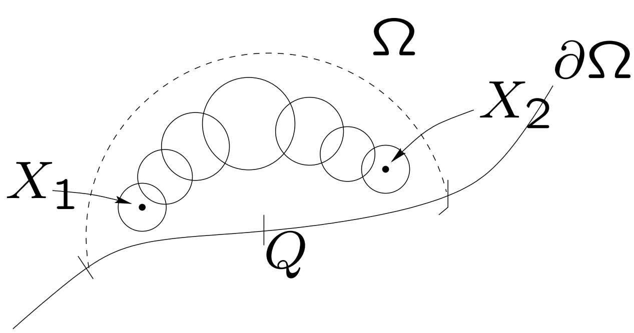
3. Ω satisfies the Harnack Chain Condition;

   if $\epsilon > 0$, and $X_1, X_2 \in B(Q, \frac{r}{4}) \cap \Omega$ with $|X_1 - X_2| \leq 2^k \epsilon$ and $d(X_i, \partial\Omega) \geq \epsilon$ for $i = 1, 2$, there exists a chain of Mk balls $B_1, \ldots, B_{Mk}$ in Ω connecting $X_1 \in B_1$ to $X_2 \in B_{Mk}$ so that $\text{diam} B_j \sim d(B_j, \partial\Omega)$ and $\text{diam} B_j \geq C^{-1} \min\{d(X_1, B_j), d(X_2, B_j)\}$ for $C > 1$.

Corkscrew condition:



Harnack Chain Condition



Condition 3 guarantees that the Harnack principle for non-negative harmonic functions holds in Ω. If

$$\Delta u = 0 \quad \text{in } \Omega, \quad \text{and } u \geq 0$$

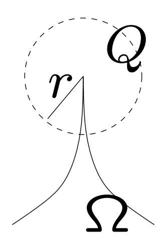
then for $X_1, X_2 \in B(Q, \frac{r}{4}) \cap \Omega$,

$$M^{-k} u(X_1) \leq u(X_2) \leq M^k u(X_1) \quad \text{for } .$$

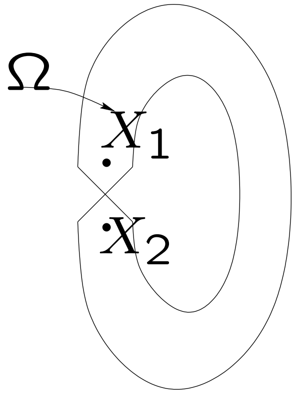
Theorem.[**JK**] NTA domains are regular.

Examples.

1. A domain with a cusp is not an NTA domain, it does not satisfy the corkscrew condition at the cusp point.



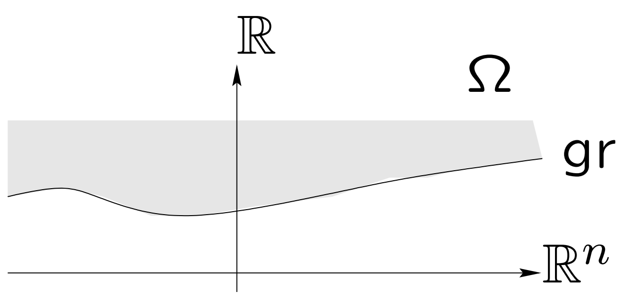
2.



Ω is not an NTA domain because X_1 and X_2 cannot be joined by a Harnack Chain.

3. $\Omega = \mathbb{R}^{n+1}_{+} = \{(x, x_{n+1}) : x \in \mathbb{R}^n, x_{n+1} > 0\}$ is an NTA domain.

4. $\Omega = \{(x,t) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n, t > \varphi(x)\}$ with φ Lipschitz (i.e. $\|\nabla\varphi\|_\infty < \infty$), is an NTA domain.



Recall that for $A, B \subset \mathbb{R}^{n+1}$,

$$D[A,B] = \sup\{d(a,B) : a \in A\} + \sup\{d(b,A) : b \in B\}.$$

denotes the Hausdorff distance between A and B.

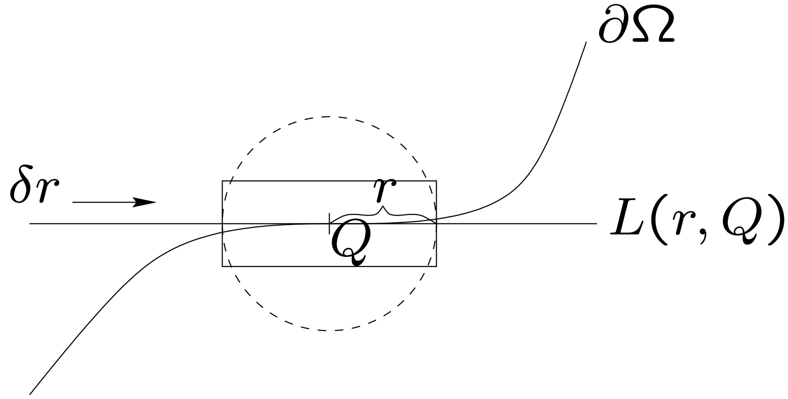
For $Q \in \partial\Omega$ we denote by

$$\theta(Q,r) = \inf_L \left\{ \frac{1}{r} D[\partial\Omega \cap B(Q,r), L \cap B(Q,r)] \right\},$$

where the infimum is taken over all n-planes containing Q.

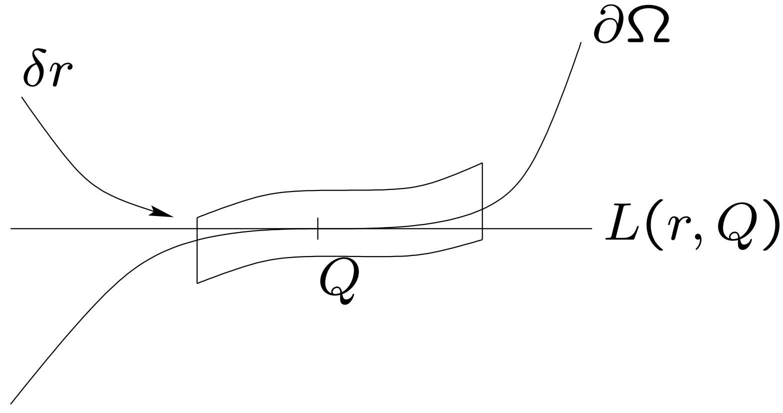
If $\theta(Q,r) \leq \delta$ there exists an n-plane $L(Q,r)$ containing $Q \in \partial\Omega$ and such that

1. $\partial\Omega \cap B(Q,r) \subset \underbrace{(L(Q,r) \cap B(Q,r), \delta r)}_{\delta r \text{ neighborhood of } L(Q,r)\cap B(Q,r)}$



and

2. $L(Q,r) \cap B(Q,r) \subset (\partial\Omega \cap B(Q,r); \delta r)$



Definition. Let $\delta \in (0, 1/8)$. $\Omega \subset \mathbb{R}^{n+1}$ is a δ-Reifenberg flat domain if for each compact set $K \subset \mathbb{R}^{n+1}$, there exists $R_K > 0$ such that

1. $\sup_{0<r<R_K} \sup_{Q \in K \cap \partial\Omega} \theta(Q, r) \leq \delta$

2. $\sup_{r>0} \sup_{Q \in \partial\Omega} \theta(Q, r) \leq 1/8$ (if Ω is unbounded)

Examples.

1. C^1 domains

2. $\Omega = \{(x, t) \in \mathbb{R}^2 : x \in \mathbb{R}^n, t > \varphi(x)\}$ with
$$\varphi(x) = \sum_{k \geq 1} \frac{\cos(2^k x)}{2^k \sqrt{k}}.$$

In both cases $\lim_{r \to 0} \theta(Q, r) = 0$.

Remark. If for each compact set $K \subset \mathbb{R}^{n+1}$ there is $R_K > 0$ such that for $r \in (0, R_K)$

$$\sup_{Q \in K \cap \partial\Omega} \theta(Q, r) \leq C_K \left(\frac{r}{R_K} \right)^{\beta},$$

then Ω is a $C^{1,\beta}$ domain.

Theorem.[R] δ-Reifenberg flat domains are have Hölder continuous boundaries, provided δ is small enough.

Theorem.[KT] δ-Reifenberg flat domains are NTA, provided δ is small enough.

Boundary behavior of harmonic functions on NTA domains.

Let Ω be an NTA domain with constants $M > 2$ and $R > 0$, and let K be a compact set. The constant C below only depends on the NTA constant and on K.

Lemma.[JK] For $Q \in \partial\Omega \cap K$, $0 < 2r < R$, and $X \in \Omega \backslash B(Q, 2Mr)$. Then for $s \in [0, r]$

$$\omega^X(B(Q, 2s)) \leq C\omega^X(B(Q, s)) \tag{9}$$

i.e. ω^X is a doubling measure.

Lemma.[JK] There exists $\beta > 0$ such that for all $Q \in \partial\Omega \cap K$, $0 < 4r < R$, and every harmonic function u in $\Omega \cap B(Q, 4r)$, if u vanishes continuously on $B(Q, 4r) \cap \partial\Omega$, then for $X \in \Omega \cap B(r, Q)$,

$$|u(X)| \leq C \left(\frac{|X - Q|}{r} \right)^{\beta} \sup_{Y \in B(Q,2r) \cap \Omega} |u(Y)|.$$

Corollary. Let $Q \in \partial\Omega \cap K$, $0 < 2r < R$ then

$$\omega^{A(Q,r)}(B(Q,r)) \geq C.$$

Lemma.[JK] Let $Q \in \partial\Omega \cap K$, and $0 < 4r < R$. If $u \geq 0$, $\Delta u = 0$ in Ω, and $u = 0$ in on $B(Q, 2r) \cap \partial\Omega$, then

$$\sup_{Y \in B(Q,r)\cap\Omega} u(Y) \leq Cu(A(Q,r)). \tag{10}$$

Lemma.[JK] Let $Q \in \partial\Omega \cap K$, $0 < 2r < R$, and $X \in \Omega \backslash B(Q, Mr)$. Then

$$C^{-1} < \frac{\omega^X(B(Q,r))}{r^{n-1}G(A(Q,r),X)} < C, \tag{11}$$

where $G(A(Q,r),-)$ is the Green's function of Ω with pole $A(Q,r)$.

Lemma.[JK] (Comparison Principle)
Let $r < R/M$. Let $u,\ v \geq 0$, $\Delta u = \Delta v = 0$ in Ω $u = v = 0$ on $B(Q, Mr) \cap \partial\Omega$ for $Q \in \partial\Omega$. Then for all $X \in B(Q, r) \cap \Omega$,

$$C^{-1}\frac{u(A(Q,r))}{v(A(Q,r))} \leq \frac{u(X)}{v(X)} \leq C\frac{u(A(Q,r))}{v(A(Q,r))}.$$

Theorem.[JK] There exists $\alpha > 0$, such that for $r < R/M$, if $u,\ v \geq 0$, $\Delta u = \Delta v = 0$ in Ω $u = v = 0$ on $B(Q, Mr) \cap \partial\Omega$ for $Q \in \partial\Omega$ then for $X, Y \in \Omega \cap B(Q, r)$,

(12)
$$\left|\frac{u(X)}{v(X)} - \frac{u(Y)}{v(Y)}\right| \leq C\frac{u(A(Q,r))}{v(A(Q,r))}\left(\frac{|X-Y|}{r}\right)^{\alpha}$$

In particular, $\lim_{X \to Q} \dfrac{u(X)}{v(X)}$ exists.

Lemma. Let Ω be an unbounded NTA domain and $Q_0 \in \partial\Omega$. There exists a unique function u such that

$$(13) \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$u(A(Q_0, 1)) = 1.$$

u is the Green function with pole ∞

Proof. Assume that $Q_0 = 0$. Let $A(0,1) = A$.

Uniqueness: Let u, v be as above. By the comparison principle for $\rho > 1$ and $X \in B(0,\rho) \cap \Omega$

$$C^{-1}\frac{u(A(0,\rho))}{v(A(0,\rho))} \leq \frac{u(X)}{v(X)} \leq C\frac{u(A(0,\rho))}{v(A(0,\rho))}.$$

Since $A \in B(0,\rho)$, and $u(A) = v(A)$ then for $X \in B(0,\rho) \cap \Omega$

$$(14) \qquad C^{-1} \leq \frac{u(X)}{v(X)} \leq C.$$

By (12) and (14) for $X \in B(0,\rho) \cap \Omega$

$$\left|\frac{u(X)}{v(X)} - 1\right| \leq C\frac{u(A(0,\rho))}{v(A(0,\rho))}\left(\frac{|X-A|}{\rho}\right)^{\alpha} \leq C\left(\frac{|X-A|}{\rho}\right)^{\alpha}.$$

Letting $\rho \to \infty$ we conclude that $u = v$ in Ω.

Existence: For $Y \in \Omega$ let

$$u_Y(X) = \frac{G(Y,X)}{G(Y,A)},$$

u_Y is a nonnegative harmonic function on $B(0,|Y|) \cap \Omega$. Let $K \subset \mathbb{R}^{n+1}$ be a fixed compact set. Fix $\rho > 0$ such that $K \cap \Omega \subset B(0,\rho) \cap \Omega$, and let $|Y| \geq 2\rho$. Let $X \in K \cap \Omega$. (10) and the Harnack Principle yield

$$G(Y,X) \leq CG(Y, A(0,\rho)) \leq C_{K,n} G(Y,A).$$

Thus for $|Y| \geq 2\rho$

$$\sup_{X \in K \cap \Omega} u_Y(X) \leq C_{K,n}.$$

Moreover by (9) and (11) the Radon measures $\frac{\omega^Y}{G(Y,A)}$ are uniformly bounded on $B(0,\rho)$. Let $\{Y_j\}_j \subset \Omega$ be such that $|Y_j| \to \infty$ as $j \to \infty$. There exists a subsequence $\{Y_{j'}\}$ such that $u_{j'}$ converges uniformly to a nonnegative harmonic function u in $B(0,\rho) \cap \Omega$ (Arzela-Ascoli) and

$$\int \phi \frac{d\omega^{Y_{j'}}}{G(Y_{j'},A)} \to \int \phi d\mu,$$

where μ is a Radon measure and $\phi \in C_c^\infty(B(0,\rho))$.

Letting $\rho \to \infty$ and taking a diagonal subsequence we conclude that there is a subsequence u_{j_k} which converges to the nonnegative harmonic function u, uniformly on compact sets of Ω. Moreover

$$\frac{\omega^{Y_{j_k}}}{G(Y_{j_k}, A)} \rightharpoonup \mu.$$

Since $u(A) = 1$ and $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, u satisfies (13). By the uniqueness proved above we conclude that $u_{|Y|} \to u$ as $|Y| \to \infty$.

By (8)

$$\int_\Omega \frac{G(Y_{j_k}, X)}{G(Y_{j_k}, A)} \Delta\varphi(X)\, dX = \int_{\partial\Omega} \varphi(Q) \frac{d\,\omega^{Y_{j_k}}(Q)}{G(Y_{j_k}, A)}.$$

Hence

$$\int_{\partial\Omega} \varphi\, d\mu = \int_\Omega u \Delta\varphi\, dX \quad \forall \varphi \in C_c^\infty(\mathbb{R}^{n+1}).$$

$\omega^\infty = \frac{\mu}{\mu(B(0,1))}$ and $v = \frac{u}{\mu(B(0,1))}$ satisfy $\omega^\infty(B(0,1)) = 1$, and

$$\int_{\partial\Omega} \varphi d\,\omega^\infty = \int_\Omega v \Delta\varphi \quad \forall \varphi \in C_c^\infty(\mathbb{R}^{n+1}).$$

Properties:

- If $u = 0$ and $v = 0$ in Ω^c, u adn v are the subharmonic on $\mathbb{R}^{n+1}$.

- For $Q \in \partial\Omega$ by (11)

$$C^{-1} < \frac{\omega^{\infty}(B(Q,r))}{r^{n-1}v(A(Q,r))} < C,$$

Corollary. Let Ω be an unbounded NTA domain, and $Q_0 \in \partial\Omega$. There exists a unique doubling Radon measure ω^∞, supported on $\partial\Omega$ satisfying:

$$\int_{\partial\Omega} \varphi d\,\omega^\infty = \int_\Omega v \Delta \varphi \quad \forall \varphi \in C_c^\infty(\mathbb{R}^{n+1})$$

where

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v > 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\omega^\infty(B(Q_0, 1)) = 1.$$

ω^∞ is the harmonic measure of Ω with pole ∞

Proof. ω^∞ is doubling. Let $K \subset \mathbb{R}^{n+1}$ be compact, $Q \in K \cap \partial\Omega$. Given $r > 0$, for j_k large, $Y_{j_k} \in \Omega \backslash B(2Mr, Q)$. Then for $s \in [0, r]$ by (9)

$$\omega^{Y_{j_k}}(B(Q, 2s)) \leq C\omega^{Y_{j_k}}(B(Q, s)).$$

Hence

$$\begin{aligned}
\omega^\infty(B(Q, 2s)) &\leq \liminf_{j_k \to \infty} \frac{\omega^{Y_{j_k}}(B(Q, 2s))}{\mu(B(0,1))G(Y_{j_k}, A)} \\
&\leq C \liminf_{j_k \to \infty} \frac{\omega^{Y_{j_k}}(\overline{B}(Q, \frac{s}{2}))}{\mu(B(0,1))G(Y_{j_k}, A)} \\
&\leq C\omega^\infty(\overline{B}(Q, \frac{s}{2})) \\
&\leq C\omega^\infty(B(Q, s)).
\end{aligned}$$

Theorem [AC]. Assume that:

1. "Ω satisfies the divergence theorem, and that the surface measure of its boundary has Euclidean growth,"

2. Ω is a unbounded δ-Reifenberg flat domain for some $\delta > 0$ small enough,

3. '$\log h \in C^{0,\beta}$ for some $\beta \in (0,1)$',

then Ω is a $C^{1,\alpha}$ domain for some $\alpha \in (0,1)$ which depends on β.

Here h denotes the Poisson kernel with pole at infinity (i.e. the Radon Nikodym derivative of the harmonic measure with pole at ∞ w.r.t. to the surface measure of $\partial\Omega$).

4. Sets of locally finite perimeter

Definition. A measurable set $\Omega \subset \mathbb{R}^{n+1}$ has locally finite perimeter if $X_\Omega \in BV_{\text{loc}}(\mathbb{R}^{n+1})$, i.e.

$$\sup\left\{ \int_\Omega \operatorname{div} \varphi \, dx \;\middle|\; \varphi \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}) \right\} < \infty$$

Theorem. Let Ω be a set of locally finite perimeter. There exist a Radon measure $\|\partial\Omega\|$ on $\mathbb{R}^{n+1}$ and a $\|\partial\Omega\|$ measurable function $\nu_\Omega : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ s.t.

1. $|\nu_\Omega| = 1 \qquad \|\partial\Omega\|$ a.e.

2. $\displaystyle\int_\Omega \operatorname{div} \varphi = \int_{\mathbb{R}^{n+1}} \varphi \cdot \nu_\Omega \, d\|\partial\Omega\|$

   $\forall\, \varphi \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$.

Example. Let $\Omega \subset \mathbb{R}^{n+1}$ be a smooth domain such that for each compact set $K \subset \mathbb{R}^{n+1}$ $\mathcal{H}^n(\partial\Omega \cap K) < \infty$. By divergence theorem if $\varphi \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ then

$$\int_\Omega \operatorname{div} \varphi dx = \int_{\partial\Omega} \varphi \cdot \nu \, d\mathcal{H}^n,$$

where ν is the outward unit normal to $\partial\Omega$.

If $|\varphi| \leq 1$ and $\operatorname{support} \varphi = K$

$$\begin{aligned} \left|\int_\Omega \operatorname{div} \varphi \, dx\right| &\leq \int_{\partial\Omega} |\varphi \cdot \nu| \, d\mathcal{H}^n \\ &\leq \mathcal{H}^n(\partial\Omega \cap K) < \infty. \end{aligned}$$

Thus Ω has locally finite perimeter and

$$\|\partial\Omega\| = \mathcal{H}^n \llcorner \partial\Omega \text{ and } \nu_\Omega = \nu \ \mathcal{H}^n \text{ a.e.} \partial\Omega$$

i.e. E Borel

$$\|\partial\Omega\|(E) = H^n(E \cap \partial\Omega).$$

Let Ω be set of locally finite perimeter.

Definition. $X \in \partial^*\Omega$, the **reduced boundary** of Ω if

i) $\|\partial\Omega\|(B(X,r)) > 0 \qquad \forall\, r > 0.$

ii) $\lim_{r\to 0} \fint_{B(X,r)} \nu_\Omega d\|\partial\Omega\| = \nu_\Omega(X)$ and

iii) $|\nu_\Omega(X)| = 1$

In particular

$$\|\partial\Omega\|(\mathbb{R}^{n+1} \setminus \partial^*\Omega) = 0.$$

Lemma. Let $\varphi \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ then

$$\int_{\Omega\cap B(X,r)} \operatorname{div}\varphi \, dY = \int_{B(X,r)} \varphi \cdot \nu_\Omega \, d\|\partial\Omega\| + \int_{\Omega\cap\partial B(X,r)} \varphi \cdot \nu \, d\mathcal{H}^n$$

for a.e. $r > 0$, ν is the outward unit normal to $B(X,r)$.

Lemma. There exist $A_1, A_2 > 0$ such that for $X \in \partial^*\Omega$

i) $\liminf_{r\to 0} \frac{\mathcal{H}^{n+1}(B(X,r)\cap\Omega)}{r^{n+1}} \geq A_1$

ii) $\liminf_{r\to 0} \frac{\mathcal{H}^{n+1}(B(X,r)\backslash\Omega)}{r^{n+1}} \geq A_2$

Definition. $X \in \partial_* \Omega$ the measure theoretic boundary of Ω if

$$\limsup_{r \to 0} \frac{\mathcal{H}^{n+1}(B(X,r) \cap \Omega)}{r^{n+1}} > 0$$

$$\limsup_{r \to 0} \frac{\mathcal{H}^{n+1}(B(X,r) \backslash \Omega)}{r^{n+1}} > 0$$

Lemma. 1) $\partial^* \Omega \subset \partial_* \Omega$

2) $\mathcal{H}^n(\partial_* \Omega \setminus \partial^* \Omega) = 0$

Remark. Let Ω be set of locally finite perimeter and NTA domain: Ω and Ω^c satisfy the corkscrew condition. Thus for $r > 0$ $Q \in \partial \Omega$

$$\mathcal{H}^{n+1}(\Omega \cap B(Q,r)) \geq c_n r^{n+1}$$

and

$$\mathcal{H}^{n+1}(B(r,Q) \setminus \Omega) \geq c_n r^{n+1}.$$

Thus $Q \in \partial_* \Omega$, and $\partial_* \Omega = \partial \Omega$,

$$\mathcal{H}^n(\partial \Omega \setminus \partial^* \Omega) = 0.$$

Theorem. [Isoperimetric inequality]

$$\min\left\{\mathcal{H}^{n+1}(B(X,r)\cap\Omega),\mathcal{H}^{n+1}(B(X,r)\setminus\Omega)\right\}^{\frac{n}{n+1}}$$
$$\leq C\|\partial\Omega\|(B(X,r))$$

For $Q\in\partial^*\Omega$ let

$$\begin{aligned} H(Q) &= \{Y\in\mathbb{R}^{n+1}:\nu_\Omega(Q)\cdot(Y-Q)=0\}\\ H^+(Q) &= \{Y\in\mathbb{R}^{n+1}:\nu_\Omega(Q)\cdot(Y-Q)\geq 0\}\\ H^-(Q) &= \{Y\in\mathbb{R}^{n+1}:\nu_\Omega(Q)\cdot(Y-Q)\leq 0\}\end{aligned}$$

Picture

Theorem. [Blow up of the reduced boundary]
If $Q \in \partial^*\Omega$ then

$$\chi_{\eta_{Q,r}(\Omega)} \underset{r\to 0}{\longrightarrow} \chi_{H^-(Q)} \text{ in } L^1_{\text{loc}}(\mathbb{R}^{n+1}),$$

where $\eta_{Q,r}(\Omega) = \frac{1}{r}(\Omega - Q)$

Corollary. If $Q \in \partial^*\Omega$ then

1. $$\lim_{r\to 0} \frac{\mathcal{H}^{n+1}(B(Q,r) \cap \Omega \cap H^+(Q))}{r^{n+1}} = 0$$

2. $$\lim_{r\to 0} \frac{\mathcal{H}^{n+1}(B(Q,r) \setminus \Omega) \cap H^-(Q))}{r^{n+1}} = 0$$

3. $$\lim_{r\to 0} \frac{\|\partial\Omega\|(B(Q,r))}{\omega_n r^n} = 1$$

Theorem. [**Structure theorem for sets of locally finite perimeter**] Let Ω be set of locally finite perimeter then

1. $\partial^*\Omega \in \bigcup_{k=1}^{\infty} \Sigma_k \cup \Sigma_0$ where

$$\|\partial\Omega\|(\Sigma_0) = 0$$

   Σ_k is a C^1 hypersurface

2. $\nu\Big|_{\partial^*\Omega\cap\Sigma_k}$ is the outer unit normal to Σ_k.

iii) $\|\partial\Omega\| = \mathcal{H}^n \llcorner \partial^*\Omega$

Corollary. If Ω is NTA and a set of locally finite perimeter

$$\|\partial\Omega\| = \mathcal{H}^n \llcorner \partial\Omega$$

Theorem.[**Generalized Gauss-Green theorem**] Let Ω be an NTA domain and a set of locally finite perimeter then

$$\int_\Omega \operatorname{div}\varphi\,dx = \int_{\partial\Omega} \varphi\cdot\nu_\Omega\,d\mathcal{H}^n \qquad \forall\,\varphi\in C_c^1(\mathbb{R}^{n+1},\mathbb{R}^{n+1}),$$

ν_Ω is the unique measure theoretic outer unit normal.

Proof.

$$\begin{aligned}\int_\Omega \operatorname{div}\varphi\,dx &= \int \varphi\cdot\nu_\Omega\,d\|\partial\Omega\| \\ &= \int \varphi\cdot\nu_\Omega\,d\mathcal{H}^n \llcorner \partial\Omega \\ &= \int_{\partial\Omega} \varphi\cdot\nu_\Omega\,d\mathcal{H}^n\end{aligned}$$

5. Free boundary regularity problem for the Poisson kernel

Definition. An domain Ω is a chord arc domain if:

- Ω is NTA
- Ω is a set of locally finite perimeter
- the surface measure of $\partial\Omega$ $\sigma = \mathcal{H}^n \llcorner \partial\Omega$ is Ahlfors regular, i.e.

$$\exists C > 1 \qquad C^{-1} \leq \frac{\sigma(B(r,Q))}{r^n} \leq C,$$

  for $r < \operatorname{diam}\Omega$.

Theorem [AC]. Assume that:

1. Ω is an unbounded chord arc domain

2. Ω is a δ-Reifenberg flat domain for some $\delta > 0$ small enough,

3. $\log h \in C^{0,\beta}$ for some $\beta \in (0,1)$,

then Ω is a $C^{1,\alpha}$ domain for some $\alpha \in (0,1)$ which depends on β. Moreover if h is identically equal to 1 then Ω is a half-space.

Here

$$\int_{\Omega} u\Delta\varphi\, dx = \int_{\partial\Omega} \varphi h\, d\mathcal{H}^n, \text{ for } \varphi \in C_c^\infty(\mathbb{R}^{n+1})$$

and

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

h is the Poisson kernel with pole at ∞

Let $Q_j \in \partial\Omega$, and $r_j > 0$, consider

$$\begin{aligned}
\Omega_j &= \frac{1}{r_j}(\Omega - Q_j) \\
\partial\Omega_j &= \frac{1}{r_j}(\partial\Omega - Q_j) \\
u_j(X) &= \frac{u(r_j X + Q_j)}{r_j \fint_{B(Q_j, r_j)} h \, d\sigma_j} \\
\omega_j(E) &= \frac{\omega(r_j E + Q_j)}{r_j^n \fint_{B(Q_j, r_j)} h \, d\sigma_j} \\
d\omega_j &= h_j \, d\sigma_j \;\; \mathcal{H}^n - a.e. \text{ in } \partial\Omega_j \\
h_j(Q) &= \frac{h(r_j Q + Q_j)}{\fint_{B(Q_j, r_j)} h \, d\sigma_j}
\end{aligned}$$

Theorem. Let Ω be a chord arc domain as above. Then

$$\begin{aligned} \Omega_j &\to \Omega_\infty \\ \partial\Omega_j &\to \partial\Omega_\infty \end{aligned}$$

where Ω_∞ is an unbounded chord arc domain. Moreover there exists u_∞ such that

$$u_j \to u_\infty \text{ uniformly on compact sets}$$

$$\begin{cases} \Delta u_\infty = 0 & \text{in} \quad \Omega_\infty \\ u_\infty = 0 & \text{on} \quad \partial\Omega_\infty \\ u_\infty > 0 & \text{in} \quad \Omega_\infty. \end{cases}$$

Furthermore

$$\omega_j \rightharpoonup \omega_\infty \quad \text{and} \quad \sigma_j \rightharpoonup \sigma_\infty$$

weakly on Radon measures. ω_∞ is the harmonic measure of Ω_∞ with pole at infinity, and σ_∞ is the surface measure of $\partial\Omega_\infty$. The Poisson kernel of Ω_∞ with pole at infinity h_∞ satisfies

$$h_\infty = \frac{d\omega_\infty}{d\sigma_\infty} = 1 \quad \mathcal{H}^n - a.e \text{ in } \partial\Omega_\infty.$$

Theorem [AC], [KT]. Assume that:

1. Ω is an unbounded chord arc domain
2. Ω is a δ-Reifenberg flat domain for some $\delta > 0$ small enough,
3. $h = 1$, $\mathcal{H}^n$-a.e. in $\partial\Omega$

Then Ω is a half-space.

Theorem [LV] Assume that:

1. Ω be a bounded chord arc domain
2. $0 \in \Omega$, and $k_0 = 1$, $\mathcal{H}^n$-a.e. in $\partial\Omega$.

Then $\Omega = B(0, R)$ with $\sigma_n R^n = 1$.

Question: *Is the flatness assumption necessary in the unbounded case?*

Examples.

- $\Omega = \mathbb{R}^{n+1}_+$, $u(x, x_{n+1}) = x_{n+1}$ and $h = 1$.

- Keldysh-Lavrentiev constructed a set of locally finite perimeter $\Omega \subset \mathbb{R}^2$ whose boundary is not Ahlfors regular, whose Poisson kernel is identically equal to 1 and Ω is not C^1.

- Kowalski-Preiss cone:

  $\Omega = \left\{(x_1, ..., x_4) \in \mathbb{R}^4 : |x_4| < \sqrt{x_1^2 + x_2^2 + x_3^2}\right\}.$

  Let $r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$, $\theta \in [\frac{\pi}{4}, \frac{3\pi}{4}]$, and $x_4 = r\cos\theta$, then $X = (x_1, x_2, x_3, x_4) \in \overline{\Omega}$.

  $$u(X) = -\frac{r}{2\sqrt{2}}\frac{\cos 2\theta}{\sin\theta}.$$

  satisfies $\Delta u = 0$ in Ω, $u > 0$ in Ω and $u = 0$ on $\partial\Omega$. $\omega^\infty = \mathcal{H}^n \llcorner \partial\Omega$, i.e $h = 1$, $\mathcal{H}^n$-a.e in $\partial\Omega$.

Main Theorem [KT]. Assume that:

1. Ω is an unbounded chord arc domain

2. Ω is a δ-Reifenberg flat domain for some $\delta > 0$ small enough,

3. $\sup_\Omega |\nabla u| \leq 1$ and $h \geq 1$, $\mathcal{H}^n$-a.e. in $\partial\Omega$

Then modulo translation and rotation $\Omega = \mathbb{R}^{n+1}_+$ and $u(x, x_{n+1}) = x_{n+1}$.

Here

$$\int_\Omega u\Delta\varphi\, dx = \int_{\partial\Omega} \varphi h\, d\mathcal{H}^n, \text{ for } \varphi \in C_c^\infty(\mathbb{R}^{n+1}),$$

and

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Definition. For $0 < \sigma_+, \sigma_< 1$, $Q_0 \in \partial\Omega$, $\rho > 0$.

$u \in F(\sigma_+; \sigma_-)$ in $B(Q_0, \rho)$ in the direction ν if

$$u(X) = 0 \text{ for } \langle X - Q_0, \nu\rangle \geq \sigma_+ \rho$$

and

$$u(X) \geq -[\langle X - Q_0, \nu\rangle + \sigma_- \rho] \quad \text{for } \langle X - Q_0, \nu\rangle \leq -\sigma_- \rho.$$

Lemma A. If $u \in F(\sigma; 1)$ in $B(Q_0, \rho)$ in the direction ν then $u \in F(2\sigma; C\sigma)$ in $B(Q_0, \frac{\rho}{2})$ in the direction ν.

$u \in F(\sigma; 1)$ in $B(Q_0, \rho)$ then $u \in F(2\sigma; C\sigma)$ in $B(Q_0, \frac{\rho}{2})$

Lemma B. Given $\theta \in (0, 1)$ there exist $\sigma_{n,\theta} > 0$ and $\eta_\theta = \eta \in (0, 1)$ so that if $\sigma \leq \sigma_{n,\theta}$ and $u \in F(\sigma; \sigma)$ in $B(Q_0, \rho)$ in the direction $\nu_{Q_0,\rho}$ for $Q_0 \in \partial\Omega$, then $u \in F(\theta\sigma; 1)$ in $B(Q_0; \eta\rho)$ in the direction $\nu_{Q_0,\eta\rho}$ and

$$|\nu_{Q_0,\rho} - \nu_{Q_0,\eta\rho}| \leq C\sigma.$$

$u \in F(\sigma; \sigma)$ in $B(Q_0, \rho)$ then $u \in F(\theta\sigma; 1)$ in $B(Q_0, \eta\rho)$

Proof of the Main Theorem. Since Ω is a δ-Reifenberg flat chord arc domain, $u \in F(\delta; 1)$ in $B(Q, 2r)$ for $r > 0$ and $Q \in \partial\Omega$. If $Q = 0$, $B(0, r) = B(r)$

(A) $u \in F(\delta; 1)$ in $B(2r)$ then $u \in F(2\delta, C\delta)$ in $B(r)$

Choosing δ so that $\max\{2\delta, C\delta\} \leq \sigma$ we have

(B) $u \in F(\sigma; \sigma)$ in $B(r)$ then $u \in F(\theta'\sigma, 1)$ in $B(2\eta r)$

(A) $u \in F(\theta'\sigma; 1)$ in $B(2\eta r)$ then $u \in F(2\theta'\sigma, C\theta'\sigma)$ in $B(\eta r)$

Choosing θ' so that $\max\{2\theta', C\theta'\} \leq \theta$ we have

(B + A) $u \in F(\sigma; \sigma)$ in $B(r)$ then $u \in F(\theta\sigma, \theta\sigma)$ in $B(\eta r)$

By iteration

$$u \in F(\theta^m\sigma; \theta^m\sigma) \text{ in } B(\eta^m r) \text{ for } r > 0.$$

Moreover if $\nu_m = \nu_{0,\eta^m r}$ then

$$|\nu_m - \nu_{m+1}| \leq C\theta^m\sigma.$$

Let $\nu_r = \lim_{m\to\infty} \nu_m$, and $\Lambda(r)$ is the n-plane orthogonal to ν_r then for $s \in (0, r)$ we have

$$\frac{1}{s} D[B(s) \cap \partial\Omega; \Lambda(r) \cap B(s)] \leq C \left(\frac{s}{r}\right)^{\beta},$$

for some $\beta > 0$. Since $\mathbb{S}^n$ is compact there exists an increasing sequence $r_i \to \infty$ and an n-plane Λ_∞ such that for $s > 0$

$$D[B(s) \cap \partial\Omega; \Lambda_\infty \cap B(s)] = 0.$$

Thus $\partial\Omega = \Lambda_\infty$ w.l.o.g $\Omega = \mathbb{R}^{n+1}_+$, $0 \leq u \leq x_{n+1}$ and $\dfrac{\partial u}{\partial x_{n+1}} = 1$ on Λ_∞. Moreover by (12)

$$\left|\frac{u(X)}{x_{n+1}} - 1\right| \leq C \frac{u(A(0,r))}{r} \left(\frac{|X|}{r}\right)^{\alpha} \leq C \left(\frac{|X|}{r}\right)^{\alpha},$$

letting $r \to \infty$ we conclude that $u(x, x_{n+1}) = x_{n+1}$.

Non-homogeneous blow-up

Lemma B is proved by contradiction. Assume that there exist $\theta \in (0,1)$ such that for every $\eta > 0$ and every non-negative decreasing sequence $\{\sigma_j\}$,

$$u \in F(\sigma_j; \sigma_j) \text{ in } B(Q_j, \rho_j) \text{ in the direction } \nu_j$$

but

$$u \notin F(\theta\sigma_j; 1) \text{ in } B(Q_j, \eta\rho_j).$$

Assume that $h(Q_j) \geq 1$, and $\nu_j = e_{n+1}$. For $X \in B(0,1)$ let

$$u_j(X) = \frac{1}{\rho_j} u(\rho_j X + Q_j).$$

Note that $\Delta u_j = 0$ in $\Omega_j = \frac{1}{\rho_j}(\Omega - Q_j)$, $u_j > 0$ in Ω_j, $u_j = 0$ on $\partial\Omega_j = \frac{1}{\rho_j}(\partial\Omega - Q_j)$, and

$$\int_{\Omega_j} u_j \Delta\varphi dX = \int_{\partial\Omega_j} \varphi h_j \, d\mathcal{H}^n \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^{n+1})$$

where

$$h_j(Q) = h(\rho_j Q + Q_j).$$

Moreover

(15)

$$\sup_{\Omega_j} |\nabla u_j| \leq 1 \quad \text{and} \quad h_j \geq 1 \quad \mathcal{H}^n \text{ a.e. in } \partial\Omega_j.$$

The hypothesis yields

$$u_j \in F(\sigma_j, \sigma_j) \text{ in } B(0,1)$$
$$\text{in the direction } e_{n+1}$$

(16) $$u_j \notin F(\theta\sigma_j; 1) \text{ in } B(0,\eta),$$

with $\sigma_j \to 0$ as $j \to \infty$.

Idea [AC]:

- Define sequences of scaled height functions corresponding to $\partial\Omega_j$.

- Prove that these sequences converge to a subharmonic Lipschitz function.

- Use this information to contradict the fact that $u_j \notin F(\theta\sigma_j; 1)$ in $B(\eta, 0)$ for j large enough.

For $y \in B(0,1) \cap \mathbb{R}^n \times \{0\} = B'$ define

$$f_j^+(y) = \sup\{h : (y, \sigma_j h) \in \partial\{u_j > 0\}\} \leq 1$$

and

$$f_j^-(y) = \inf\{h; (y, \sigma_j h) \in \partial\{u_j > 0\}\} \geq -1$$

Lemma. There exists a subsequence k_j such that for $y \in B'$

$$f(y) = \limsup_{\substack{k_j\to\infty \\ z\to y}} f^+_{k_j}(z) = \liminf_{\substack{k_j\to\infty \\ z\to y}} f^-_{k_j}(z).$$

Corollary. f is a continuous function in B', $f(0) = 0$; and $f^+_{k_j}$ and $f^-_{k_j}$ converge uniformly to f on compact sets of B'.

Lemma.* f is subharmonic in B'.

Lemma. There is a constant $C > 0$ such that for $y \in B'_{\frac{1}{2}}$

$$0 \le \int_0^{\frac{1}{4}} \frac{1}{r^2}(f_{y,r} - f(y))dr \le C$$

where

$$f_{y,r} = \fint_{\partial B'(y,r)} f \, d\mathcal{H}^{n-1}.$$

Lemma.* f is Lipschitz in $B'_{\frac{1}{16}}$.

Lemma. There exists $C > 0$ such that for any given $\theta \in (0,1)$ there exist $\eta = \eta(\theta) > 0$ and $l \in \mathbb{R}^n \times \{0\}$ with $|l| \leq C$ so that

$$f(y) \leq \langle l, y \rangle + \frac{\theta}{4}\eta \quad \text{for} \quad y \in B'_\eta.$$

Contradiction in the proof of Lemma B For $\theta \in (0,1)$ there exists $\eta = \eta(\theta) > 0$ such that for j large enough that

$$f_j^+(y) \leq \langle l, y\rangle + \frac{\theta}{2}\eta \quad \text{for} \quad y \in B'_\eta.$$

Since $f_j^+(y) = \sup\{h : (y, \sigma_j h) \in \partial\{u_j > 0\}\}$

(17)

$$\begin{aligned} u_j(X) \;&=\; 0 \text{ for } \; X \in B(0,\eta) \\ &\qquad \text{with} \;\; x_{n+1} \geq \sigma_j \langle l, x\rangle + \theta\eta\sigma_j. \end{aligned}$$

Let $\overline{\nu} = (1 + \sigma_j^2 |l|^2)^{-\frac{1}{2}}(-\sigma_j l, 1)$, (17) implies that

(18)

$$\begin{aligned} u_j(X) \;&=\; 0 \text{ for } X \in B(0,\eta) \\ &\qquad \text{with } \langle X; \overline{\nu}\rangle \geq \frac{\theta\eta\sigma_j}{2(1+\sigma_j^2|l|^2)^{\frac{1}{2}}} \geq \theta\eta\sigma_j, \end{aligned}$$

for j large enough. (18) states that for every $\theta \in (0,1)$ there is $\eta > 0$ so that $u_j \in F(\theta\sigma_j, 1)$ in $B(0,\eta)$ in the direction $\overline{\nu}$, which contradicts (16).

6. Weiss monotonicity formula

Assume that:

1. $\Omega \subset \mathbb{R}^{n+1}$ is an unbounded chord arc domain
2.
$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$
3. $h = 1$, $\mathcal{H}^n$-a.e. in $\partial\Omega$ i.e
$$\int_\Omega u \Delta\varphi = \int_{\partial\Omega} \varphi d\mathcal{H}^n \quad \forall \varphi \in C_c^\infty(\mathbb{R}^{n+1})$$

For $Q \in \partial\Omega$ and $r > 0$ the quantity

$$\phi(Q,r) = \frac{1}{r^{n+1}} \int_{B(Q,r)} |\nabla u|^2 - \frac{1}{r^{n+2}} \int_{\partial B(Q,r)} u^2 + \frac{\mathcal{H}^{n+1}(\Omega \cap B(Q,r))}{r^{n+1}}$$

is monotone and

$$\phi(Q,r) - \phi(Q,s) = 2 \int_s^r t^{-n-1} \int_{\partial B(Q,r)} \left(\nabla u \cdot \frac{P-Q}{|P-Q|} - \frac{u}{r} \right)^2 d\mathcal{H}^n dt$$

This monotonicity formula yields that the blow up limits of u are homogeneous functions of degree 1.

Theorem. [W] Assume that:

1. $\Omega \subset \mathbb{R}^{n+1}$ is an unbounded chord arc domain
2. $h = 1$, $\mathcal{H}^n$-a.e. in $\partial\Omega$ i.e

$\partial^*\Omega$ is C^∞. $\Sigma = \partial\Omega \backslash \partial^*\Omega$ the singular set of $\partial\Omega$, satisfies:

- If $n = 1$, $\Sigma = \emptyset$

- If $n = 2$, Σ consists of isolated points.

- Σ is a closed set of Hausdorff dimension at most $n - 2$.

Question: *Does there exist a characterization of Σ in terms $\phi(Q) = \lim_{r\to 0} \phi(Q, r)$?*

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