Let \( R \) be a ring and \( I \) an ideal of \( R \). The set of associated primes of the monomial ring \( R/I \) is the set

\[
\text{Ass}(R/I) := \{ p \mid p \text{ is prime and } p = (I : c) \text{ for some } c \text{ in } R \},
\]

where \((I : c)\) is a colon ideal, that is it consists of the elements \( r \) of \( R \) such that \( rc \) is in \( I \). Let \( p \) belong to the associated primes \( \text{Ass}(R/I) \) of \( R/I \) and \( c \) be such that \( p = (I : c) \). Let \( k \) be a field and \( R := k[x_1, x_2, \ldots, x_m] \) be a polynomial ring over \( m \) indeterminates. The main result that I want to prove is the following theorem.

**Theorem 1.** The associated primes of any square-free monomial ideal of \( R \) has the persistence property, that is

\[
\text{Ass}(R/I^k) \subset \text{Ass}(R/I^{k+1}),
\]

for all \( k \geq 1 \).

This theorem is known to not hold in the case of monomial ideals that are not square-free (for an example see [11], Example 4.18).

This problem can be aided by graph theory. To see this, we start with some definitions: A *simple graph* \( G \) is a pair of sets \( G := (V, E) \) such that \( E \) belongs to the power set \( \mathcal{P}(V) \) and every element of \( E \) has a cardinality of two. The elements of the set \( V \) are called the *vertices* of \( G \) and the elements of \( E \) are the *edges* of \( G \). Although edges are elements of the power set of \( V \), with an abuse of language they may also be seen as a product of vertices, for example, the edge \( e := \{v_1, v_2\} \) may be referred to as \( v_1v_2 \). Let \( W \) be a subset of the vertex set of a graph \( G \). The induced graph on \( W \), denoted by \( G[W] \), is the graph with the vertex set \( W \) and all edges of \( G \) that contain only vertices in \( W \). The duplication of a vertex \( v \) in a graph \( G \) is adding an addition vertex \( v' \) to the vertex set of \( G \) and by adding all edges that contain \( v \) with the vertex \( v \) replaced with \( v' \) to the edge set of \( G \).
Graphs provide a visualization: for example, the graph \( G := (\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\}) \) can be represented by the following diagram.

\[
\begin{array}{c}
\text{v}_1 \\
v_1v_2 \quad v_2v_3 \quad v_3 \\
v_1v_3
\end{array}
\]

A hypergraph is the same as a simple graph except that the elements of its edge set are not restricted to cardinality of two, but can have cardinality of any positive integer. From this, one can see that any square-free monomial ideal \( I \) of a polynomial ring can be represented as a hypergraph.

We will call such a graph the associated graph of the ideal \( I \) in the polynomial ring \( R \). One can also go the other way from a graph to create an ideal \( I \), which is called the edge ideal of the graph, by taking the ideal generated by the product edges of the graph in the polynomial ring over the vertices of the graph. We will not distinguish much between a square-free monomial ideal and its graph representation.

A walk is an ordered multi-set of vertices \( \{v_1, \ldots, v_q\} \) together with an ordered set of edges \( \{v_1v_2, \ldots, v_{q-1}v_q, v_qv_1\} \). If the starting and ending vertices of a walk are the same, it is said to be closed. A cycle is a closed walk whose vertices do not repeat. The length of a walk is the cardinality of its vertex multi-set. Two vertices \( v_1, v_2 \) are adjacent if \( v_1v_2 \) is an edge of the graph. An edge \( e \) is incident to a vertex \( v \) if \( v \) belongs to \( e \). Two edges are independent if they have empty intersection. A graph has a \( t \)-matching if it contains \( t \) independent edges. A graph is bipartite if there exists a partition of the vertices of the graph into two sets such that no two vertices of a single one of these sets is adjacent to one another.

The case of Theorem 1 where the square-free monomials are restricted to have degree two, that is its associated graph is simple, was first proved by José Martínez-Bernal, Susan Morey and Rafael Villarreal in 2011 ([1], Theorem 2.15). I have already been able to provide two new proofs of this case, and a third in a slightly more restricted case. I will mention the two most interesting.

Before this above mentioned proof of the case of degree two was published, there were several attempts. One being by Christopher Francisco, Huy Hà and Adam Tuyl in 2010 [8]. Before seeing what they did, some terminology is required. A graph has a \( s \)-coloring if each vertex of the graph can be assigned a color among \( s \) colors such that any two adjacent vertices have a different color. The chromatic number of a graph is the
smallest value of $s$ such that an $s$-coloring of the graph exists. A graph is critically $s$-chromatic if it is $s$ chromatic and the subgraph obtained by removing any vertex is critically $(s + 1)$-chromatic. They posed the following conjecture, which they showed implies Theorem 1 in the case of degree two and the edge ideal is unmixed and of height two ([8], Conjecture 1.1 and Theorem 1.2) (these definitions do not matter at the time being).

**Theorem 2.** Let $G$ be a simple graph that is critically $s$-chromatic, where $s$ is a positive integer. Then there exists a subset of vertices of the vertex set of $G$ such that the induced graph on this set is critically $(s + 1)$-chromatic.

I have been able to prove this conjecture. Furthermore, the proof that the authors provided can be easily extend to the case of any degree of Theorem 1 for a hypergraph analog of this conjecture. But such a generalization is much more difficult to prove, and I have not made any progress on it.

We now look at some basic properties that these associated primes have, and then outline a construction for the case of such associated primes of degree two, which naturally leads to another proof of this case. It has already been shown that the set of associated primes of $R/I$ is the set of all minimal covers of the graph $G$, which are minimal sets of vertices of $G$ such that every edge of $G$ is incident to a vertex in this set ([13], Proposition 7.2.3). It is also known that if the graph is bipartite, the sets of associated primes for powers of $I$ stabilize at one, that is $Ass(R/I) = Ass(R/I^k)$ for any $k$. A simple graph is bipartite if and only if it does not contain a cycle of odd length. This leads one to consider what role odd cycles have in the proof of Theorem 1.

M. Brodmann, in 1979, showed that the sets of associated primes of $R/I^N$ stabilize for large $N$ [2]. In 2002, Janet Chen, Susan Morey and Anne Sung published a paper that was able to take odd cycles of a given length in the graph in question and construct all associated primes that were in the stable set by a step process, although it was not able to answer what were the smallest powers of the edge ideal that they were an associated primes for [3]. They were then able to provide a bound of the minimal value of $N$ such that the sets of associated primes of $R/I^N$ stabilize.

Before reading this paper by Chen, Morey and Sung, I was able to obtain a more generalized result. I was able to characterize all associated primes of Theorem 1 in the case of degree two. To explain this, we
will first need the concept of what I have been calling an \textit{odd closed multi-walk}. An \textit{odd closed multi-walk} is a graph with an odd number \( r \) of vertices such that each of these vertices is on an odd walk, say of length \( q \), such that such that the subgraph of this odd closed multi-walk induced on the complement of this odd walk contains a matching of size \( \frac{r-q}{2} \). The following is an example of an odd closed multi-walk represented by solid edges, with dotted edges as additional edges in the graph.

\begin{center}
\begin{tikzpicture}
\draw[thick] (0,0) -- (1,1) -- (2,0) -- (0,-1) -- (1,0) -- (2,1);
\draw[dotted] (0,0) -- (1,-1) -- (2,1) -- (0,1) -- (1,0) -- (2,-1);
\fill (0,0) circle (2pt);
\fill (1,1) circle (2pt);
\fill (2,0) circle (2pt);
\fill (0,-1) circle (2pt);
\fill (1,0) circle (2pt);
\fill (2,1) circle (2pt);
\end{tikzpicture}
\end{center}

It is easy to see that duplicating any vertex in an odd closed multi-walk induces a matching that is greater in cardinality than any matching in the the odd closed multi-walk by itself. It is worth noting that that singles vertices are also odd close multi-walks of length one, but this can be put to the side for now, although in the general case of Theorem 1 this plays more of a role.

By taking a set of these odd closed multi-walks \( \{M_1, \ldots, M_n\} \) that are disjoint and not adjacent to one another, an associated prime of \( R/I^k \), where \( k := (\sum_{i=1}^n |M_i| - 1) + 1 \), can be found by taking the vertex sets of them unioned with all incident vertices to them and with a minimal cover of the graph all these previous vertices removed, and then taking the ideal \( p \) generated by this set of vertices. One sees that \( p = (I^k : c) \), where \( c \) is all the product of all vertices of each odd closed multi-walk, the vertices in the minimal cover that is not incident to these odd closed multi-walks and all vertices adjacent to them, and an arbitrary number of vertices not in \( p \).

From each associated prime \( p \) found in this way for \( R/I^k \), one can add an additional two vertices that make up an edge of one the odd closed multi-walks to the odd closed multi-walk to create a new and see this this new ideal is still found by the same process, with just one of the odd closed multi-walks extended. If the \( p \) is not made from any odd closed multi-walks but just a minimal cover, one can still show that this associated prime for \( R/I \) is an associated prime for \( R/I^k \) for any \( k \) by taking by multiplying the element \( c \) in \( p = (I : c) \) by an edge of \( G \) with only one vertex in the minimal generating set of the associated prime \( k - 1 \) times. This shows that the associated primes found in this way form an ascending chain.
We will now see why all associated primes can be found in this way. Suppose that there exists an associated prime \( \mathfrak{p} = (I^k : c) \). From \( c \), we make a graph whose vertices are all vertices of \( c \) that are in \( \mathfrak{p} \) and whose edges are the edges in the original graph with all vertices that are not in \( \mathfrak{p} \) removed. One can go one to find that this new graph is decomposable into these odd closed multi-walks and vertices that were in the minimal cover.

A survey of this field [11] asks some main open questions in this field. One of which (Question 4.3) asks what is the smallest \( N \) such that given an associated prime \( \mathfrak{p} \), \( \mathfrak{p} \) is an associated prime of \( R/I^N \). The above is able answer this question by breaking up \( \mathfrak{p} \) into these odd closed multi-walks of the smallest order possible. Also from the above method, one can go on to find which primes are associated primes for some power of \( I \) by seeing which ones decompose into small enough odd closed multi-walks, which gives some sort of answer to Question 4.2 of the survey.

Another question (Question 4.1) asks what is an effective upper-bound on the index of stability. As mentioned above Chen, Morey and Sung were able to provide an upper-bound, and also Lê Hoa was able to provide another bound in 2006 [9], but both of these are not minimal and not very effective. I am not sure if I can provide an “effective” upper-bound, but the characterization above is able to provide a better bound than the ones already given. It also may explain why it is difficult to find an effective upper-bound because there is the problem that for two associated primes of that the stable set of the monomial ring \( R/I \), it that the associated prime with fewer generators may only occur for higher powers of the edge ideal \( I \) than the minimal power that the associated prime with more generators appears at.

When expanding to the more general case of square-free monomial ideals of any degree, the situation becomes rapidly more complicated. By employing a similar characterization, which seems very likely to consist of all associated primes, I have been able to show that primes found by this construction form an ascending chain by finding a bound of two on the minimal length of an edge is any such construction. From this, I can then use the same method used above to prove that they have the persistence property (or I could use a variation of the method used to prove the case of degree two that was not explained).

Problems then arise that I have not been able to overcome when trying to reconstruct the odd closed multi-walks from the induced graph as was outlined for the case of degree two, but I have made some
progress. Furthermore, to find a reasonable upper-bound on the index of stability in this more general case may be a little bit far-fetched, but the bound in this case does not seem to be very important in the current literature.

\[ \cdots \]

We now change directions and look at another problem that I plan on investigating during this research project, if time allows, which is the Conforti-Cornu\'ejols conjecture. As usual, some definitions are in order.

A clutter is a graph such that no edge is a subset of another edge. The vertex covering number of a graph is the minimal cardinality of any cover of the graph. Let \( p \) be a prime ideal. The height of the prime ideal \( p \) is the supremum of the length of any strictly increasing chain of prime ideals such that the largest ideal of the chain is \( p \), that is the largest value of \( n \) such that

\[ p_0 \subset p_1 \subset \cdots \subset p_n = p. \]

The height of an ideal \( I \) in general, denoted by \( ht(I) \), is

\[ ht(I) := \min\{ht(p) | I \subset p \text{ and } p \text{ is prime}\}. \]

One can see that the height of an edge ideal is equal to its corresponding vertex covering number ([13], Lemma 6.3.37). A graph is said to have the König property if the cardinality of the largest matching of the graph is equal to the vertex covering number. One can show that there exists a minimal generating set of an edge ideal. A minor of such an edge ideal is made by removing some products of vertices from all edges that are divisible by one of these products in the minimal generating set of the ideal. A parallelization of a graph is obtained from the graph by duplicating vertices in the graph a finite number of times.

A clutter has the packing property if it has the König property and so does every minor of it as well. To properly express what it means for a clutter to have the min-cut max-flow property takes some effort through the Theory Linear Programming. For simplicity we will use an equivalent definition of the min-cut max-flow property of a clutter, which is the condition that the clutter is mengerian, that is the clutter and all parallelizations of it have the König property ([13], Theorem 14.3.6).
It was shown by G. Cornuéjols, in 1993, that if a clutter has the max-flow min-cut property, then it also has the packing property [5]. The converse of this is known as the Conforti-Cornuéjols conjecture [4], which is still an open problem.

**Conjecture 3.** A clutter has the packing property if and only if it has the max-flow min-cut property.

**References**


