Standard Closure Operations on Ideals of Hypersurface Rings

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Abstract

The study of standard closure operations on the ideals of a ring is a fairly new area of abstract algebra. In one specific case, a list of 24 standard closure operations has been defined on the quotient of a multivariate power series ring $R$. We show that this ring is isomorphic to another ring by a change of basis which in turn is equivalent to the completion of an integral domain. In this domain setting, almost all of the 24 standard closures become undefined. However, if we restrict $R$ to only finite degree polynomials, we find that all of the 24 standard closures have extensions in the corresponding quotient of the polynomial ring. To get an idea of what we are dealing with, we extend one of the standard closure operations in the polynomial ring and show that it is, in fact, standard.
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# Contents

1 Preliminaries ........................................... 4  
   1.1 Basic Concepts ........................................ 4  
   1.2 Polynomial and Power Series Rings ....................... 5  

2 Isomorphism .............................................. 8  

3 Ideals in $A$ ............................................... 12  
   3.1 The Ideals $(x^n)$ and $(y^n)$ .............................. 12  
   3.2 The Ideals $(x^n, y^m)$ and $(x^n + ay^m)$ .................. 13  

4 Closure Operations ........................................ 15  
   4.1 Definition and Examples .................................... 15  
   4.2 Standard Closure Operations ............................... 16  

5 Polynomial Rings ........................................... 20  
   5.1 Standard Closures in $A'$ .................................. 20  
   5.2 Standard Closures in $R'$ .................................. 21  
   5.3 Potential Future Work ..................................... 27
Chapter 1

Preliminaries

We use this section to get the reader who is a bit rusty on abstract algebra up to speed. We give definitions and some helpful examples of the algebraic structures that will be used in this paper.

1.1 Basic Concepts

Definition [5] A commutative ring \((R, +, \cdot)\) is a set \(R\) with two binary operations \(+\) and \(\cdot\) which we associate with addition and multiplication, defined on \(R\) so that the following are satisfied.

1. The set \(R\), together with the addition operation \(+\), is an abelian group.
2. For every \(a, b,\) and \(c\) in \(R\), \((a \cdot b) \cdot c = a \cdot (b \cdot c)\). In other words, multiplication is associative.
3. For every \(a, b\) in \(R\), \(a \cdot b = b \cdot a\). In other words, multiplication is commutative.
4. For every \(a, b,\) and \(c\) in \(R\), \(a \cdot (b + c) = (a \cdot b) + (a \cdot c)\), and because \(R\) is commutative, \((a + b) \cdot c = (a \cdot c) + (b \cdot c)\). In other words, distributivity holds.

In particular, a commutative ring with unity is a ring, \(R\), with an element called the unity and denoted by \(1\), which acts a multiplicative identity for all of the elements of \(R\).

Example The most common example of a ring is the set of integers with the usual addition and multiplication. \((\mathbb{Z}, +, \cdot)\). It is well known that addition is commutative, and associativity and distributivity hold.

Definition A unit is an element \(a\) in a ring \(R\) with unity such that \(a\) has a multiplicative inverse in \(R\). In other words, the units of \(R\) are any elements \(a, b\) in \(R\) such that \(a \cdot b = 1\).

Definition An ideal, \(I\), in a commutative ring \(R\) is a subring which is closed under the addition and multiplication operations of \(R\), and which also absorbs multiplication by elements not in \(I\).

Integral Domains

Before we establish the definition of an integral domain, we must first define the notion of a zero-divisor.

Definition [5] Zero-divisors are non-zero elements \(a\) and \(b\) of a ring \(R\) such that \(a \cdot b = 0\). A ring with no zero-divisors is called an integral domain.

Example Consider the set \(\mathbb{Z}_8 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}\}\) of residue classes modulo 8 with addition and multiplication mod 8. \((\mathbb{Z}_8, +_8, \cdot_8)\) is indeed a ring with unity. Consider the elements \(\overline{3}\) and \(\overline{5}\). Both of these are units, since \(\overline{3} \cdot_8 \overline{3} = \overline{1}\) and similarly for \(\overline{5}\). However, \(\overline{2}\) and \(\overline{4}\) are zero divisors in \(\mathbb{Z}_8\) since \(\overline{2} \cdot_8 \overline{4} = \overline{8} = \overline{0}\).
Example Consider now the ring \( \mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\} \) with addition and multiplication modulo 7. Because 7 is prime, there are clearly no zero-divisors and a quick check will show us that every non-zero element is a unit in \( \mathbb{Z}_7 \). A commutative ring with unity with the property that every element is a unit is called a field.

Every field is an integral domain. However, an integral domain is not necessarily a field. The ring of integers, \((\mathbb{Z}, +, \cdot)\) is a domain, but the only units are 1 and \(-1\), so it can not be a field. Some classic examples of fields include \( \mathbb{Q}, \mathbb{R}, \mathbb{C} \), and \( \mathbb{Z}_p \) where \( p \) is prime.

\[ \begin{align*}
\text{1.2 Polynomial and Power Series Rings} \\
\text{Definition} \quad \text{The polynomial ring in } x \text{ and } y \text{ with coefficients in the field } K, \text{ denoted } K[x, y] \text{ is the ring of finite degree polynomials}
\end{align*} \]

\[ \sum_{i,j=0}^{n,m} a_{ij}x^i y^j = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \cdots + a_{nm}x^n y^m \]

with coefficients \( a_{ij} \) in \( K \), for all \( i, j \in \mathbb{N} \). Because \( K \) is commutative, \( K[x, y] \) is also commutative, and addition and multiplication are defined as the usual function addition and multiplication.

Most who have taken a course in algebra are familiar with polynomial rings over fields. \( K[x, y] \) is an integral domain, whose only units are those elements of \( K \). A somewhat less familiar concept to an undergraduate student is when the degree of these polynomials is allowed to go to infinity. The result is a formal power series ring.

\[ \begin{align*}
\text{Definition} \quad \text{The formal power series ring in } x \text{ and } y \text{ with coefficients in the field } K, \text{ denoted } K[[x, y]] \text{ is defined as the ring with elements}
\end{align*} \]

\[ \sum_{i,j=0}^{\infty} a_{ij}x^i y^j = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \cdots + a_{nm}x^n y^m + \ldots \tag{1.1} \]

with coefficients \( a_{ij} \) in \( K \), for all \( i, j \in \mathbb{N} \). Multiplication and addition are defined as they are for the polynomial ring. \( K[[x, y]] \) is commutative because \( K \) is commutative.

In calculus, the primary concern when dealing with power series is convergence. In this setting however, these series will be manipulated algebraically with no regard as to whether they converge or not. We begin by rearranging the terms of (1.1)

\[ a_{00} + (a_{10}x + a_{20}x^2 + \ldots) + (a_{01}y + a_{02}y^2 + \ldots) + (a_{11}xy + a_{21}x^2y + a_{12}xy^2 + a_{31}x^3y + \ldots) \]

Which can then be written compactly again as the sum of the three infinite series,

\[ a_{00} + \sum_{i=1}^{\infty} a_{i0}x^i + \sum_{j=1}^{\infty} a_{0j}y^j + \sum_{i,j=1}^{\infty} a_{ij}x^i y^j, \tag{1.2} \]

Proposition 1.2.1 An element in the form (1.2) of \( K[[x, y]] \) is invertible if the \( a_{00} \) term is invertible.

In other words, if the constant term of an element in \( K[[x, y]] \) is invertible, then we can find an inverse. Because \( a_{00} \) is an element of the field \( K \), we can trust that it is invertible, as long as it is not zero. To see why this is true, consider an element in \( K[[x, y]] \) of the form

\[ a - f(x, y), \]
where \( a \) is in \( K \) and \( f(x, y) \) is a power series with no constant term. Informally, to invert this element, we would be looking for another infinite polynomial that can be written as

\[
\frac{1}{a - f(x, y)}.
\]

We will expand this as a geometric series,

\[
\sum_{n=0}^{\infty} \left( \frac{1}{a} \right)^{n+1} f(x, y)^n.
\]

In \( K[[x,y]] \), this is a perfectly acceptable element.

**Example** Consider the element \( f \) of \( R \),

\[
\sqrt{y+1} = 1 + \frac{y}{2} - \frac{y^2}{8} + \frac{y^3}{16} - \frac{5y^4}{128} + \frac{7y^5}{256} - \ldots
\]

This element has a non-zero constant term, and thus must be invertible, and whose inverse is

\[
\frac{1}{\sqrt{y+1}} = \frac{1}{1 - (-\frac{y}{2} + \frac{y^2}{8} - \frac{y^3}{16} + \frac{5y^4}{128} - \frac{7y^5}{256} - \ldots)}
\]

which itself can be expanded as an infinite geometric series.

Consider an element of the form \((1.2)\) in \( K[[x,y]] \). The last sum of the element can be written as the product

\[
(xy) \left( \sum_{i,j=0}^{\infty} a_{ij}x^iy^j \right) = (xy)g(x,y).
\]

(1.3)

Now consider the principle ideal \((xy)\). All of the elements of this ideal are of the form \((1.3)\), where \(g(x,y)\) is any element in \( K[[x,y]] \). In other words, the elements of this ideal will be the elements of \( K[[x,y]] \) in which any positive power of \( xy \) can be factored out. Recall that the quotient of a ring \( R \) by an ideal \( I \) is the set of additive cosets of \( I \). If we take the quotient of \( K[[x,y]] \) mod \((xy)\), (1.3) will be absorbed into the ideal of the general coset representation of the elements in

\[
\frac{K[[x,y]]}{(xy)},
\]

so that (1.2) will become

\[
a_{00} + \sum_{i=1}^{\infty} a_{i0}x^i + \sum_{j=1}^{\infty} a_{0j}y^j + (xy).
\]

(1.5)

In general, any element of the form (1.3) will be representative of the zero element in (1.4), so when referring to elements in (1.4) (or the polynomial ring counterpart), the ideal \((xy)\) will be dropped from the notation henceforth. Since the ring (1.4) will be the focus of a large portion of this paper, we will call it \( R \). The polynomial ring in \( x \) and \( y \) mod \((xy)\) will be called \( R' \).

Now that it has been established what the elements of \( R \) look like, we will investigate the zero-divisors. Recall that zero-divisors are nonzero elements that when multiplied by another nonzero element, results in the zero element. In the case of the ring \( R \), the zero element is any element contained in the ideal \((xy)\).

Very easily it can be seen that if given the element \( x \) in \( R \), when multiplied by \( y \), we get \( xy = 0 \). So \( x \) and \( y \) are zero divisors in \( R \). This can be extended to see that any element is of the form

\[
f(x) = \sum_{i=1}^{\infty} a_ix^i \text{ or } g(y) = \sum_{j=1}^{\infty} b_jy^j,
\]

(1.6)
since \( f(x)g(y) = 0 \). Elements of the form (1.6) are the only zero-divisors in \( R \).

One of the the distinctive differences between \( R \) and \( R' \) is the form of the ideals in each. In a manner of speaking, because the units of \( R' \) are restricted to only the elements of the field \( \mathbb{K} \), there are many more ideals in \( R' \) than in \( R \). What is meant by that is that all of the ideals in \( R \) can be sorted into a small number of categories, listed below. For \( n, m \in \mathbb{N} \) and \( a \) a unit in \( R \).

(\( x^n \)), \hspace{1cm} (1.7)
(\( y^n \)), \hspace{1cm} (1.8)
(\( x^n, y^m \)), \hspace{1cm} (1.9)
(\( x^n + ay^m \)), \hspace{1cm} (1.10)

and of course, (0) = (\( xy \)) and \( R \).

**Example** Consider the ideal in \( R \) generated by the element \( 2x + x^3 \). This element can be factored into \( x(2 + x^2) \). Recall from above that \( 2 + x^2 \) is invertible in \( R \), so an ideal generated by \( 2x + x^3 \) is equivalent to the principle ideal (\( x \)), an ideal of form (1.7).

**Example** Consider the ideal \((x^2 + 5x^3 + y + 3y^5)\). Factoring, we attain \((x^2(1 + 5x) + y(1 + 3y^4))\). \(1 + 5x \) and \(1 + 3y^4 \) are units in \( R \), so multiplying the generators by their inverses will not change the ideal. We get

(\(1 + 5x\))\(^{-1}\)(\(1 + 3y^4\))\(^{-1}\)(\(x^2(1 + 5x) + y(1 + 3y^4)\)) = (\(x^2(1 + 3y^4)^{-1} + y(1 + 5x)^{-1}\))

= \(x^2(1 - 3y^4 + 9y^8 - \ldots) + y(1 - 5x + 25x^2 - \ldots)\)

= \(x^2 + y\).

so the ideal \((x^2 + 5x^3 + y + 3y^5)\) is equivalent to the more simply generated \((x^2 + y)\).
Chapter 2

Isomorphism

Before showing the isomorphism between two rings, we state a fundamental Theorem in abstract algebra that plays a key role in the proof below.

**Theorem 2.0.1** [5] (First Isomorphism Theorem for Rings) Let $\phi : R \rightarrow A$ be a ring homomorphism with kernel $N$. Then $\phi[R]$ is a ring, and the map $\mu : R/N \rightarrow \phi[R]$ given by $\mu(x+N) = \phi(x)$ is an isomorphism. If $\varphi : R \rightarrow R/N$ is the canonical homomorphism given by $\varphi(x) = x+N$, then for each $x$ in $R$, $\phi(x) = \mu \circ \varphi(x)$.

**Lemma 2.0.2** There exists an isomorphism between the two formal power series rings $\mathbb{K}[[x,y]]/(xy)$ and $\mathbb{K}[[x+y\sqrt{y+1}, x-y\sqrt{y+1}]/(x^2-(y^2+y^3))].$

**Proof** Let $R = \mathbb{K}[[x,y]]/(xy)$ and $A = \mathbb{K}[[x+y\sqrt{y+1}, x-y\sqrt{y+1}]/(x^2-(y^2+y^3))]$. We want to show

$$\frac{\mathbb{K}[[x,y]]}{(xy)} \cong \frac{\mathbb{K}[[x+y\sqrt{y+1}, x-y\sqrt{y+1}]]}{(x^2-(y^2+y^3))}.$$ 

As discussed in the previous section, the elements in $\mathbb{K}[[x,y]]$ look like

$$f(x,y) = \sum_{i,j=0}^{\infty} a_{ij}x^i y^j = a_{00} + \sum_{i=1}^{\infty} a_{i0}x^i + \sum_{j=1}^{\infty} a_{0j}y^j + \sum_{i,j=1}^{\infty} a_{ij}x^i y^j,$$

and

$$\sum_{i,j=1}^{\infty} a_{ij}x^i y^j = (xy)h(x,y) \in (xy).$$

So elements in $R$ are of the form

$$f(x,y) = a_{00} + \sum_{i=1}^{\infty} a_{i0}x^i + \sum_{j=1}^{\infty} a_{0j}y^j + (xy).$$

To prove that $R$ and $A$ are isomorphic, the plan is to construct a homomorphism $\phi = \mu \circ \varphi : \mathbb{K}[[x,y]] \rightarrow A$, where $\mu : \mathbb{K}[[x,y]]/(xy) \rightarrow \mathbb{K}[[x,y]]/(xy)$ is the canonical homomorphism and $\varphi : \mathbb{K}[[x,y]]/(xy) \rightarrow \mathbb{K}[[x+y\sqrt{y+1}, x-y\sqrt{y+1}]/(x^2-(y^2+y^3))]$ is the homomorphism given by $\varphi(x) = x+N$. The details of this construction are beyond the scope of this section and will be explored in a later chapter.
Since the elements of $\phi$ is surjective will support that $f$ is a surjection. Then $f(x, y)$ in $\mathbb{K}[[x, y]]$ define

$$\varphi(f(x, y)) = f(x + y\sqrt{y + 1}, x - y\sqrt{y + 1}),$$

and

$$\mu : \mathbb{K}[[x + y\sqrt{y + 1}, x - y\sqrt{y + 1}] \to A,$$

where $\mu$ is defined so that $K$ and $\phi$ is a surjection. Notice that because $\mu$ is a natural projection, it is known that $\mu$ is a surjective homomorphism. So we really only need to be concerned with showing that $\varphi$ is also a homomorphism. Once this has been accomplished, by the First Isomorphism Theorem it can be seen that $R \simeq \phi(R)$ where

$$\psi : R \to \phi(\mathbb{K}[[x, y]]) \quad \psi(f + (x)y) = \phi(f).$$

If $\varphi$ is a surjection, then

$$\phi(\mathbb{K}[[x, y]]) = A.$$

Let $f(x, y), g(x, y)$ be elements of $\mathbb{K}[[x, y]]$. Then we have

$$\varphi(f(x, y) + g(x, y)) = \varphi((f + g)(x, y)) = (f + g)(x + y\sqrt{y + 1}, x - y\sqrt{y + 1}) = f(x + y\sqrt{y + 1}, x - y\sqrt{y + 1}) + g(x + y\sqrt{y + 1}, x - y\sqrt{y + 1}) = \varphi(f(x, y)) + \varphi(g(x, y)).$$

So $\varphi$ preserves addition. Similarly,

$$\varphi(f(x, y)g(x, y)) = \varphi((fg)(x, y)) = (fg)(x + y\sqrt{y + 1}, x - y\sqrt{y + 1}) = f(x + y\sqrt{y + 1}, x - y\sqrt{y + 1})g(x + y\sqrt{y + 1}, x - y\sqrt{y + 1}) = \varphi(f(x, y))\varphi(g(x, y)).$$

So $\varphi$ is definitely a homomorphism. It is not necessary to show that $\varphi$ is injective. However, showing that $\varphi$ is surjective will support that $\phi$ is indeed an isomorphism between $R$ and $S$.

Since the elements of $\mathbb{K}[[x + y\sqrt{y + 1}, x - y\sqrt{y + 1}]$ are of the form

$$f(x + y\sqrt{y + 1}, x - y\sqrt{y + 1})$$

and $\varphi$ is defined so that

$$\varphi(x) = x + y\sqrt{y + 1} \text{ and } \varphi(y) = x - y\sqrt{y + 1}.$$

All elements $f(x + y\sqrt{y + 1}, x - y\sqrt{y + 1})$ in $\mathbb{K}[[x + y\sqrt{y + 1}, x - y\sqrt{y + 1}]$ are mapped to by $f(x, y)$ in $\mathbb{K}[[x, y]]$ so that $\phi$ is a surjection.

Reiterating that $\mu$ is the natural projection,

$$\mu : \mathbb{K}[[x + y\sqrt{y + 1}, x - y\sqrt{y + 1}] \to \mathbb{K}[[x + y\sqrt{y + 1}, x - y\sqrt{y + 1}] / (x^2 - (y^2 + y^3))$$
mapping \( \mathbb{K}[x+y\sqrt{y+1}, x-y\sqrt{y+1}] \) into its quotient ring with the ideal generated by \((x^2 - y^2(y+1))\). This is a known surjective homomorphism. Furthermore,

\[
\ker(\mu) = (x^2 - y^2(y+1)).
\]

Since the zero element in \( S \) is the ideal \((x^2 - y^2(y+1))\), to acquire \( \ker(\phi) \), we need to look at \( \varphi^{-1}((x^2 - y^2(y+1)) \). Since \((x^2 - y^2(y+1)) = ((x + y\sqrt{y+1})(x - y\sqrt{y+1}))\), we know that all of the elements in the ideal are of the form

\[
((x + y\sqrt{y+1})(x - y\sqrt{y+1}))f(x + y\sqrt{y+1}, x - y\sqrt{y+1}).
\]

It is clear from the definition of \( \varphi \) that the elements of \( \mathbb{K}[x,y] \) that map to these elements must be the elements

\[
(xy)f(x,y),
\]

Which are precisely the elements of the ideal \((xy)\) in the ring \( \mathbb{K}[x,y] \). So the kernel of \( \phi \) is the ideal \((xy)\).

Now we can conclude, by the first isomorphism theorem that there is an isomorphism \( \psi \) between \( R \) and \( \phi(\mathbb{K}[x,y]) \) where

\[
\psi(f(x,y) + (xy)) = \phi(f(x,y))
\]

and since \( \phi \) is a surjection, \( \varphi(\mathbb{K}[x,y]) = A. \)

Now that the isomorphism has been established, the behaviors of the ring \( R \) apply to the ring \( A \). Lemma 2.0.2 can be extended farther to include the following equality.

**Proposition 2.0.3** The rings,

\[
\frac{\mathbb{K}[x+y\sqrt{y+1}, x-y\sqrt{y+1}]}{(x^2 - (y^2 + y^3))} \quad \text{and} \quad \frac{\mathbb{K}[x,y]}{(x^2 - (y^2 + y^3))}
\]

are equivalent.

**Proof** It will be shown in the usual way that these two sets are equal. First we will show

\[
\mathbb{K}[x,y] \subseteq \mathbb{K}[x+y\sqrt{y+1}, x-y\sqrt{y+1}].
\]

Observe that the element \( x \) can easily be expressed as a power series of \( x + y\sqrt{y+1} \) and \( x - y\sqrt{y+1} \):

\[
x = \frac{1}{2}(x + y\sqrt{y+1}) + \frac{1}{2}(x - y\sqrt{y+1}).
\]

Similarly \( y \) can be expressed as a power series:

\[
y = (\sqrt{y+1})^{-1}\left(\frac{1}{2}(x + y\sqrt{y+1}) + \frac{1}{2}(x - y\sqrt{y+1})\right).
\]

where \((\sqrt{y+1})^{-1}\) is also a power series, as was shown earlier.

Because \( x \) and \( y \) are elements of \( \mathbb{K}[x+y\sqrt{y+1}, x-y\sqrt{y+1}] \), any power series in \( x \) and \( y \), (precisely the elements of \( \mathbb{K}[x,y] \))

\[
\sum_{i,j=0}^{\infty} a_{ij}x^i y^j
\]

10
will also be an element of \( \mathbb{K}[x + y\sqrt{y + 1}, x - y\sqrt{y + 1}] \). So because every element of \( \mathbb{K}[x, y] \) is an element of \( \mathbb{K}[x + y\sqrt{y + 1}, x - y\sqrt{y + 1}] \), we have that
\[
\mathbb{K}[x, y] \subseteq \mathbb{K}[x + y\sqrt{y + 1}, x - y\sqrt{y + 1}].
\]

Now we will go in the other direction. The elements \( x + y\sqrt{y + 1} \) and \( x - y\sqrt{y + 1} \) can be expressed as power series in \( x \) and \( y \):
\[
\begin{align*}
x + y\sqrt{y + 1} &= x + y \left(1 + \frac{y^2}{2} - \frac{y^3}{8} + \frac{y^3}{16} - \ldots\right) \\
x - y\sqrt{y + 1} &= x - y \left(1 + \frac{y^2}{2} - \frac{y^3}{8} + \frac{y^3}{16} - \ldots\right)
\end{align*}
\]
Both of these are elements, (which make up the elements of \( \mathbb{K}[x + y\sqrt{y + 1}, x - y\sqrt{y + 1}] \)), of the ring \( \mathbb{K}[x, y] \), so any combination of these, or in other words, any element of \( \mathbb{K}[x + y\sqrt{y + 1}, x - y\sqrt{y + 1}] \) must also be an element of \( \mathbb{K}[x, y] \), so we have
\[
\mathbb{K}[x + y\sqrt{y + 1}, x - y\sqrt{y + 1}] \subseteq \mathbb{K}[x, y]
\]
And we have shown that
\[
\mathbb{K}[x + y\sqrt{y + 1}, x - y\sqrt{y + 1}] = \mathbb{K}[x, y].
\]
Taking the quotient of both rings by the element \( x^2 - (y^2 + y^3) \) will preserve the equality.
\[\square\]

We conclude this section by stating the isomorphism,
\[
\mathbb{K}[x, y] \cong \frac{\mathbb{K}[x, y]}{(xy)} \cong \frac{\mathbb{K}[x, y]}{(x^2 - (y^2 + y^3))}.
\]
Henceforth, the ring (2.1) will be referred to as \( A \). We are primarily concerned with this isomorphism when considering the corresponding polynomial rings. Most of the work in \( A \), however, will be expressed in terms of the generators \( x + y\sqrt{y + 1} \) and \( x - y\sqrt{y + 1} \). Because \( R \) and \( A \) are isomorphic, we expect that the elements \( A \) will behave in exactly the same manner as those in \( R \). For this reason, we will not go over what the units and zero-divisors of \( A \) look like.
Chapter 3

Ideals in $A$

It can be seen that from the ring $R = \mathbb{K}[x, y]/(xy)$ to $A = \mathbb{K}[[x + y\sqrt{y + 1}, x - y\sqrt{y + 1}]/(x^2 - (y^2 + y^3))$, $x \mapsto x + y\sqrt{y + 1}$ and $y \mapsto x - y\sqrt{y + 1}$. Using this mapping, we can develop some intuition for behavior of the ideals in $A$. Since an isomorphism exists, $A$ must have ideals equivalent to those in $R$. Unfortunately most of these ideals are unattractive and tedious to work with. Some, however can be written more compactly. Below we will focus on these special cases of ideals in $A$ and compare them to their analogues in $R$. The reason we single out the following ideals in $A$ is so that when we move to the polynomial ring $\mathbb{K}[x, y]/(x^2 - (y^2 + y^3))$, these ideals (and linear combinations of them) remain expressible in finite terms. It will become apparent that there is no nice way to express pure powers of the generators of $A$, in particular, as a finite sum. As a consequence, in the polynomial ring, there will be no analogue in $A$ to those ideals generated by pure powers of $x$ and $y$ in $R$.

3.1 The Ideals $(x^n)$ and $(y^n)$

In $A$ it has been shown that the element $x$ can be written as $\frac{1}{2}(x + y\sqrt{y + 1}) + \frac{1}{2}(x - y\sqrt{y + 1})$ which is of the form $((x + y\sqrt{y + 1})^n + a(x - y\sqrt{y + 1})^m)$. So $(x)$ is an ideal in both $R$ and $A$, but in $A$, $(x)$ behaves more like an ideal of the form $(x^n + ay^m)$ in $R$.

If in $A$, $x$ can be expressed as the linear combination $\frac{1}{2}(x + y\sqrt{y + 1}) + \frac{1}{2}(x - y\sqrt{y + 1})$, then it clearly follows that

$$x^2 = \left(\frac{1}{2}(x + y\sqrt{y + 1}) + \frac{1}{2}(x - y\sqrt{y + 1})\right)^2$$

Carrying out the computations,

$$x^2 = \left(\frac{1}{2}(x + y\sqrt{y + 1}) + \frac{1}{2}(x - y\sqrt{y + 1})\right)^2$$
$$= \left(\frac{1}{2}(x + y\sqrt{y + 1}) + \frac{1}{2}(x - y\sqrt{y + 1})\right)\left(\frac{1}{2}(x + y\sqrt{y + 1}) + \frac{1}{2}(x - y\sqrt{y + 1})\right)$$
$$= \frac{1}{4}(x + y\sqrt{y + 1})^2 + \frac{1}{2}(x + y\sqrt{y + 1})(x - y\sqrt{y + 1}) + \frac{1}{4}(x - y\sqrt{y + 1})^2$$
$$= \frac{1}{4}(x + y\sqrt{y + 1})^2 + \frac{1}{4}(x - y\sqrt{y + 1})^2$$
Similarly it can be found that $x^3 = \left(\frac{1}{2}(x + y\sqrt{y+1})\right)^3 + \left(\frac{1}{2}(x - y\sqrt{y+1})\right)^3$ and so on so that any power of $x$ in $A$ can be expressed as the linear combination

$$x^n = \frac{1}{2^n}(x + y\sqrt{y+1})^n + \frac{1}{2^n}(x - y\sqrt{y+1})^n \quad (3.1)$$

The case with ideals generated by powers of $y$ in $A$ is essentially identical to the $x$ case.

$$y^2 = \left(\frac{x + y\sqrt{y+1}}{2\sqrt{y+1}} - \frac{x - y\sqrt{y+1}}{2\sqrt{y+1}}\right)^2$$

$$= \left(\frac{x + y\sqrt{y+1}}{2\sqrt{y+1}} - \frac{x - y\sqrt{y+1}}{2\sqrt{y+1}}\right)\left(\frac{x + y\sqrt{y+1}}{2\sqrt{y+1}} - \frac{x - y\sqrt{y+1}}{2\sqrt{y+1}}\right)$$

$$= \left(\frac{x + y\sqrt{y+1}}{2\sqrt{y+1}}\right)^2 + \left(\frac{x - y\sqrt{y+1}}{2\sqrt{y+1}}\right)^2$$

Multiplying again by $y$,

$$y^3 = \left(\frac{x + y\sqrt{y+1}}{2\sqrt{y+1}}\right)^3 + \left(\frac{x - y\sqrt{y+1}}{2\sqrt{y+1}}\right)^3$$

$$= \left(\frac{x + y\sqrt{y+1}}{2\sqrt{y+1}}\right)^3 - \left(\frac{x - y\sqrt{y+1}}{2\sqrt{y+1}}\right)^3$$

It can therefore be seen that powers of $y$ can be expressed as

$$y^n = \left(\frac{1}{2\sqrt{y+1}}\right)^n(x + y\sqrt{y+1})^n + \left(\frac{-1}{2\sqrt{y+1}}\right)^n(x - y\sqrt{y+1})^n \quad (3.2)$$

From the calculations above, it is seen that pure powers of $x$ and $y$ are represented as linear combinations of the generators of the ring $A$. Looking back at the adapted list of standard closure operations, the ideals $(x^n)$ and $(y^n)$ in $A$ must behave in the same way the ideals of the form $(x^n + ay^m)$ in $R$ behave.

There unfortunately is not an aesthetically pleasing way to represent the pure powers of the generators of $A$. Therefore, once reverting to the polynomial ring corresponding to $A$, it will be seen that there are no ideals that will behave in the same way that the ideals of the pure powers of $x$ and $y$ behave in $R$.

### 3.2 The Ideals $(x^n, y^m)$ and $(x^n + ay^m)$

Another concern is the form of ideals in $A$ that correspond to the ideals in $R$ of the form $(x^n, y^m)$. Since $x^n$ looks like $x^n + ay^n$ and $y^m$ looks like $x^m + ay^m$, an ideal of the form $(x^n, y^m)$ in $R$ will look like $(x^n + ay^n, x^m + by^m)$ in $A$. In our list of the forms of principle ideals, this particular ideal is not included. In the ring $A$, the ideal $(x^n, y^m)$ can be expressed as

$$(v^n + aw^n, v^m + bw^m) \quad (3.3)$$

where

$$v = x + y\sqrt{y+1} \quad \text{and} \quad w = x - y\sqrt{y+1}$$

Looking back at the list of ideals in the ring $R$, there is no ideal that corresponds to (3). However it can be shown that in the case that $m = n,$
(v^n + aw^n, v^m + bw^m) = (v^n, w^m)
\quad = ((x + y\sqrt{y + 1})^n, (x - y\sqrt{y + 1})^m)
\quad = ((x + y\sqrt{y + 1})^n, (x - y\sqrt{y + 1})^m)

So, in this case, the ideal \((x^n, y^m)\) in \(A\) behaves again like \((x^n, y^m)\) in \(R\).

**Proposition 3.2.1** In the ring \(R = \frac{K[[x,y]]}{(xy)}\), the ideal \((x^n + ay^n, x^m + by^m)\) is equivalent to the ideal \((x^n, y^n)\) for \(m = n\).

**Proof** Showing that the ideal \((x^n + ay^n, x^m + by^m)\), where \(a\) and \(b\) are units in \(K^\times\), is contained in \((x^n, y^n)\) is trivial. To show that the latter is contained in the former, first note that \(b - a\) is invertible. Then \(b(b - a)^{-1}(x^n + ay^n)\) and \(a(b - a)^{-1}(x^n + by^n)\) are in the first ideal as well as
\[
b(b - a)^{-1}(x^n + ay^n) - a(b - a)^{-1}(x^n + by^n) = x^n.
\]

Similarly, \((a - b)^{-1}(x^n + ay^n)\) and \((a - b)^{-1}(x^n + by^n)\) are both in the first ideal, as well as
\[
(a - b)^{-1}(x^n + ay^n) - (a - b)^{-1}(x^n + by^n) = y^n.
\]

Since the generators of \((x^n, y^n)\) are in \((x^n + ay^n, x^m + by^m)\), then it must be the case that \((x^n, y^n) \subset (x^n + ay^n, x^m + by^m)\) and hence that \((x^n, y^n) = (x^n + ay^n, x^m + by^m)\).

\(\Box\)

**Proposition 3.2.2** In the ring \(\frac{K[[x,y]]}{(xy)}\), the ideal \((x^n + ay^n, x^m + by^m)\) is equivalent to the ideal \((x^n + ay^n)\) for \(m > n\).

**Proof** Obviously every element in the ideal \((x^n + ay^n)\) must be in \((x^n + ay^n, x^m + by^m)\).

Since \(m > n\), it can be seen that
\[
(x^m + by^m) = (x^{m-n} + \frac{b}{a}y^{m-n})(x^n + ay^n)
\]

So any element of the form \(f(x, y)(x^m + by^m)\) is also of the form \(f(x, y)(x^{m-n} + \frac{b}{a}y^{m-n})(x^n + ay^n)\) which is in the ideal \((x^n + ay^n)\).

\(\Box\)

So it should be observed that for \(m \neq n\), the ideal \((x^n, y^m)\) behaves in \(A\) the same way the ideal \((x^n + ay^m)\) behaves in \(R\). Because \(R \simeq A\), the same applies to ideals of the same form in \(A\).

**Example** Naturally, we are interested in how the behavior of the ideal \((x^n, y^m)\) in \(A\) compares to \((x^n, y^m)\) in \(R\). In \(A\), let \(I = (x, y^2)\). From the calculations above, we see that this ideal can be written as
\[
I = \left(\frac{1}{2}(x + y\sqrt{y + 1}) + \frac{1}{2}(x - y\sqrt{y + 1}), \left(\frac{x + y\sqrt{y + 1}}{2\sqrt{y + 1}}\right)^2 + \left(\frac{x - y\sqrt{y + 1}}{2\sqrt{y + 1}}\right)^2\right).
\]

Letting \(v = x + y\sqrt{y + 1}, w = x - y\sqrt{y + 1}\), and multiplying the first generator of \(I\) by the unit \(b = 2\), and the second generator by the unit \(f = 4(y + 1)^{-1}\),
\[
I = \left(2\left(\frac{1}{2}v + \frac{1}{2}w\right), 4(y + 1)^{-1}\left(\frac{v}{2\sqrt{y + 1}}\right)^2 + \left(\frac{w}{2\sqrt{y + 1}}\right)^2\right) = (v + w, v^2 + w^2).
\]

By Proposition 4.2.2, we see that we can write this ideal as
\[
I = (v + w).
\]

14
Chapter 4

Closure Operations

4.1 Definition and Examples

For the time being, let $R$ denote an arbitrary commutative ring.

**Definition** [3] A closure operation on a set of ideals $I$ of a commutative ring is a mapping $c : I \rightarrow I$ satisfying all of the following three properties:

1. (Extension) $I \subseteq I^c$ for all $I$ in $I(R)$.
2. (Order Preservation) If $I \subseteq J$, then $I^c \subseteq J^c$ for $I, J$ in $I(R)$.
3. (Idempotence) $(I^c)^c = I^c$ for all $I$ in $I(R)$.

The set of all ideals in a ring will be denoted $I(R)$.

**Example** Consider the finite abelian ring $\mathbb{Z}_8$. The ideals are $\{0\}$, $\mathbb{Z}_8$, $(2) = \{0, 2, 4, 6\}$, and $(4) = \{0, 4\}$. These ideals form the chain $\{0\} \subset (2) \subset \mathbb{Z}_8$. Referring to the properties of closure operations, it can be deduced that if $c : I(\mathbb{Z}_8) \rightarrow I(\mathbb{Z}_8)$ is a closure operation on the set $I(\mathbb{Z}_8) = \{\{0\}, (2), \mathbb{Z}_8\}$, then $\mathbb{Z}_8^c = \mathbb{Z}_8$, $(2)^c = (2)$ or $(2)^c = \mathbb{Z}_8$. Depending on the choice of $(2)^c$, $(4)^c = (4)$, $(4)^c = (2)$, or $(4)^c = \mathbb{Z}_8$. Similarly, depending on the choice of $(4)$, $\{0\}$ could be mapped to any of the 4 ideals. Using this information, we can map out all of the possible closure operations.

Notice that there are $8 = 2^3$ distinct closure operations that can be defined on $\mathbb{Z}_8$. In fact, if $R$ is a commutative ring with $n < \infty$ non-zero ideals, then $2^n$ is an upper bound on the number of closure operations on $I(R)$. If all of the ideals form a chain, as they do in this example, then there are exactly $2^n$ possible closure operations. [7]
**Example** A trivial example of a closure operation in any ring is the identity closure, \( id \), which sends each ideal to itself, i.e. \( I^{id} = I \). It is easy to see that the identity closure satisfies the above properties of a closure operation. An ideal is called \( c \)-closed if the closure of \( I \) is itself. In the case of the identity closure, every ideal is \( id \)-closed.

**Example** Let \( R = \mathbb{Z} \). The radical closure, denoted \( \sqrt{I} \), is a closure operation on \( \mathcal{I}(R) \), defined by

\[
\sqrt{I} := \{ r \in R : \text{there is an } n \in \mathbb{N} \text{ such that } r^n \in I \}.
\]

All of the ideals of \( \mathbb{Z} \) are of the form

\[
m\mathbb{Z} = \{ \ldots, -2m, -m, 0, m, 2m, 3m, \ldots \}.
\]

Consider the ideal \( 4\mathbb{Z} = \{ \ldots, -8, -4, 0, 4, 8, 12, 16, \ldots \} \). Clearly all of the elements of \( 4\mathbb{Z} \) will be in \( \sqrt{4\mathbb{Z}} \) (for \( n = 1 \)). Furthermore, \( 2, 6, 10, \ldots \) will be in \( \sqrt{4\mathbb{Z}} \) since \( 2^2 = 4 \), \( 6^2 = 36 \), and \( 10^2 = 100 \), and so on, are all in \( 4\mathbb{Z} \). Every even integer is in \( \sqrt{4\mathbb{Z}} \). So the radical of the ideal generated by \( 4 \), is \( 2\mathbb{Z} \). Next, consider the ideal \( 12\mathbb{Z} \). It can easily be checked that the radical is \( 6\mathbb{Z} \). If we were to proceed, it would be seen that for every ideal \( m\mathbb{Z} \), the radical will be the ideal generated by the product of the distinct prime factors of \( m \), a property which certainly ensures the satisfaction of the 3 properties of closure operations. In general, this property actually holds in an arbitrary commutative ring, so we can define the radical as a closure operation for any such ring.

### 4.2 Standard Closure Operations

**Definition** A closure \( * : \mathcal{I}(R) \to \mathcal{I}(R) \) is **standard** if

\[
((xI)^* : x) = I^*
\]

for any regular element \( x \) in \( R \) where the **ideal quotient** \( (I : J) \) is defined to be the set

\[
(I : J) = \{ x \in R | xJ \subseteq I \}.
\]

\( (I : J) \) is again an ideal in \( R \).

G. Morre and J. Vassilev showed that for a commutative ring with \( n < \infty \) ideals, the maximum number of closure operations is no more than \( 2^n \). [7] In addition, they classified a set of 24 standard closures on the ring \( R \) as in (2.4). For natural numbers \( n \) and \( m \) and \( a \) in \( \mathbb{K} - \{0\} \),

1. \( *_1 \): The identity \( I = I \) for all \( I \) in \( \mathcal{I}(R) \).
2. \( *_2 \): \( I^{*_2} = I \) for all \( I \neq (x^n + ay^m) \) and \( (x^n + ay^m)^{*_2} = (x^n, y^m) \).
3. \( *_3 \): \( (0)^{*_3} = (0) \); \( (x^n)^{*_3} = (x^n) \); \( (y^n)^{*_3} = (y^n) \); \( (x^n, y^m)^{*_3} = (x^n + ay^m)^{*_3} = (x, y^m) \) and \( R^{*_3} = R \).
4. \( *_4 \): \( (0)^{*_4} = (0) \); \( (x^n)^{*_4} = (x) \); \( (y^n)^{*_4} = (y^n) \); \( (x^n, y^m)^{*_4} = (x^n + ay^m)^{*_4} = (x, y^m) \) and \( R^{*_4} = R \).
5. \( *_5 \): \( (0)^{*_5} = (0) \); \( (x^n)^{*_5} = (x^n) \); \( (y^n)^{*_5} = (x^n, y^m)^{*_5} = (x^n + ay^m)^{*_5} = (x, y^m) \) and \( R^{*_5} = R \).
6. \( *_6 \): \( (0)^{*_6} = (0) \); \( (x^n)^{*_6} = (x) \); \( (y^n)^{*_6} = (x^n, y^m)^{*_6} = (x^n + ay^m)^{*_6} = (x, y^m) \) and \( R^{*_6} = R \).
7. \( *_7 \): \( (0)^{*_7} = (x^n)^{*_7} = (x) \); \( (y^n)^{*_7} = (x^n, y^m)^{*_7} = (x^n + ay^m)^{*_7} = (x, y^m) \) and \( R^{*_7} = R \).
8. \( *_8 \): \( (0)^{*_8} = (0) \); \( (y^n)^{*_8} = (y^n) \); \( (x^n)^{*_8} = (x^n) \); \( (x^n, y^m)^{*_8} = (x^n + ay^m)^{*_8} = (x^m, y) \) and \( R^{*_8} = R \).
9. \( *_9 \): \( (0)^{*_9} = (0) \); \( (y^n)^{*_9} = (y) \); \( (x^n)^{*_9} = (x^n) \); \( (x^m, y^n)^{*_9} = (x^m, y^n)^{*_9} = (x^m, y) \) and \( R^{*_9} = R \).
10. $*_{10} : (0)^{*10} = (0); (y^n)^{*10} = (y^n); (x^m)^{*10} = (x^m, y^n)^{*10} = (x^m + ay^n)^{*10} = (x^m, y)$ and $R^{*10} = R$.

11. $*_{11} : (0)^{*11} = (0); (y^n)^{*11} = (y); (x^m)^{*11} = (x^m, y^n)^{*11} = (x^m + ay^n)^{*11} = (x^m, y)$ and $R^{*11} = R$.

12. $*_{12} : (0)^{*12} = (y^n)^{*12} = (y); (x^m)^{*12} = (x^m, y^n)^{*12} = (x^m + ay^n)^{*12} = (x^m, y)$ and $R^{*12} = R$.

13. $*_{13} : (0)^{*13} = (0); (x^n)^{*13} = (x^n); (y^n)^{*13} = (y^n)$ and $I^{*13} = R$ for all other $I$.

14. $*_{14} : (0)^{*14} = (0); (x^n)^{*14} = (x); (y^n)^{*14} = (y^n)$ and $I^{*14} = R$ for all other $I$.

15. $*_{15} : (0)^{*15} = (0); (x^n)^{*15} = (x^n); (y^n)^{*15} = (y)$ and $I^{*15} = R$ for all other $I$.

16. $*_{16} : (0)^{*16} = (0); (x^n)^{*16} = (x); (y^n)^{*16} = (y)$ and $I^{*16} = R$ for all other $I$.

17. $*_{17} : (0)^{*17} = (0); (x^n)^{*17} = (x^n)$ and $I^{*17} = R$ for all other $I$.

18. $*_{18} : (0)^{*18} = (0); (x^n)^{*18} = (x)$ and $I^{*18} = R$ for all other $I$.

19. $*_{19} : (0)^{*19} = (x^n)^{*19} = (x)$ and $I^{*19} = R$ for all other $I$.

20. $*_{20} : (0)^{*20} = (0); (y^n)^{*20} = (y^n)$ and $I^{*20} = R$ for all other $I$.

21. $*_{21} : (0)^{*21} = (0); (y^n)^{*21} = (y)$ and $I^{*21} = R$ for all other $I$.

22. $*_{22} : (0)^{*22} = (y^n)^{*22} = (y)$ and $I^{*22} = R$ for all other $I$.

23. $*_{23} : (0)^{*23} = (0)$ and $I^{*23} = R$ for all other $I$.

24. $*_{24} : I^{*24} = R$ for all $I$.

we can find analogues of the classified standard closures listed above in $A$. In this new ring, we have ideals that are of the form

- $\{0\}$
- $R$
- $((x + y\sqrt{y+1})^n)$
- $((x - y\sqrt{y+1})^n)$
- $((x + y\sqrt{y+1})^n, (x - y\sqrt{y+1})^n)$
- $((x + y\sqrt{y+1})^n + a(x - y\sqrt{y+1})^m)$

$*: I(A) \to I(A)$.

Then for natural numbers $n$ and $m$ and $a$ is a unit in $\mathbb{K} - \{0\}$, the following is the set of 24 standard closures that are classified on the ring $A$.

1. $*_{1} :$ The identity $I = I$ for all $I$ in $I(A)$.

2. $*_{2} : I^{*2} = I$ for all $I \neq ((x + y\sqrt{y+1})^n + a(x - y\sqrt{y+1})^m)$ and $((x + y\sqrt{y+1})^n + a(x - y\sqrt{y+1})^m)^{*2} = ((x + y\sqrt{y+1})^n, (x - y\sqrt{y+1})^m)$.

3. $*_{3} : (0)^{*3} = (0); ((x + y\sqrt{y+1})^n)^{*3} = ((x + y\sqrt{y+1})^n); ((x - y\sqrt{y+1})^n)^{*3} = ((x - y\sqrt{y+1})^n); ((x + y\sqrt{y+1})^n, (x - y\sqrt{y+1})^m)^{*3} = ((x + y\sqrt{y+1})^n + a(x - y\sqrt{y+1})^m)^{*3} = (x + y\sqrt{y+1}, (x - y\sqrt{y+1})^m)$ and $A^{*3} = A$. 

17
4. \(*_4 : (0)^{*4} = (0)\); \((x + y\sqrt{y+1})^n\)^{*4} = (x + y\sqrt{y+1}); \((x - y\sqrt{y+1})^n\)^{*4} = (x - y\sqrt{y+1});
\((x + y\sqrt{y+1})^n, (x - y\sqrt{y+1})^n\)^{*4} = ((x + y\sqrt{y+1})^n + a(x - y\sqrt{y+1})^n)^{*4} = (x + y\sqrt{y+1}, (x - y\sqrt{y+1})^n) and \(A^{*4} = A\).

5. \(*_5 : (0)^{*5} = (0)\); \((x + y\sqrt{y+1})^n\)^{*5} = (x + y\sqrt{y+1});
\((x - y\sqrt{y+1})^n\)^{*5} = (x - y\sqrt{y+1}); \((x + y\sqrt{y+1})^n, (x - y\sqrt{y+1})^n\)^{*5} = ((x + y\sqrt{y+1})^n + a(x - y\sqrt{y+1})^n)^{*5} = (x + y\sqrt{y+1}, (x - y\sqrt{y+1})^n) and \(A^{*5} = A\).

6. \(*_6 : (0)^{*6} = (0)\); \((x + y\sqrt{y+1})^n\)^{*6} = (x + y\sqrt{y+1});
\((x - y\sqrt{y+1})^n\)^{*6} = (x - y\sqrt{y+1}); \((x + y\sqrt{y+1})^n, (x - y\sqrt{y+1})^n\)^{*6} = ((x + y\sqrt{y+1})^n + a(x - y\sqrt{y+1})^n)^{*6} = (x + y\sqrt{y+1}, (x - y\sqrt{y+1})^n) and \(A^{*6} = A\).

7. \(*_7 : (0)^{*7} = (0)\); \((x + y\sqrt{y+1})^n\)^{*7} = (x + y\sqrt{y+1});
\((x - y\sqrt{y+1})^n\)^{*7} = (x - y\sqrt{y+1}); \((x + y\sqrt{y+1})^n, (x - y\sqrt{y+1})^n\)^{*7} = ((x + y\sqrt{y+1})^n + a(x - y\sqrt{y+1})^n)^{*7} = (x + y\sqrt{y+1}, (x - y\sqrt{y+1})^n) and \(A^{*7} = A\).

8. \(*_8 : (0)^{*8} = (0)\); \((x + y\sqrt{y+1})^n\)^{*8} = (x + y\sqrt{y+1});
\((x - y\sqrt{y+1})^n\)^{*8} = (x - y\sqrt{y+1}); \((x + y\sqrt{y+1})^n, (x - y\sqrt{y+1})^n\)^{*8} = ((x + y\sqrt{y+1})^m + a(x - y\sqrt{y+1})^n)^{*8} = (x + y\sqrt{y+1}, (x - y\sqrt{y+1})^m) and \(A^{*8} = A\).

9. \(*_9 : (0)^{*9} = (0)\); \((x + y\sqrt{y+1})^n\)^{*9} = (x + y\sqrt{y+1});
\((x - y\sqrt{y+1})^n\)^{*9} = (x - y\sqrt{y+1}); \((x + y\sqrt{y+1})^n, (x - y\sqrt{y+1})^n\)^{*9} = ((x + y\sqrt{y+1})^m + a(x - y\sqrt{y+1})^n)^{*9} = (x + y\sqrt{y+1}, (x - y\sqrt{y+1})^m) and \(A^{*9} = A\).

10. \(*_{10} : (0)^{*10} = (0)\); \((x + y\sqrt{y+1})^n\)^{*10} = (x + y\sqrt{y+1});
\((x + y\sqrt{y+1})^m\)^{*10} = (x + y\sqrt{y+1}); \((x + y\sqrt{y+1})^m, (x - y\sqrt{y+1})^n\)^{*10} = ((x + y\sqrt{y+1})^m + a(x - y\sqrt{y+1})^n)^{*10} = (x + y\sqrt{y+1}, (x - y\sqrt{y+1})^m) and \(A^{*10} = A\).

11. \(*_{11} : (0)^{*11} = (0)\); \((x + y\sqrt{y+1})^n\)^{*11} = (x - y\sqrt{y+1});
\((x + y\sqrt{y+1})^m\)^{*11} = (x - y\sqrt{y+1}); \((x + y\sqrt{y+1})^m, (x - y\sqrt{y+1})^n\)^{*11} = ((x + y\sqrt{y+1})^m + a(x - y\sqrt{y+1})^n)^{*11} = (x + y\sqrt{y+1}, (x - y\sqrt{y+1})^m) and \(A^{*11} = A\).

12. \(*_{12} : (0)^{*12} = (0)\); \((x + y\sqrt{y+1})^n\)^{*12} = (x - y\sqrt{y+1});
\((x + y\sqrt{y+1})^m\)^{*12} = (x - y\sqrt{y+1}); \((x + y\sqrt{y+1})^m, (x - y\sqrt{y+1})^n\)^{*12} = ((x + y\sqrt{y+1})^m + a(x - y\sqrt{y+1})^n)^{*12} = (x + y\sqrt{y+1}, (x - y\sqrt{y+1})^m) and \(A^{*12} = A\).

13. \(*_{13} : (0)^{*13} = (0)\); \((x + y\sqrt{y+1})^n\)^{*13} = (x + y\sqrt{y+1});
\((x + y\sqrt{y+1})^m\)^{*13} = (x + y\sqrt{y+1}) and \(I^{*13} = A\) for all other \(I\).

14. \(*_{14} : (0)^{*14} = (0)\); \((x + y\sqrt{y+1})^n\)^{*14} = (x + y\sqrt{y+1});
\((x - y\sqrt{y+1})^n\)^{*14} = (x - y\sqrt{y+1}) and \(I^{*14} = A\) for all other \(I\).

15. \(*_{15} : (0)^{*15} = (0)\); \((x + y\sqrt{y+1})^n\)^{*15} = (x + y\sqrt{y+1});
\((x - y\sqrt{y+1})^n\)^{*15} = (x - y\sqrt{y+1}) and \(I^{*15} = A\) for all other \(I\).

16. \(*_{16} : (0)^{*16} = (0)\); \((x + y\sqrt{y+1})^n\)^{*16} = (x + y\sqrt{y+1});
\((x - y\sqrt{y+1})^n\)^{*16} = (x - y\sqrt{y+1}) and \(I^{*16} = A\) for all other \(I\).

17. \(*_{17} : (0)^{*17} = (0)\); ((x + y\sqrt{y+1})^n)^{*17} = (x + y\sqrt{y+1}) and \(I^{*17} = A\) for all other \(I\).

18. \(*_{18} : (0)^{*18} = (0)\); ((x + y\sqrt{y+1})^n)^{*18} = (x + y\sqrt{y+1}) and \(I^{*18} = A\) for all other \(I\).
19. \( *_{19} : (0)^{*_{19}} = ((x + y\sqrt{y + 1})^n)^{*_{19}} = (x + y\sqrt{y + 1}) \) and \( I^{*_{19}} = A \) for all other \( I \).

20. \( *_{20} : (0)^{*_{20}} = (0); ((x - y\sqrt{y + 1})^n)^{*_{20}} = ((x - y\sqrt{y + 1})^n) \) and \( I^{*_{20}} = A \) for all other \( I \).

21. \( *_{21} : (0)^{*_{21}} = (0); ((x - y\sqrt{y + 1})^n)^{*_{21}} = (x - y\sqrt{y + 1}) \) and \( I^{*_{21}} = A \) for all other \( I \).

22. \( *_{22} : (0)^{*_{22}} = ((x - y\sqrt{y + 1})^n)^{*_{22}} = (x - y\sqrt{y + 1}) \) and \( I^{*_{22}} = A \) for all other \( I \).

23. \( *_{23} : (0)^{*_{23}} = (0) \) and \( I^{*_{23}} = R \) for all other \( I \).

24. \( *_{24} : I^{*_{24}} = A \) for all \( I \).
Chapter 5

Polynomial Rings

When pulling back to the polynomial rings, we see that the isomorphism between $R$ and $A$ no longer holds and the list of standard closures in each diverge from each other. The work becomes significantly more complicated in the polynomial rings since we now have to consider ideals with more complex generating elements. A main contributing factor of this has to do with the fact that the quotient of $\mathbb{K}[x, y] \mod (x^2 - (y^2 + y^3))$ forms an integral domain [1]. $\mathbb{K}[x, y] \mod (xy)$ is clearly not a domain. Furthermore, we can no longer consider elements of the form $a - f(x, y)$ to be units in either polynomial ring, so we must also explore how closure operations will extend to ideals generated by elements of this form.

Rather than exhausting the list in each polynomial ring, some examples and a general idea of how the standard closures will change will be given. We start by quickly addressing the standard closures in the polynomial counterpart to $A$.

5.1 Standard Closures in $A'$

Briefly, it should be mentioned what the significance of introducing the ring $A = \mathbb{K}[[x, y]]/(x^2 - (y^2 + y^3))$ is. As it was shown earlier, $A$ is isomorphic to the ring $R = \mathbb{K}[[x, y]]/(xy)$, and that guarantees that all 24 standard closure operations on $R$, are also defined on $A$. However, once reverting to the polynomial quotient ring analogues, the list of standard closure operations diverge in each case. The ideals in $A$ whose generators can be expressed nicely as monomials, or a sum of an $x$ and a $y$ monomial, will be preserved when observing the polynomial ring $\mathbb{K}[x, y]/(x^2 - (y^2 + y^3))$. However, because many of the ideals in $A$ do not survive in the polynomial analogue, the list of standard closures will suffer a large reduction.

Let $A' = \mathbb{K}[x, y] / (x^2 - (y^2 + y^3))$. (5.1)

The generators $x + y\sqrt{y + 1}$ and $x - y\sqrt{y + 1}$ of $A$ can be expressed as infinite series in $A$. However, when restricting the elements to only polynomials of finite degree, we find that we can no longer express these generators. For this reason, there are no ideals of the form $(v^n)$ or $(w^n)$, where $v = x + y\sqrt{y + 1}$ and $w = x - y\sqrt{y + 1}$. Correspondingly, most of the standard closure operations listed above become undefined in $A'$. Recall that the ideals $(x^n)$ and $(y^n)$ behave in $A$ in the same way as ideals of the form $(x^n + ay^n)$ do in $R$. As a consequence, we can immediately reduce the list of standard closures before considering other ideals that might occur in $A'$. In the polynomial ring corresponding to $R$, we unfortunately do not have that luxury.
5.2 Standard Closures in $R'$

Unlike in the case of $A'$, we cannot reduce the list of standard closures in the polynomial ring

$$R' = \frac{\mathbb{K}[x, y]}{(xy)},$$

(5.2)

before tackling the task of generalizing it. Instead, we extend just one of the standard closure operations, $\star_{13}$, to all of the ideals in $R'$, and show that it is, in fact, standard.

A Standard Closure in $R'$

**Proposition 5.2.1** The closure operation $c : \mathcal{I}(R') \to \mathcal{I}(R')$ is a standard closure operation on the ideals of the ring

$$R' = \frac{\mathbb{K}[x, y]}{(xy)}.$$

For natural numbers $n$ and $m$ and for $a_j, b_k$, and $c_l$ in $\mathbb{K}$, define the standard closure $c : \mathcal{I}(R') \to \mathcal{I}(R')$ as follows

$$(0)c = (0);$$

$$(x^n)c = (x^n);$$

$$(y^n)c = (y^n);$$

$$(x^n, y^m)c = (x^n + ay^m)c = R'$$

$$R'c = R'$$

For $a_n \neq 0$, 

$$\left( \sum_{i=1}^{n} a_i x^i \right)c = \left( \sum_{i=1}^{n} a_i x^i \right);$$

$$\left( \sum_{i=1}^{n} a_i y^i \right)c = \left( \sum_{i=1}^{n} a_i y^i \right);$$

(5.3)

$$(f_1(x), \ldots, f_r(x)c) = (f_1(x), \ldots, f_r(x)) \text{ with } f_j(x) = \sum_{i=1}^{n} a_{ij} x^i, \ 1 \leq j \leq s, \ a_{nj} \neq 0;$$

(5.4)

$$(g_1(y), \ldots, g_r(y))c = (g_1(y), \ldots, g_r(y)) \text{ with } g_j(y) = \sum_{i=1}^{n} a_{ij} y^i, \ 1 \leq j \leq s, \ a_{nj} \neq 0;$$

(5.5)

$$(f_1(x), \ldots, f_r(x), g_1(y), \ldots, g_s(y))c = R', \text{ with } f_j(x), g_k(y) \text{ defined as above, } 1 \leq j \leq r, 1 \leq k \leq s;$$

(5.6)

For $a_n, b_m \neq 0$, 

$$\left( \sum_{i=1}^{n} a_i x^i + \sum_{i=1}^{m} b_i y^i \right)c = R';$$

(5.7)

$$(f_1(x, y), \ldots, f_r(x, y)c) = R' \text{ where for } 1 \leq j \leq r, \ f_j(x, y) = \sum_{i=1}^{n} a_{ij} x^i + \sum_{i=1}^{m} b_{ij} y^i, \ a_{nj}, b_{mj} \neq 0;$$

(5.8)

$$\left( 1 + \sum_{i=1}^{n} a_i x^i \right)c = R'; \quad \left( 1 + \sum_{i=1}^{n} a_i y^i \right)c = R'; \quad \left( 1 + \sum_{i=1}^{n} a_i x^i + \sum_{i=1}^{m} b_i y^i \right)c = R'$$

(5.9)
\[(f_1(x), \ldots, f_r(x))^c = R', \quad f_j(x) = 1 + \sum_{i,j=1}^{n_j} a_{ij} x^{ij}, \quad 1 \leq j \leq r; \quad (5.10)\]

\[(g_1(y), \ldots, g_r(y))^c = R', \quad g_j(y) = 1 + \sum_{i,j=1}^{n_j} a_{ij} y^{ij}, \quad 1 \leq j \leq r; \quad (5.11)\]

\[(f_1(x), \ldots, f_r(x), g_1(y), \ldots, g_s(y))^c = R' \text{ with } f_j \text{ and } g_j \text{ defined as above;} \quad (5.12)\]

\[(h_1(x, y), \ldots, h_r(x, y))^c = R', \quad h_j(x, y) = 1 + \sum_{i,j=1}^{n_j} b_{ij} x^{ij} + \sum_{i,j=1}^{m_j} b_{ij} y^{ij} \quad (5.13)\]

Before showing that this is a standard closure operation on the ideals of \(R'\), it should be mentioned that in the setting \(A' = K[x, y]/(x^2 - (y^2 + y^3)), \ast_{13}\) reduces down to the trivial closure operation that maps \((0)\) to itself and all other ideals to the entire ring.

**Proof** We start by showing that (6.3) satisfies the standard property. We will only show this for \(I = (\sum_{i=1}^{n} a_i x^i)\), as the case with \(y\) is proven in exactly the same way.

Let \(I = (\sum_{i=1}^{n} a_i x^i)\)

i.) let \(f = 1 + \sum_{i=1}^{m} b_i x^i\). Suppose that \(m > n\). To compute the ideal quotient:

\[((fI)^c : f) = \left(\left(1 + \sum_{i=1}^{m} b_i x^i\right)\left(\sum_{i=1}^{n} a_i x^i\right)^c\right) : 1 + \sum_{i=1}^{m} b_i x^i\]

We first expand \(fI\):

\[\left(1 + \sum_{i=1}^{m} b_i x^i\right)\left(\sum_{i=1}^{n} a_i x^i\right) = \sum_{i=1}^{n} a_i x^i + \left(\sum_{i=1}^{m} b_i x^i\right)\left(\sum_{i=1}^{n} a_i x^i\right)\]

\[= a_1 x + (b_1 a_1 + a_2) x^2 + (b_1 a_2 + b_2 a_1 + a_3) x^3 + \cdots + (b_1 a_{n-1} + \cdots + b_{n-1} a_1 + a_n) x^n\]

\[+ b_1 a_1 + \cdots + b_{n-1} a_1) x^{n+1} + \cdots + (b_m a_n + b_m a_{n-1}) x^{m+n-1} + b_m a_n x^{m+n}\]

By definition, the closure of the above ideal \((fI)\) is itself. So we need to find all of the elements \(g\) such that \(gf\) is in \((fI)^c\). In other words, we need to find all of the elements \(g\) such that \(gf = h(fI)^c = (fI)\), where \(h(x, y)\) is any element in \(R'\).

Immediately, it is clear that \(g\) won’t be an element of the form \(1 + \varphi(x, y)\), since \((fI)^c\) does not contain any elements with non-zero constant terms. \(g\) also won’t be an element made up purely of powers of \(y\), since \((fI)\) obviously has no elements with \(y\) terms. For the same reason, \(g\) also won’t be of the form \(\varphi(x) + \psi(y)\).

This leaves us only with elements of the form \(g = \varphi(x) = \sum_{i=1}^{k} c_i x^i\). Already, it seems that this closure might possibly satisfy the standard property for this particular ideal and element.

We are looking for the element \(g(x)\) such that

\[\left(\sum_{i=1}^{k} c_i x^i\right)\left(1 + \sum_{i=1}^{m} b_i x^i\right) = (a_1 x + (b_1 a_1 + a_2) x^2 + (b_1 a_2 + b_2 a_1 + a_3) x^3 + \cdots + (b_1 a_{n-1} + \cdots + b_{n-1} a_1 + a_n) x^n + \cdots + (b_m a_n + b_m a_{n-1}) x^{m+n-1} + b_m a_n x^{m+n}\]

Then every other element will just be a multiple of \(g(x)\). Expanding the left-hand side,

\[\left(\sum_{i=1}^{k} c_i x^i\right)\left(1 + \sum_{i=1}^{m} b_i x^i\right) = c_1 x^1 + (c_1 b_1 + c_2) x^2 + \cdots + (b_{m-1} c_k + b_m c_{k-1}) x^{m+k-1} + b_m c_k x^{k+m}\]
It is immediately clear that \( k = n \). So we get the system of equations
\[
\begin{align*}
c_1 &= a_1 \\
c_2 + b_1c_1 &= b_1a_1 + a_2 \\
c_3 + b_2c_2 + b_1c_1 &= b_1a_2 + b_2a_1 + a_3 \\
& \vdots \\
c_n + b_1c_{n-1} + \cdots + b_{n-1}c_1 &= b_1a_{n-1} + \cdots + b_{n-1}a_n
\end{align*}
\]

Then using the first equality, \( c_1 = a_1 \), we substitute into the second
\[
c_2 = b_1a_1 + a_2 - b_1a_1 = a_2
\]
Likewise for the third equation:
\[
c_3 = b_1a_{n-1} + b_2a_1 + a_3 - b_1a_2 + b_2a_1 = a_3
\]
And so on such that for each \( i = 1, \ldots, n \), \( c_i = a_i \). We immediately see that, the elements \( g : gf \in (fI)^c \) are the elements of the form
\[
g(x) = h(x, y) \left( \sum_{i=1}^{n} a_ix^i \right)
\]
Elements of this form are exactly the elements of the ideal \( I \), and we can conclude that for elements \( f \) of the form \( 1 + \sum_{i=1}^{n} b_ix^i \), \( I \) satisfies the standard property of a closure operation.

ii.) If \( f = 1 + \sum_{i=1}^{m} b_iy^i \), then
\[
fI = \left( 1 + \sum_{i=1}^{m} b_iy^i \right) \left( \sum_{i=1}^{n} a_ix^i \right) = \sum_{i=1}^{n} a_ix^i
\]
This is a trivial case, as the ideal quotient will clearly be only those elements of the ideal \( \left( \sum_{i=1}^{n} a_ix^i \right) \).

iii.) Let \( f = \sum_{i=1}^{m} b_ix^i + \sum_{i=1}^{k} d_iy^i \). We can again compute the ideal quotient. First, we find \( fI \):
\[
\left( \left( \sum_{i=1}^{m} b_ix^i + \sum_{i=1}^{k} d_iy^i \right) \left( \sum_{i=1}^{n} a_ix^i \right) \right)^{c} = \left( \sum_{i=1}^{m} b_ix^i \right)^{c} \cdot \left( \sum_{i=1}^{n} a_ix^i \right)^{c} + \left( \sum_{i=1}^{k} d_iy^i \right)^{c} \cdot \left( \sum_{i=1}^{n} a_ix^i \right)^{c}
\]
\[
= \left( \sum_{i=1}^{m} b_ix^i \right)^{c} \cdot \left( \sum_{i=1}^{n} a_ix^i \right)^{c} + \left( \sum_{j+k=1}^{m+n} x^i \right) \cdot \left( \sum_{j+k=1}^{m+n} a_{j+k}b_k \right)
\]
Again, to compute the ideal quotient, the only eligible candidates for elements \( g \) in \( R' \) such that \( gf \in (fI)^c \) are the elements that are multiples of an element of the form \( \sum_{i=1}^{n} c_ix^i \). Then,
\[
gf = \left( \sum_{i=1}^{n} c_ix^i \right) \left( \sum_{i=1}^{m} b_ix^i + \sum_{i=1}^{k} d_iy^i \right) = \left( \sum_{i=1}^{n} c_ix^i \right) \left( \sum_{i=1}^{m} b_ix^i \right) = \sum_{i=2}^{m+n} x^i \left( \sum_{j+k=i} c_{j+k}b_k \right)
\]
We again get a system of equations
\[
c_1 b_1 = a_1 b_1 \\
c_2 b_1 = a_1 b_2 + a_2 b_1 - c_1 b_2 \\
c_3 b_1 = a_1 b_3 + a_2 b_2 + a_3 b_1 - c_2 b_3 - c_1 b_3 \\
\vdots \\
c_n b_1 = a_1 b_n + \cdots + a_n b_1 - (c_{n-1} b_2 + \cdots + c_1 b_n)
\]
The same applies for \( n > m \). If \( b_1 \neq 0 \), we can divide by \( b_1 \) to find that \( c_1 = a_1 \) and solve for each of the rest of the \( c_i \) in the same way as we did for the previous case. If \( b_1 = 0 \) then we get no information about \( c_1 \) from the first equation. In this case, we have
\[
0 = \sum_{i=1}^n a_i x^i + \sum_{i=1}^m b_i y^i + \sum_{i=1}^n c_i x^i + \sum_{i=1}^m b_i y^i
\]
We simply adjust our system accordingly:
\[
c_1 b_2 = a_1 b_2 \\
c_2 b_2 = a_1 b_3 + a_2 b_2 - c_1 b_3 \\
c_3 b_2 = a_1 b_4 + a_2 b_3 + a_3 b_2 - (c_2 b_3 + c_1 b_4) \\
\vdots \\
c_n b_2 = a_1 b_{n+1} + \cdots + a_n b_2 - (c_{n-1} b_3 + \cdots + c_1 b_{n+1})
\]
And proceed as was stated above, assuming that \( b_2 \neq 0 \). If \( b_2 = 0 \), we repeat the readjustment process until we arrive at a nonzero \( b_i \) term. Obviously, we are not allowing all of the \( b_i, 1 \leq i \leq m \) to be zero. However, even if all of the first \( m - 1 \) \( b_i \) terms are zero, we will still have the appropriate number of equations needed to find all of the \( c_i \).

The cases where \( f \) is of the form \( 1 + \sum_{i=1}^m b_i x^i + \sum_{i=1}^m d_i y^i \) or \( 1 + \sum_{i=1}^m b_i y^m \) reduce to one of the cases above, or a trivial case. For the ideal \( I = (\sum_{i=1}^n a_i x^i) \), the proof is identical.

Let \( I = (\sum_{i=1}^{n_1} a_i x^i + \sum_{i=1}^{n_2} b_i y^i) \).

i.) If \( f = 1 + \sum_{i=1}^m c_i x^i \), then
\[
fI = \left(1 + \sum_{i=1}^m c_i x^i\right) \left(\sum_{i=1}^{n_1} a_i x^i + \sum_{i=1}^{n_2} b_i y^i\right)
\]
\[
= \sum_{i=1}^{n_1} a_i x^i + \sum_{i=1}^{n_2} b_i y^i + \left(\sum_{i=1}^m c_i x^i\right) \left(\sum_{i=1}^{n_2} b_i y^i\right)
\]
Whose closure, by definition, is the entire ring \( R' \). This is a trivial case, since the quotient ideal \((R' : f) = R' = I^c\).

ii.) If \( f = 1 + \sum_{i=1}^m c_i y^i \), then the proof is identical to i.).

iii.) Let \( f = \sum_{i=1}^{m_1} c_i x^i + \sum_{i=1}^{m_2} d_i y^i \), then
\[
fI = \left(\sum_{i=1}^{m_1} c_i x^i + \sum_{i=1}^{m_2} d_i y^i\right) \left(\sum_{i=1}^{n_1} a_i x^i + \sum_{i=1}^{n_2} b_i y^i\right)
\]
\[
= \left(\sum_{i=1}^{m_1} c_i x^i\right) \left(\sum_{i=1}^{n_1} a_i x^i\right) + \left(\sum_{i=1}^{m_2} d_i y^i\right) \left(\sum_{i=1}^{n_2} b_i y^i\right)
\]
Whose closure is again the entire ring. So this also reduces down to the trivial case.

Now consider $I = \left( 1 + \sum_{i=1}^{n} a_{i}x^{i} \right)$ with closure defined as in (6.9).

i.) If $f = 1 + \sum_{i=1}^{m} b_{i}x^{i}$ or $f = 1 + \sum_{i=1}^{m} b_{i}y^{i}$, then the proof is easy, as

$$((fI)^{c} : f) = (R')^{c} = I^{c}.$$  

ii.) If $f = \sum_{i=1}^{m} b_{i}x^{i} + \sum_{i=1}^{k} d_{i}y^{i}$, then

$$fI = \left( \sum_{i=1}^{m} b_{i}x^{i} + \sum_{i=1}^{k} d_{i}y^{i} \right) \left( 1 + \sum_{i=1}^{n} a_{i}x^{i} \right)$$

$$= \sum_{i=1}^{m} b_{i}x^{i} + \sum_{i=1}^{k} d_{i}y^{i} + \left( \sum_{i=1}^{m} b_{i}x^{i} \right) \left( \sum_{i=1}^{n} a_{i}x^{i} \right)$$

However, the closure operation of an ideal of this form is defined to again be $R'$, so this case is again easy. The case with $I = \left( 1 + \sum_{i=1}^{n} a_{i}y^{i} \right)$ is proven in exactly the same way.

Let $I = 1 + \sum_{i=1}^{m_{1}} a_{i}x^{i} + \sum_{i=1}^{m_{2}} b_{i}y^{i}$.

i.) If $f = 1 + \sum_{i=1}^{m} c_{i}x^{i}$, then

$$fI = \left( 1 + \sum_{i=1}^{m} c_{i}x^{i} \right) \left( 1 + \sum_{i=1}^{m_{1}} a_{i}x^{i} + \sum_{i=1}^{m_{2}} b_{i}y^{i} \right)$$

$$= 1 + \sum_{i=1}^{m_{1}} a_{i}x^{i} + \sum_{i=1}^{m_{2}} b_{i}y^{i} + \sum_{i=1}^{m} c_{i}x^{i} + \left( \sum_{i=1}^{m} c_{i}x^{i} \right) \left( \sum_{i=1}^{m_{1}} a_{i}x^{i} \right)$$

Whose closure is the entire ring. Therefore this reduces to the trivial case.

ii.) For $f = 1 + \sum_{i=1}^{m} c_{i}y^{i}$, it is the same as above.

iii.) If $f = \sum_{i=1}^{m_{1}} c_{i}x^{i} + \sum_{i=1}^{m_{2}} d_{i}y^{i}$, then

$$fI = \left( \sum_{i=1}^{m_{1}} c_{i}x^{i} + \sum_{i=1}^{m_{2}} d_{i}y^{i} \right) \left( 1 + \sum_{i=1}^{m_{1}} a_{i}x^{i} + \sum_{i=1}^{m_{2}} b_{i}y^{i} \right)$$

$$= \sum_{i=1}^{m_{1}} c_{i}x^{i} + \sum_{i=1}^{m_{2}} d_{i}y^{i} + \left( \sum_{i=1}^{m_{1}} c_{i}x^{i} \right) \left( \sum_{i=1}^{m_{1}} a_{i}x^{i} \right) + \left( \sum_{i=1}^{m_{2}} d_{i}y^{i} \right) \left( \sum_{i=1}^{m_{2}} b_{i}y^{i} \right)$$

Whose closure is again $R'$ and the rest of the proof for this case is trivial. We now move on to ideals with multiple generators. Using the results above, we can show that the same notions hold in the case that $I$ has a finite number of generating elements.

First, let $I$ be an ideal as in (5.4).

i.) Let $f = 1 + \sum_{i=1}^{m} c_{i}x^{i}$. We first consider the case when $r = 2$. Then

$$I = \left( \sum_{i=1}^{n_{1}} a_{i}x^{i}, \sum_{i=1}^{n_{2}} b_{i}x^{i} \right).$$  

(5.14)
Computing \( fI \):
\[
fI = \left( \sum_{i=1}^{m} c_i x^i \right) \left( \sum_{i=1}^{n_1} a_i x^i, \sum_{i=1}^{n_2} b_i x^i \right)
\]
\[
= \left( \left( 1 + \sum_{i=1}^{m} c_i x^i \right) \left( \sum_{i=1}^{n_1} a_i x^i \right) \right) \left( 1 + \sum_{i=1}^{n_1} a_i x^i \right) \left( \sum_{i=1}^{n_2} b_i x^i \right)
\]
\[
= \left( \sum_{i=1}^{n_1} a_i x^i + \left( \sum_{i=1}^{m} c_i x^i \right) \left( \sum_{i=1}^{n_1} a_i x^i \right) \right) \left( \sum_{i=1}^{n_2} b_i x^i \right)
\]
ii.) Let \( f = \sum_{i=1}^{m} c_{i}y^{i} \) and \( r = 2 \). Then

\[
fI = \left( 1 + \sum_{i=1}^{m} c_{i}y^{i} \right) \left( \sum_{i=1}^{n_1} a_{i}x^{i}, \sum_{i=1}^{n_2} b_{i}x^{i} \right)
= \left( 1 + \sum_{i=1}^{n_1} a_{i}x^{i} + \sum_{i=1}^{m} c_{i}y^{i}, 1 + \sum_{i=1}^{n_2} b_{i}x^{i} + \sum_{i=1}^{m} c_{i}y^{i} \right).
\]

Whose closure is again \( R' \) and inductively it can be shown that this will be the case for any \( r \in \mathbb{N} \).

iii.) Let \( f = \sum_{i=1}^{m_1} c_{i}x^{i} + \sum_{i=1}^{m_2} d_{i}y^{i} \) and \( r = 2 \).

\[
fI = \left( \sum_{i=1}^{m_1} c_{i}x^{i} + \sum_{i=1}^{m_2} d_{i}y^{i} \right) \left( 1 + \sum_{i=1}^{n_1} a_{i}x^{i}, 1 + \sum_{i=1}^{n_2} b_{i}x^{i} \right)
= \left( \sum_{i=1}^{m_1} c_{i}x^{i} + \sum_{i=1}^{m_2} d_{i}y^{i} + \left( \sum_{i=1}^{m_1} c_{i}x^{i} \right) \left( \sum_{i=1}^{n_1} a_{i}x^{i} \right), \sum_{i=1}^{m_1} c_{i}x^{i} + \sum_{i=1}^{m_2} d_{i}y^{i} + \left( \sum_{i=1}^{m_1} c_{i}x^{i} \right) \left( \sum_{i=1}^{n_2} b_{i}x^{i} \right) \right).
\]

Once again, the closure is the entire ring, so showing that the ideal quotient in this case, and in general for any \( r \in \mathbb{N} \), is equal to \( I^{c} \) is easy. The remaining cases are shown in a similar fashion.

\[\square\]

### 5.3 Potential Future Work

Above, only the standard closure operation \( \ast_{13} \) was extended to the ideals of \( \mathbb{K}[x, y]/(xy) \). Any of the other standard closures following \( \ast_{13} \) can be defined similarly on \( \mathcal{I}(R') \), and the proofs would be similar to the one given above. Alternatively, it is not required that most of these extra ideals that arise in \( R' \) must go to the entire ring. Our extension was only one of several possible. In the future, one could explore these alternative extensions as well as consider the generalizations of the more complicated standard closures near the beginning of the list.
Bibliography


