PERTURBATIONS WITH ZERO SPECTRAL SHIFT FUNCTIONS

ŠÁRKA BLAHNIK

ABSTRACT. We explore necessary and sufficient conditions for spectral shift functions of various orders between two finite \( n \times n \) Hermitian matrices \( H, V \) to be identically zero. We resolve the general even order case. We obtain partial results for the odd order case.

1. INTRODUCTION

With respect to two bounded linear operators \( H \) and \( V \), the first order spectral shift function \( \eta_1(t) \) measures how the number of eigenvalues increases past the real argument \( t \) when \( H \) is perturbed to \( H + V \). This function was first introduced by physicist Ilya M. Lifshits around 1950 as a computational tool for investigating the effects of introducing small impurities on the free energy of otherwise pure crystalline solids [1].

Further spectral shift functions later emerged and have become objects of broader interest in mathematics and quantum physics. The second order spectral shift function \( \eta_2(t) \) provides a measure of how much shift the eigenvalues have undergone during perturbation.

Higher order functions \( \eta_p(t) \) can provide additional insight into the nature of such perturbations and are particularly useful in the case of infinite-dimensional operators when one or more of the lower order functions may become undefined. This is an important point in a broader context as most current and potential future applications of spectral theory in physics and other sciences do indeed involve infinite-dimensional operators.

In this undergraduate thesis we explore properties of identically zero \( \eta_p \) for finite-dimensional Hermitian matrices.

2. PRELIMINARIES

We assume familiarity with the basic theory of eigenvalues and eigenvectors. We further recall from linear algebra that the adjoint of a matrix \( H \), denoted \( H^* \), is formed by taking the transpose of \( H \) and replacing all entries with their complex conjugates. A matrix \( H \) is called Hermitian or self-adjoint if \( H = H^* \), unitary if \( H^* = H^{-1} \), and normal if \( HH^* = H^*H \). We note that all Hermitian or unitary matrices are normal, though the converse is false.
Notations 2.1. Let $M_n(\mathbb{C})$ be the Hilbert space of all $n \times n$ matrices with complex entries, and let $M_n(\mathbb{C})_{sa} \subset M_n(\mathbb{C})$ be the subset of all self-adjoint matrices $H = H^*$ in this space. Let $\sigma(H)$ denote the spectrum of $H \in M_n(\mathbb{C})_{sa}$.

Proposition 2.2. (Well-known from linear algebra). If $H \in M_n(\mathbb{C})_{sa}$, then $\sigma(H)$ is a subset of $\mathbb{R}$.

That is, the eigenvalues of Hermitian matrices are always real numbers (even if the matrix itself contains some complex entries).

Theorem 2.3 (The Spectral Theorem). A matrix $H \in M_n(\mathbb{C})$ is normal if and only if it is diagonalizable by a unitary matrix $S$.

As a result of the famous spectral theorem, it is well-known that all Hermitian matrices can be diagonalized by some unitary matrix. This becomes a useful fact in the study of spectral shift functions.

Proposition 2.4. (See, for example, [2, Proposition 4.9]). Let $H \in M_n(\mathbb{C})_{sa}$ with $\sigma(H) = \{\lambda_i\}_{i=1}^n$. Then the decomposition $H = SDS^*$ can be written as the linear combination

$$H = \sum_{i=1}^n \lambda_i S E_{ii} S^*,\quad (2.1)$$

where $E_{ij}$ represents the elementary matrix with a single nonzero entry of 1 as its $ij$th element.

Example 2.5. Decomposition of a diagonal $H \in M_3(\mathbb{C})_{sa}$.

Let $H = \begin{pmatrix} 9 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 2 \end{pmatrix} = SDS^*$, $\lambda_1 = 9$, $\lambda_2 = -5$, $\lambda_3 = 2$.

In this case $D = H$ and $S = I = S^*$ is the $3 \times 3$ identity matrix. We can write this decomposition of $H$ as the following linear combination.

$$H = 9 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - 5 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$H = 9E_{11} - 5E_{22} + 2E_{33} = \sum_{i=1}^3 \lambda_i E_{ii} = \sum_{i=1}^3 \lambda_i S E_{ii} S^*.$$
Example 2.6. Decomposition of a non-diagonal $H \in M_2(\mathbb{C})_{sa}$.

Let $H = \begin{pmatrix} 7 & i \\ -i & 3 \end{pmatrix}$, and note that $H$ is Hermitian with $\sigma(H) = \{5 + \sqrt{5}, 5 - \sqrt{5}\}$.

Note also that $S = \begin{pmatrix} \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \\ \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \end{pmatrix}$ is a unitary element of $M_2(\mathbb{C})_{sa}$ because:

$$SS^* = \begin{pmatrix} \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \\ \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \\ \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = S^*S.$$

We have $H = SDS^*$ as follows:

$$H = SDS^* = \begin{pmatrix} \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \\ \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \end{pmatrix} \begin{pmatrix} 5 + \sqrt{5} & 0 \\ 0 & 5 - \sqrt{5} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \\ \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \end{pmatrix}.$$

Knowing this, we can decompose $H$ as in Proposition 2.4:

$$H = \sum_{i=1}^{n} \lambda_i SE_{ii}S^* = (5 + \sqrt{5})SE_{11}S^* + (5 - \sqrt{5})SE_{22}S^*$$

$$= (5 + \sqrt{5}) \begin{pmatrix} \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \\ \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \\ \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \end{pmatrix} + (5 - \sqrt{5}) \begin{pmatrix} \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \\ \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \\ \frac{1}{\sqrt{10 - 4\sqrt{5}}} & \frac{1}{\sqrt{10 + 4\sqrt{5}}} \end{pmatrix}.$$ 

Definition 2.7. (See, for example, [2, Definition 4.11]). Let $H \in M_n(\mathbb{C})_{sa}$, with $\sigma(H) = \{\lambda_i\}_{i=1}^{n}$, and let $f$ be a bounded, scalar-valued function defined on $\mathbb{R}$. Then $f$ can be defined on $M_n(\mathbb{C})_{sa}$ as follows:

$$f(H) = SF(D)S^* = \sum_{i=1}^{n} f(\lambda_i)SE_{ii}S^*.$$ (2.2)
Definition 2.8. (See, for example [3, Definition 2.16]). The characteristic function \( \chi_{(a,b)}(t) \) is defined for \( t \in \mathbb{R} \) by:

\[
\chi_{(a,b)}(t) = \begin{cases} 
1 & \text{if } t \in (a,b) \\
0 & \text{otherwise}
\end{cases}.
\] (2.3)

The definition is extended to \( \chi_{(a,b)}(H) \) for \( H \in M_n(\mathbb{C})_{sa} \) by:

\[
\chi_{(a,b)}(H) = \sum_{i=1}^{n} \chi_{(a,b)}(\lambda_i)SE_{ii}S^*,
\] (2.4)

where \( H \) is as in Equation (2.1).

When \( \chi_{(a,b)}(H) \) is applied to a vector \( g \) using the inner product with the set of eigenvectors of \( H \), \( \{f_j\} \), \( j = 1, 2, \ldots, n \), we write:

\[
\chi_{(a,b)}(H)g = \sum_{\lambda \in (a,b)} \langle g, f_j \rangle f_j.
\] (2.5)

Example 2.9. Calculation of \( \chi_{(a,b)}(H) \) with a diagonal \( H \in M_3(\mathbb{C})_{sa} \).

Let \( H \) be as in Example 2.5. Then:

\[
\chi_{(1,9.1)}(H) = \sum_{i=1}^{n} \chi_{(1,9.1)}(\lambda_i)SE_{ii}S^* = \sum_{i=1}^{n} \chi_{(1,9.1)}(\lambda_i)E_{ii}
\]

\[
= \chi_{(1,9.1)}(9) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \chi_{(1,9.1)}(-5) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \chi_{(1,9.1)}(2) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
= (1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + (0) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + (1) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
= (1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
Example 2.10. Calculation of \( \chi_{(a,b)}(H) \) with a non-diagonal \( H \in M_2(\mathbb{C})_{sa} \).

Let \( H \) be as in Example 2.6. Then:

\[
\chi_{(5,8)}(H) = \sum_{i=1}^{n} \chi_{(5,8)}(\lambda_i)SE_{ii}S^*,
\]

\[
\chi_{(5,8)}(H) = \chi_{(5,8)}(5 + \sqrt{5}) \begin{pmatrix}
\frac{1}{\sqrt{10-4\sqrt{5}}} & \frac{1}{\sqrt{10+4\sqrt{5}}} \\
\frac{1}{\sqrt{10-4\sqrt{5}}} & \frac{1}{\sqrt{10+4\sqrt{5}}}
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
\frac{1}{\sqrt{10-4\sqrt{5}}} & \frac{(-2+\sqrt{5})i}{\sqrt{10+4\sqrt{5}}} \\
\frac{1}{\sqrt{10-4\sqrt{5}}} & \frac{(-2-\sqrt{5})i}{\sqrt{10+4\sqrt{5}}}
\end{pmatrix},
\]

\[
+ \chi_{(0,8)}(5 - \sqrt{5}) \begin{pmatrix}
\frac{1}{\sqrt{10-4\sqrt{5}}} & \frac{1}{\sqrt{10+4\sqrt{5}}} \\
\frac{1}{\sqrt{10-4\sqrt{5}}} & \frac{(-2+\sqrt{5})i}{\sqrt{10+4\sqrt{5}}}
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\frac{1}{\sqrt{10-4\sqrt{5}}} & \frac{(-2-\sqrt{5})i}{\sqrt{10+4\sqrt{5}}} \\
\frac{1}{\sqrt{10-4\sqrt{5}}} & \frac{9-\sqrt{5}}{10-\sqrt{5}}
\end{pmatrix}.
\]

Noting \( \chi_{(0,8)}(5 + \sqrt{5}) = 1 \) and \( \chi_{(0,8)}(5 - \sqrt{5} = 0) \), we have:

\[
\chi_{(5,8)}(H) = \begin{pmatrix}
\frac{1}{\sqrt{10-4\sqrt{5}}} & 0 \\
\frac{1}{\sqrt{10-4\sqrt{5}}} & 0
\end{pmatrix} \begin{pmatrix}
\frac{1}{\sqrt{10-4\sqrt{5}}} & \frac{(-2+\sqrt{5})i}{\sqrt{10+4\sqrt{5}}} \\
\frac{1}{\sqrt{10+4\sqrt{5}}} & \frac{(-2-\sqrt{5})i}{\sqrt{10+4\sqrt{5}}}
\end{pmatrix}.
\]

Definition 2.11. (See, for example, [3, Definition 3.1]). Let \( H, V \in M_n(\mathbb{C})_{sa} \). The first order spectral shift function is defined as follows:

\[
\eta_1(t) := \text{Tr} \left( \chi_{(-\infty,t)}(H) - \chi_{(-\infty,t)}(H + V) \right).
\]

(2.6)

Proposition 2.12. [3, Proposition 3.3] Let \( H, V \in M_n(\mathbb{C})_{sa} \), \( \sigma(H) = \{\lambda_i\}_{i=1}^{n} \), and \( \sigma(H + V) = \{\mu_j\}_{j=1}^{n} \). Then the following representation of \( \eta_1(t) \) is equivalent to that of Definition 2.11:

\[
\eta_1(t) = \text{Card}\{i : \lambda_i < t\} - \text{Card}\{j : \mu_j < t\},
\]

(2.7)

where Card refers to the cardinality of the sets.

Definition 2.13. (See, for example, [3, Definition 4.1]). The second order spectral shift function can be defined in terms of the first order spectral shift function as follows:

\[
\eta_2(t) := \text{Tr} \left( \chi_{(-\infty,x)}(H)V \right) - \int_{-\infty}^{t} \eta_1(s)ds.
\]

(2.8)

Definition 2.14. (See, for example, [8, Section 9.7]). Let \( H \in M_n(\mathbb{C})_{sa} \). The spectral measure of \( H \), denoted \( E \), is a function defined for any (Borel) subset of \( \mathbb{R} \) by:
\[ E(A) = \sum_{\lambda_k \in A} P_k, \]  

(2.9)

where \( \sigma(H) = \{ \lambda_i \}_{i=1}^n \) with associated orthonormal eigenvectors \( \{ f_i \}_{i=1}^n \), and for any \( g \in \mathbb{C}^n \) we have that \( P_k g = \langle g, f_k \rangle f_k \) is the orthogonal projection of \( g \) onto the subspace \( \text{span}\{ f_k \} \).

**Proposition 2.15.** The following properties of the spectral measure can be derived directly from Definition 2.14 and well-known properties of orthogonal projections:

\[
E(A \cap B) = E(A)E(B),
\]

\[
E(A \cup B) = E(A) + E(B) \text{ if } A \cap B = \emptyset,
\]

\[
E(\emptyset) = 0 \text{ and } E(\mathbb{R}) = I.
\]

**Definition 2.16.** (Standard in analysis). We define the function \( x^k_+ \) as follows:

\[
x^k_+ = \begin{cases} x^k & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}, \quad \text{for } k \in \mathbb{N}.
\]

**Definition 2.17.** (See, for example, [4, Equation (1.2)]). Let \( E \) be the spectral measure of \( H \in M_n(\mathbb{C})_{sa} \). We let \( \omega^{(m)} \) denote the function defined as follows, where \( \lambda_i \in \sigma(H) \):

\[
\omega^{(m)}(\lambda_1, \lambda_2, \ldots, \lambda_m) := \text{Tr} \left( E(\lambda_1)VE(\lambda_2)(V) \cdots E(\lambda_m)V \right). \tag{2.10}
\]

**Definition 2.18.** (See, for example, [6, Definition 2.1]). We let \( f^{[n]} \) denote the divided difference of \( f \) of order \( n \). The divided difference of the \( 0 \)th order \( f^{[0]} \) is the function \( f \) itself. Let \( \lambda_1, \ldots, \lambda_n, \lambda_{n+1} \) be points in \( \mathbb{R} \) and let \( f \in C^n(\mathbb{R}) \). The divided difference \( f^{[n]} \) of order \( n \) is defined recursively by:

\[
f^{[n]}(\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_{n+1}) = \begin{cases} f^{[n-1]}(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n) - f^{[n-1]}(\lambda_1, \ldots, \lambda_{n-1}, \lambda_{n+1}) & \text{if } \lambda_n \neq \lambda_{n+1}, \\ \frac{\partial}{\partial \lambda} \bigg|_{\lambda=\lambda_n} f^{[n-1]}(\lambda_1, \ldots, \lambda_{n-1}, \lambda) & \text{if } \lambda_n = \lambda_{n+1}. \end{cases}
\]

**Proposition 2.19.** [5, Section 4.7] Basic properties of the divided difference.

(i) A divided difference is invariant with respect to permutation of its variables.

(ii) \( f^{[n]} \geq 0 \) whenever \( f^{(n)} \geq 0 \).
Example 2.20. Calculation of a divided difference.

Calculate the divided difference \(((\lambda - t)_+)^{[1]}(\lambda_1, \lambda_2)\) for the following values of \(\lambda_1, \lambda_2\).

(i) \(\lambda_1 = \lambda_2 = 1\):

\[
((\lambda - t)_+)^{[1]}(1, 1) = \frac{\partial}{\partial \lambda}|_{\lambda=1}(\lambda - t)_+
\]

\[
= \begin{cases} 
1 & \text{if } t \leq 1 \\
0 & \text{if } t > 1
\end{cases}.
\]

(ii) \(\lambda_1 = \lambda_2 = -1\):

\[
((\lambda - t)_+)^{[1]}(-1, -1) = \frac{\partial}{\partial \lambda}|_{\lambda=-1}(\lambda - t)_+
\]

\[
= \begin{cases} 
1 & \text{if } t \leq -1 \\
0 & \text{if } t > -1
\end{cases}.
\]

(iii) \(\lambda_1 = -1, \lambda_2 = 1\):

\[
((\lambda - t)_+)^{[1]}(-1, 1) = \frac{(-1 - t)_+ - (1 - t)_+}{(-1) - (1)}
\]

\[
= \begin{cases} 
1 & \text{if } t \leq -1 \\
\frac{1-t}{2} & \text{if } -1 < t < 1 \\
0 & \text{if } t \geq 1
\end{cases}.
\]

(iv) \(\lambda_1 = 1, \lambda_2 = -1\):

By Proposition 2.19, \(((\lambda - t)_+)^{[1]}(1, -1) = ((\lambda - t)_+)^{[1]}(-1, 1)\) since the divided difference is always invariable under permutation of its variables.

Theorem 2.21. [6, Theorem 5.1 (ii)] Let \(H, V \in M_n(\mathbb{C})_{sa}, n \in \mathbb{N}\). With \(\eta_1(t)\) and \(\eta_2(t)\) defined as in Definitions 2.11 and 2.13, respectively, the spectral shift functions of higher orders \(p\) satisfy the following recursive formulas:

\[
\eta_p(t) = \frac{\text{Tr} \left( V^2 \right)}{2!} - \int_{-\infty}^{t} \eta_2(s) \, ds - \frac{1}{2} \sum_{\lambda_1, \lambda_2 \in \sigma(H)} ((\lambda - t)_+)^{[1]}(\lambda_1, \lambda_2) \omega^{(2)}(\lambda_1, \lambda_2), \quad (2.11)
\]

\[
\ldots,
\]
\[
\eta_p(t) = \frac{\text{Tr} (V^{p-1})}{(p-1)!} - \int_{-\infty}^{t} \eta_{p-1}(s) \, ds \tag{2.12} \\
= \frac{1}{(p-1)!} \sum_{\lambda_1, \ldots, \lambda_{p-1} \in \sigma(H)} ((\lambda - t)^{p-2})^{[p-2]}(\lambda_1, \ldots, \lambda_{p-1}) \omega^{(p-1)}(\lambda_1, \ldots, \lambda_{p-1}).
\]

3. Zero Even Order Spectral Shift for \( H, V \in M_n(\mathbb{C})_{sa} \)

It is desirable, particularly for purposes of solving certain inverse problems in applications of spectral theory, to understand what conditions on \( V \) are required to produce identically zero spectral shift functions of various orders. In first studying this question, it is simplest to restrict the situation to such functions of even order.

**Proposition 3.1.** [6, Lemma 6.2] Given any \( H, V \in M_n(\mathbb{C})_{sa} \) the following formula is valid for their associated spectral shift functions of all orders \( p \):

\[
\int_{\mathbb{R}} \eta_p(t) \, dt = \frac{\text{Tr} (V^p)}{p!}. \tag{3.1}
\]

**Proposition 3.2.** Let \( H, V \in M_n(\mathbb{C})_{sa} \) and suppose that at least one associated even order spectral shift function \( \eta_{2k}(t) \) is identically zero, i.e. \( \eta_{2k}(t) \equiv 0 \) for some \( k \in \mathbb{N} \). Then \( V = 0 \).

**Proof.** By the spectral theorem, \( V \) is diagonalizable by some unitary matrix \( S \). Thus for any \( p \in \mathbb{N} \) we can write:

\[
V^p = (SDS^{-1})^p = SD^pS^{-1}.
\]

Then, by the cyclic property of the trace:

\[
\text{Tr} (V^p) = \text{Tr} (SD^pS^{-1}) = \text{Tr} (S^{-1}SD^p) = \text{Tr} (D^p).
\]

Combining this with equation (3.1) we have now that, if \( \eta_{2k}(t) \equiv 0 \), then:

\[
\text{Tr} (V^{2k}) = \text{Tr} (D^{2k}) = 0.
\]

In other words, if \( \sigma(V) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) (with all multiplicities included) then:

\[
\sum_{i=1}^{n}(\lambda_i)^{2k} = \sum_{i=1}^{n}(\lambda_i^2)^k = 0.
\]

Since \( V \) is known to have only real eigenvalues, this implies \( \lambda_i = 0 \ \forall \ i = 1, 2, \ldots, n \). Thus \( V \) is the zero matrix.

\( \square \)
It was conjectured by K. Harriger [3, Conjecture 4.12] that if \( \eta_1(t) \equiv 0 \) and \( \eta_2(t) \equiv 0 \) then the matrices \( H \) and \( H + V \) must be such that \( \sigma(H) = \sigma(H + V) \) and have the same eigenvectors. We note that this conjecture is indeed true since Proposition 3.2 applies to the assumption \( \eta_2(t) \equiv 0 \), forcing \( H + V = H \).

**Proposition 3.3.** Let \( H, V \in M_n(\mathbb{C})_{sa} \) with \( V = 0 \). Then \( \eta_p(t) \equiv 0 \forall p \in \mathbb{N} \).

**Proof.** We proceed by induction on the order of \( \eta_p \). In light of Definition 2.12, clearly \( \eta_1(t) \equiv 0 \) whenever \( V = 0 \). Now suppose \( V = 0 \Rightarrow \eta_k(t) \equiv 0 \) for some fixed \( k \in \mathbb{N} \). Referring back to Equation (2.10), note that \( \omega_0^{(p-1)} \equiv 0 \) when \( V = 0 \) since \( \text{Tr}(0) = 0 \) regardless of the values \( \lambda_i \). Then by Equation (2.12) we have:

\[
\eta_{k+1}(t) = -\int_{-\infty}^{t} \eta_k(s) \, ds = -\int_{-\infty}^{t} 0 \, ds \equiv 0, \quad \forall t \in \mathbb{R}.
\]

□

We can now strengthen and prove the result of [3, Conjecture 4.11] as follows.

**Corollary 3.4.** If an even order spectral shift function \( \eta_{2k}(t) \equiv 0 \) for at least one fixed value of \( k \in \mathbb{N} \), then \( \eta_p(t) \equiv 0 \forall p \in \mathbb{N} \).

**Proof.** By Proposition 3.2, we know \( \eta_{2k}(t) \equiv 0 \Rightarrow V = 0 \). Thus \( \eta_p(t) \equiv 0 \) by Proposition 3.3.

□

We can also now prove K. Harriger’s conjecture regarding the effect of a zero first order spectral shift function \( \eta_1(t) \equiv 0 \) on its related even order \( \eta_2(t) \) function.

**Proposition 3.5.** [3, Conjecture 4.13] Let \( H, V \in M_n(\mathbb{C})_{sa} \). If \( \sigma(H) = \sigma(H + V) \), but \( H \) and \( H + V \) have different eigenvectors, then \( \eta_1(t) \equiv 0 \) and \( \eta_2(t) \) is a piecewise constant function.

**Proof.** It has already been rigorously shown in [3, Proposition 3.11] that \( \eta_1(t) \equiv 0 \) under these conditions. Intuitively, the argument is that \( \sigma(H) = \sigma(H + V) \) should lead to \( \eta_1(t) \equiv 0 \) via Definition 2.12 since neither ordering of elements nor their associated eigenvectors have any effect on the relevant set cardinalities. Now when \( \eta_1(t) \equiv 0 \) we see that Equation 2.8 reduces to:

\[
\eta_2(t) = \text{Tr} \left( \chi_{(-\infty,t)}(H)V \right).
\]

Using Equation (2.2) we can rewrite this as:

\[
\eta_2(t) = \text{Tr} \left( \sum_{i=1}^{n} \chi_{(-\infty,t)}(\lambda_i) \mathbf{S} \mathbf{E}_{ii} \mathbf{S}^* V \right).
\]

Consider \( \sigma_o(H) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) arranged in increasing order so that \( \lambda_i \leq \lambda_j \) whenever \( i < j \). This relabeling has no effect on the values of \( \eta_2(t) \) when calculated as in Equation (3.2) since matrix addition is commutative. Then \( \eta_2(t) = 0 \) for
$t \in (-\infty, \lambda_1]$ because $\chi_{(-\infty,t)}(\lambda_i) = 0$ in this range. Moreover, $\chi_{(-\infty,t)}(\lambda_i)$ remains constant throughout any individual interval $t \in (\lambda_k, \lambda_{k+1}]$ as well as for $t \in (\lambda_n, \infty)$. Thus $\eta_2(t)$ is a piecewise constant function.

Note that Proposition 3.5 does not reveal whether it is actually possible for $H$ and $H + V$ to have different eigenvectors when $\eta_1(t) \equiv 0$. The answer to this previously unresolved question is addressed for the $n = 2$ case below in Proposition 4.2 and Example 4.4.

**Theorem 3.6.** [6, Theorem 5.1(ii)] Let $H, V \in M_n(\mathbb{C})_{sa}$, and let $\lambda_{\text{max}}$ be the greatest element of $\sigma(H)$. Then $\eta_p(t) = 0$ for all $t > \lambda_{\text{max}}$.

### 4. Zero First Order Spectral Shift for $H, V \in M_2(\mathbb{C})_{sa}$

We next investigate what it might mean for an odd order spectral shift function to be identically zero, i.e. $\eta_{2k+1}(t) \equiv 0$ for some $k \in \mathbb{N}$. Having established in Section 3 that setting $V = 0$ is sufficient to force spectral shift functions of all orders to be identically zero, we know this is one possibility. However, the general case of odd order zero spectral shift functions is considerably more complicated than that of even order. To achieve $\eta_{2k+1}(t) \equiv 0$ for some $k \in \mathbb{N}$ it is in general sufficient but not necessary for $V$ to be the zero matrix.

**Notations 4.1.** Unless otherwise specified, let $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

**Proposition 4.2.** Let $H = \pm P$. Then $\eta_1(t) \equiv 0$ if and only if $V \in M_2(\mathbb{C})_{sa}$ is of the form:

$$V = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & -v_{11} \end{pmatrix}$$  \hspace{1cm} (4.1)

where

$$v_{11} = \begin{cases} -1 \pm \sqrt{1 - |v_{12}|^2} & \text{for } H = P \\ 1 \pm \sqrt{1 - |v_{12}|^2} & \text{for } H = -P \end{cases} \hspace{1cm} (4.2)$$

and

$$|v_{12}| \leq 1. \hspace{1cm} (4.3)$$

Proof. $\Rightarrow$ Let $H = P$. We know $v_{21} = \overline{v_{12}}$ since $V$ is Hermitian. Further, since $\eta_1(t) \equiv 0$ implies $\text{Tr}(V) = 0$ by Equation (3.1), we must have $v_{22} = -v_{11}$. Then:

$$P + V = \begin{pmatrix} 1 + v_{11} & v_{12} \\ v_{12} & -1 - v_{11} \end{pmatrix},$$

with characteristic equation:
\[ \det(P + V - \lambda I) = (1 + v_{11} - \lambda)(-1 - v_{11} - \lambda) - |v_{12}|^2 = 0, \]

which simplifies to:

\[ v_{11}^2 + 2v_{11} + |v_{12}|^2 + 1 - \lambda^2 = 0. \]

Under the assumption that \( \eta_1(t) \equiv 0 \) we have \( \sigma(P) = \sigma(P + V) = \{-1, 1\} \), so \( 1 - \lambda^2 = 0 \). Thus:

\[ v_{11}^2 + 2v_{11} + |v_{12}|^2 = 0, \]

\[ v_{11} = \frac{-2 \pm \sqrt{4 - 4|v_{12}|^2}}{2} = -1 \pm \sqrt{1 - |v_{12}|^2}. \]

The result for \( v_{11} \) in the case of \( H = -P \) can be similarly verified.

Finally, in both cases the fact that \( V \) must have only real-valued entries on its diagonal results in restriction \( |v_{12}|^2 \leq 1 \).

\[ \iff \] In the other direction, if \( V \) is known to be of the specified form then, in either case of \( H = P \) or \( H = -P \), we have:

\[ H + V = \left( \begin{array}{cc} \sqrt{1 - |v_{12}|^2} & v_{12} \\ \frac{v_{12}}{v_{12}} & -\sqrt{1 - |v_{12}|^2} \end{array} \right) \text{ or } \left( \begin{array}{cc} -\sqrt{1 - |v_{12}|^2} & v_{12} \\ \frac{v_{12}}{v_{12}} & \sqrt{1 - |v_{12}|^2} \end{array} \right). \]

Both cases yield the same characteristic equation:

\[ \det(H + V - \lambda I) = (\sqrt{1 - |v_{12}|^2} - \lambda)(-\sqrt{1 - |v_{12}|^2} - \lambda) - |v_{12}|^2 = 0, \]

\[ -1 + |v_{12}|^2 + \lambda^2 - |v_{12}|^2 = 0, \]

\[ \lambda^2 - 1 = 0. \]

Thus \( \sigma(H + V) = \{-1, 1\} = \sigma(H) \) and \( \eta_1(t) \equiv 0. \)

\[ \Box \]

**Remark 4.3.** The reader may find it peculiar, or at least extremely restrictive, that we have considered only two out of infinitely many matrices \( H \in M_2(\mathbb{C})_{sa} \) to which one may wish to apply the theory of spectral shift functions. However, the author was advised by her thesis mentor Dr. Anna Skripka that obtaining results for these special choices of \( H \) is in fact sufficient to treat the general case. Dr. Skripka is currently refining as-yet unpublished methods of extending such formulas to work for any given \( H \) once appropriate results for these special \( H \) are known.
Example 4.4. The $n = 2$ case of $\eta_1(t) \equiv 0$ with special $H$, different eigenvectors before vs. after perturbation.

Let $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = P$, and apply Proposition 4.2.

$\sigma(H) = \{1, -1\}$

Eigenbasis of $H$: $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$

By Equation (4.3), we are free to assign any value to $v_{12}$ so long as it has squared magnitude less than or equal to 1. Here we arbitrarily choose $v_{12} = \frac{1}{2} + \frac{2}{3}i$ so that $|v_{12}|^2 = \frac{1}{4} + \frac{4}{9} = \frac{25}{36} < 1$. We then calculate $V$ from Equations (4.1) and (4.2) as follows.

$V = \begin{pmatrix} -1 + \sqrt{1 - \frac{25}{36}} & \frac{1}{2} + \frac{2}{3}i \\ \frac{1}{2} - \frac{2}{3}i & 1 - \sqrt{1 - \frac{25}{36}} \end{pmatrix} = \begin{pmatrix} -1 + \sqrt{\frac{11}{6}} & \frac{1}{2} + \frac{2}{3}i \\ \frac{1}{2} - \frac{2}{3}i & 1 - \sqrt{\frac{11}{36}} \end{pmatrix}$

$H + V = \begin{pmatrix} \sqrt{\frac{11}{36}} & \frac{1}{2} + \frac{2}{3}i \\ \frac{1}{2} - \frac{2}{3}i & -\sqrt{\frac{11}{36}} \end{pmatrix}$

$\sigma(H + V) = \{1, -1\}$

Eigenbasis of $H + V$: $\left\{ \begin{pmatrix} \frac{1}{6-\sqrt{11}} \\ 3+4i \end{pmatrix}, \begin{pmatrix} -\frac{1}{6-\sqrt{11}} \\ 3+4i \end{pmatrix} \right\}$

Corollary 4.5. Let $V \in M_2(\mathbb{C})_{sa}$, $H = P$, $\eta_1(t) \equiv 0$, and suppose $H + V$ has the same eigenbasis as $H$. Then $V = 0$ or $V = -2H$.

Proof. If $H + V$ and $H$ share their eigenbasis (in which $H$ is already represented here), then $H + V$ is diagonal and so is $V$, i.e. $v_{12} = \overline{v_{12}} = 0$. Therefore by Proposition 4.2 $v_{11} = -1 \pm 1 = 0, -2$. Thus:

$V = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $V = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$.

It is worth noting here that under the given special conditions of Corollary 4.5 setting $V = 0$ corresponds to the unique way to achieve the same eigenvalues in the same order for $H$ and $H + V$, and setting $V = -2H$ is the unique way to achieve the same two eigenvalues permuted from the original order.

We can immediately extend these results to calculate $V$ such that $\eta_1(t) \equiv 0$ given any $H$, retaining the restriction that $H$ and $H + V$ should share an eigenbasis.

Notations 4.6. In reference to a given matrix $H \in M_2(\mathbb{C})_{sa}$ with $\sigma(H) = \{\lambda_1, \lambda_2\}$, let $D$ be as follows so that $H = SDS^*$ for some unitary $S \in M_2(\mathbb{C})$. 

\[
D = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}.
\]

In most cases the context of the matrix \( D \) being associated specifically with the matrix \( H \) will be clear. If not, the notation \( D_H \) will be used.

Further, let:
\[
x_1 = \frac{\lambda_1 + \lambda_2}{2}, \quad x_2 = \frac{\lambda_1 - \lambda_2}{2}.
\]

**Remark 4.7.** It is interesting to note that any diagonal matrix with \( n = 2 \) can be represented as a linear combination of the identity matrix \( I \) and the matrix \( P \) as in Notations 4.1. Here we can write \( D = x_1 I + x_2 P \). Thus in general for any normal \( H \in M_2(\mathbb{C}) \) with \( H = SDS^* \) we can write:
\[
H = S(x_1 I + x_2 P)S^* = x_1 I + x_2 P S S^*.
\]

**Proposition 4.8.** Let \( H, V \in M_2(\mathbb{C})_{sa} \) be such that \( \eta_1(t) \equiv 0 \), with \( S \) being the unitary matrix consisting of the normalized eigenvectors of \( H \) so that \( H = SDS^* \), and let \( H \) and \( H + V \) be known to share these eigenvectors. Then \( V = 0 \) or \( V = (\lambda_2 - \lambda_1) P S S^* \).

**Proof.** Since \( H \) and \( H + V \) share their eigenbasis, we know \( H + V = SMS^* \) for some diagonal \( M \in M_2(\mathbb{C})_{sa} \). If the eigenvectors are paired to the same eigenvalues, then \( H + V = SDS^* = H \) so \( V = 0 \). If the eigenvectors are paired to the opposite eigenvalues, then
\[
H + V = S \begin{pmatrix}
\lambda_2 & 0 \\
0 & \frac{\lambda_1 - \lambda_2}{2}
\end{pmatrix} S^* = S \left[ \begin{pmatrix}
\lambda_1 & 0 \\
0 & \frac{\lambda_1 - \lambda_2}{2}
\end{pmatrix} + \begin{pmatrix}
\lambda_2 - \lambda_1 & 0 \\
0 & \lambda_1 - \lambda_2
\end{pmatrix} \right] S^*
= SDS^* + (\lambda_2 - \lambda_1) P S S^* = H + (\lambda_2 - \lambda_1) P S S^*.
\]

**Corollary 4.9.** Let \( H, V \in M_2(\mathbb{C})_{sa} \) be such that \( \eta_1(t) \equiv 0 \), with \( \sigma(H) = \{\lambda_1, \lambda_2\} \). Then \( V = 0 \) if \( \lambda_1 = \lambda_2 \). If \( \lambda_1 \neq \lambda_2 \), then there are exactly \( 2! = 2 \) distinct permissible choices of \( V \) (including \( V = 0 \)) such that \( H \) and \( H + V \) share an eigenbasis.

**Proof.** By Proposition 4.8, we already know that there are exactly two (not necessarily distinct) permissible choices of \( V \). If \( \lambda_1 = \lambda_2 \), then the second option of \( V = (\lambda_2 - \lambda_1) P S S^* \) is identical to the first option of \( V = 0 \). Otherwise, the two choices are distinct.
While Corollary 4.9 may not seem particularly exciting at first glance, it is important for later insight into extending the theory from this case of $H \in M_2(\mathbb{C})_{sa}$ to the more general case of $H \in M_n(\mathbb{C})_{sa}$.

**Example 4.10.** The $n = 2$ case of $\eta_1(t) \equiv 0$ with same eigenvectors, permuted eigenvalues.

Let $H = \begin{pmatrix} 7 & i \\ -i & 3 \end{pmatrix}$ and apply Proposition 4.8.

$$\sigma(H) = \{5 + \sqrt{5}, 5 - \sqrt{5}\}$$

$$H = SDS^* = \begin{pmatrix} \sqrt{10 - 4\sqrt{5}} & \sqrt{10 + 4\sqrt{5}} \\ (2 - \sqrt{5})i & (2 + \sqrt{5})i \end{pmatrix} \begin{pmatrix} 5 + \sqrt{5} & 0 \\ 0 & 5 - \sqrt{5} \end{pmatrix} \begin{pmatrix} \sqrt{10 - 4\sqrt{5}} & (-2 + \sqrt{5})i \\ \sqrt{10 + 4\sqrt{5}} & \sqrt{10 + 4\sqrt{5}} \end{pmatrix}$$

Eigenbasis of $H$: $\left\{ \begin{pmatrix} 1 \\ (2 - \sqrt{5})i \end{pmatrix}, \begin{pmatrix} 1 \\ (2 + \sqrt{5})i \end{pmatrix} \right\}$

$$V = (\lambda_2 - \lambda_1)SPS^* = -2\sqrt{5} \begin{pmatrix} \sqrt{10 - 4\sqrt{5}} & \sqrt{10 + 4\sqrt{5}} \\ (2 - \sqrt{5})i & (2 + \sqrt{5})i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{10 - 4\sqrt{5}} & (-2 + \sqrt{5})i \\ \sqrt{10 + 4\sqrt{5}} & \sqrt{10 + 4\sqrt{5}} \end{pmatrix}$$

$$\Rightarrow V = \begin{pmatrix} -4 & -2i \\ 2i & 4 \end{pmatrix}$$

$$H + V = \begin{pmatrix} 3 & -i \\ i & 7 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{10 - 4\sqrt{5}} & \sqrt{10 + 4\sqrt{5}} \\ (2 - \sqrt{5})i & (2 + \sqrt{5})i \end{pmatrix} \begin{pmatrix} 5 - \sqrt{5} & 0 \\ 0 & 5 + \sqrt{5} \end{pmatrix} \begin{pmatrix} \sqrt{10 - 4\sqrt{5}} & (-2 + \sqrt{5})i \\ \sqrt{10 + 4\sqrt{5}} & \sqrt{10 + 4\sqrt{5}} \end{pmatrix}$$

$$\sigma(H + V) = \{5 - \sqrt{5}, 5 + \sqrt{5}\}$$

Eigenbasis of $H + V$: $\left\{ \begin{pmatrix} 1 \\ (2 + \sqrt{5})i \end{pmatrix}, \begin{pmatrix} 1 \\ (2 - \sqrt{5})i \end{pmatrix} \right\}$

Example 4.10 suggests the following alternative way to view a special case of Proposition 4.8.

**Proposition 4.11.** Let $H = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in M_2(\mathbb{C})_{sa}$ with Re$(b) = 0$, $V \in M_2(\mathbb{C})_{sa}$ be such that $\eta_1(t) \equiv 0$, with $\sigma(H) = \{\lambda_1, \lambda_2\}$ and $H = SDS^*$. Further let $H$ and $H + V$
be known to have the same eigenvectors. Then \( H + V = H \) or \( H + V = \begin{pmatrix} d & \overline{b} \\ b & a \end{pmatrix} \).

That is, \( V = 0 \) or \( V = \begin{pmatrix} d - a & \overline{b} - b \\ b - \overline{b} & a - d \end{pmatrix} \).

Proof. By Proposition 4.8 there is at most one nonzero choice of \( V \) satisfying these conditions. Thus it suffices to show that the proposed choice of \( V \) is a valid nonzero option to force \( \eta_1(t) \equiv 0 \) without a shared eigenbasis. Since \( H \) is Hermitian, we know that \( a, d \in \mathbb{R} \). It is then easily verified that our proposed choice of \( V \) is also Hermitian, since \( \mathbb{R} \) is closed under subtraction and \( \overline{b} - b = -2 \text{Im}(b)i = 2 \text{Im}(b)i = b - \overline{b} \). (Note that \( V \) is nonzero unless \( a = d \) and \( b = \overline{b} = 0 \), i.e. \( V = 0 \) if and only if \( H \) is diagonal with repeated eigenvalues. In that case we have already shown that there is no nonzero choice as the factor of \( (\lambda_2 - \lambda_1) \) from Proposition 4.8 becomes zero.) In order to verify that \( \eta_1(t) \equiv 0 \) we note that \( H \) and \( H + V \) share the same characteristic polynomial:

\[
(a - \lambda)(d - \lambda) - |b|^2 = 0.
\]

This leads to \( \sigma(H) = \sigma(H + V) \Rightarrow \eta_1(t) \equiv 0 \) with shared eigenvalues

\[
\lambda = \frac{1}{2}[a + d \pm \sqrt{(a - d)^2 + 4|b|^2}].
\]

All that remains to be shown now is that \( H \) and \( H + V \) have a shared eigenbasis. A standard calculation of the associated eigenvectors for each matrix yields

Eigenbasis of \( H = \left\{ \begin{pmatrix} 1 \\ d - a + \sqrt{(a - d)^2 + 4|b|^2} \\ 2b \end{pmatrix}, \begin{pmatrix} 1 \\ d - a - \sqrt{(a - d)^2 + 4|b|^2} \\ 2b \end{pmatrix} \right\}, \right. \)

Eigenbasis of \( H + V = \left\{ \begin{pmatrix} 1 \\ a - d + \sqrt{(a - d)^2 + 4|b|^2} \\ 2b \end{pmatrix}, \begin{pmatrix} 1 \\ a - d - \sqrt{(a - d)^2 + 4|b|^2} \\ 2b \end{pmatrix} \right\}. \right.

These two eigenbases are in fact the same by the following manipulations:

\[
\frac{a - d + \sqrt{(a - d)^2 + 4|b|^2}}{2b} = -\frac{b}{\overline{b}} \times \frac{d - a - \sqrt{(a - d)^2 + 4|b|^2}}{2b},
\]

\[
\frac{a - d - \sqrt{(a - d)^2 + 4|b|^2}}{2b} = -\frac{b}{\overline{b}} \times \frac{d - a + \sqrt{(a - d)^2 + 4|b|^2}}{2b},
\]

\[
-\frac{b}{\overline{b}} = -\frac{b}{-b} = 1. \quad (4.4)
\]

Thus the original eigenvectors have simply been permuted in relation to the shared eigenvalues.

\[ \square \]
Remark 4.12. For the proof of Proposition 4.11, the restriction that $\text{Re}(b) = 0$ was necessary only to justify (4.4). All prior steps would remain valid with respect to any $H \in M_2(\mathbb{C})_{sa}$. Therefore the suggested choice of $V$ is still valid to guarantee $\eta_1(t) \equiv 0$ even if $\text{Re}(b) \neq 0$. However, in that case this $V$ would be just one of the infinitely many allowable choices of $V$ with $H$ and $H + V$ not sharing an eigenbasis.

Example 4.13. Simplified calculation of $V$ when $\text{Re}(b) = 0$.

Let $H = \begin{pmatrix} 35 & 22i \\ -22i & 547 \end{pmatrix}$, and apply Proposition 4.11.

$$V = \begin{pmatrix} 547 - 35 & -22i - 22i \\ 22i - (-22i) & 35 - 547 \end{pmatrix} = \begin{pmatrix} 512 & -44i \\ 44i & -512 \end{pmatrix}.$$ 

$$H + V = \begin{pmatrix} 547 & -22i \\ 22i & 35 \end{pmatrix}.$$ 

We are guaranteed by Proposition 4.11 that for this choice of $V$, $\eta_1(t) \equiv 0$ and $H$ and $H + V$ share an eigenbasis (with permutation). We could of course verify exactly what the shared eigenvalues and eigenvectors are after the initial calculation of $V$, if desired. Those results are shown below:

$$\sigma(H) = \sigma(H + V) = \{291 \pm 2\sqrt{16,505}\},$$

Shared Eigenbasis: \[ \left\{ \left( \begin{array}{c} 128 + \sqrt{16,505} \\ 11i \end{array} \right), \left( \begin{array}{c} 128 - \sqrt{16,505} \\ 11i \end{array} \right) \right\}. \]

In Example 4.13 on the special case satisfying $\text{Re}(b) = 0$, we performed a much simpler calculation than using the more general formula $V = (\lambda_1 - \lambda_2)SPS^*$ which requires the extra work of diagonalizing $H$ as in Example 4.10.

Determining the unitary matrix $S$ for which $H = SDS^*$, $V = (\lambda_1 - \lambda_2)SPS^*$ then requires yet another step of normalizing the above eigenvectors. Thus we see that use of Proposition 4.11 is indeed useful whenever applicable.

Definition 4.14. Let $M \in M_n(\mathbb{C})$. For this finite situation the norm of $M$, denoted $||M||$ can be defined as the greatest eigenvalue of $M$:

$$||M|| = \lambda \in \sigma(M) \text{ such that } \lambda \geq \lambda_i \forall \lambda_i \in \sigma(M). \quad (4.5)$$

Conjecture 4.15. Let $H \in M_2(\mathbb{C})_{sa}$. Then $\eta_1(t) \equiv 0 \Rightarrow |v_{12}|^2 \leq ||H||$ (where $V$ has the same form as in Equation (4.1)).

This conjecture is a logical potential generalization of Equation (4.3). We note that it does hold for the specific $V$ calculated in Example 4.13:
\[ |v_{12}|^2 = 22^2 < 291 + 2\sqrt{16,505} = ||H||. \]

5. Zero First Order Spectral Shift for \( H, V \in M_n(\mathbb{C})_{sa} \)

**Proposition 5.1.** For \( H \in M_n(\mathbb{C})_{sa} \) there are at most \( n! \) distinct choices of \( V \) for which \( \eta_1(t) \equiv 0 \) such that \( H \) and \( H + V \) share an eigenbasis. There are exactly \( n! \) choices of such \( V \) if and only if \( H \) has \( n \) distinct eigenvalues.

*Proof.* Let \( S \) be the unitary matrix such that \( H = S(D_H)S^* \). Then, since \( H \) and \( H + V \) share an eigenbasis, \( H + V = S(D_{H+V})S^* \). Moreover, since \( \sigma(H) = \sigma(H+V) \), the nonzero entries of the diagonal matrix \( D_{H+V} \) are some permutation of the \( n \) diagonal entries of \( D_H \). Thus the permissible choices of \( V \), which are
\[
V = S(D_{H+V})S^* - S(D_H)S^* = S(D_{H+V} - D_H)S^* ,
\]
correspond directly to the ways in which it is possible to form permutations of a list of the elements of \( \sigma(H) \). It is a well-known combinatorial fact that there are \( n! \) (not necessarily distinct) ways to permute such a list. If no two elements are the same, then all of these permutations are distinct. If there are any repeated elements, then fewer than \( n! \) permutations are distinct.

\[ \square \]

Given a finite \( H \in M_n(\mathbb{C})_{sa} \), the up to \( n! \) matrices \( V \) such that \( \eta_1 \equiv 0 \) and \( H \) and \( H + V \) share an eigenbasis can be explicitly calculated.

**Example 5.2.** Calculation of appropriate \( V \) to satisfy \( \eta_1(t) \equiv 0 \) with shared eigenbasis, complete treatment of \( H \in M_3(\mathbb{C})_{sa} \).

Let \( H \in M_3(\mathbb{C})_{sa} \) with eigenvalues originally ordered as \( \lambda_1 = a, \lambda_2 = b, \lambda_3 = c \). The choice \( V = 0 \) corresponds to the permutation which leaves this order unchanged. The other five possible permutations are \( \lambda_1, \lambda_2, \lambda_3 \) equal, respectively, to:

(i) \( a, c, b; \)
(ii) \( b, a, c; \)
(iii) \( b, c, a; \)
(iv) \( c, a, b; \)
(v) \( c, b, a. \)

Each of (i) through (v) corresponds to a formula for an allowable choice of \( V \). We calculate these formula associated with the permutation as follows, making use of the decomposition form of Proposition 2.4, i.e. \( H = \sum_{i=1}^{n} \lambda_i S E_{ii} S^* \).

(i) \( a, b, c \rightarrow a, c, b \)
\[ V = (H + V) - H \]

\[ = [a(SE_{11}S^*) + c(SE_{22}S^*) + b(SE_{33}S^*)] - [a(SE_{11}S^*) + b(SE_{22}S^*) + c(SE_{33}S^*)] \]

\[ = [(a - a)(SE_{11}S^*) + (c - b)(SE_{22}S^*) + (b - c)(SE_{33}S^*)] \]

\[ = [(c - b)(SE_{22}S^*) - (c - b)(SE_{33}S^*)] \]

\[ = [(c - b)(SE_{22}S^*) - (c - b)(SE_{33}S^*)] \]

\[ = (c - b)S \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right) S^* = S \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & c - b & 0 \\ 0 & 0 & b - c \end{array} \right) S^*. \]

(ii) \( a, b, c \rightarrow b, a, c \)

\[ V = (H + V) - H \]

\[ = [b(SE_{11}S^*) + a(SE_{22}S^*) + c(SE_{33}S^*)] - [a(SE_{11}S^*) + b(SE_{22}S^*) + c(SE_{33}S^*)] \]

\[ = [(b - a)(SE_{11}S^*) + (a - b)(SE_{22}S^*) + (c - c)(SE_{33}S^*)] \]

\[ = [(b - a)(SE_{11}S^*) - (b - a)(SE_{22}S^*)] \]

\[ = [(b - a)(SE_{11}S^*) - (b - a)(SE_{22}S^*)] \]

\[ = (b - a)S \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) S^* = S \left( \begin{array}{ccc} b - a & 0 & 0 \\ 0 & a - b & 0 \\ 0 & 0 & 0 \end{array} \right) S^*. \]

(iii) \( a, b, c \rightarrow b, c, a \)

\[ V = (H + V) - H \]

\[ = [b(SE_{11}S^*) + c(SE_{22}S^*) + a(SE_{33}S^*)] - [a(SE_{11}S^*) + b(SE_{22}S^*) + c(SE_{33}S^*)] \]

\[ = [(b - a)(SE_{11}S^*) + (c - b)(SE_{22}S^*) + (a - c)(SE_{33}S^*)] \]

\[ = S \left[ (b - a) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) + (c - b) \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) + (a - c) \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) \right] S^* \]
\[ V = S \begin{pmatrix} c - a & 0 & 0 \\ 0 & a - b & 0 \\ 0 & 0 & b - c \end{pmatrix} S^*. \]

(v) \( a, b, c \rightarrow c, b, a \)

\[ V = S \begin{pmatrix} c - a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a - c \end{pmatrix} S^*. \]

Again, these five choices are all distinct from each other and from the sixth choice of \( V = 0 \) only when there are no eigenvalues of algebraic multiplicity greater than one contained in \( \sigma(H) \) (i.e., the characteristic polynomial of \( H \) has no repeated roots). Otherwise, degeneracies occur.

**Proposition 5.3.** Let \( H \in M_n(\mathbb{C})_{sa}, \eta_1(t) \equiv 0, \) and suppose \( H+V \) has the same eigenbasis as \( H. \) Then all entries of each of the up to \( n! \) possible choices of \( V \) can be explicitly calculated and is of the form:

\[ V = V_{[\mu_1, \mu_2, \ldots, \mu_n]} = S \begin{pmatrix} \mu_1 - \lambda_1 & 0 & \cdots & 0 \\ 0 & \mu_2 - \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n - \lambda_n \end{pmatrix} S^*, \quad (5.1) \]

where

\[ H = SDS^* = S \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} S^*, \]

and \( \mu_1, \mu_2, \ldots, \mu_n \) is a permutation of \( \lambda_1, \lambda_2, \ldots, \lambda_n. \)

**Proof.** If \( H \) is as above, then \( (H + V)_{[\mu_1, \mu_2, \ldots, \mu_n]} = S \begin{pmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n \end{pmatrix} S^* \) is the
matrix which results from perturbing $H$ such that $\eta_1(t) \equiv 0$, $H$ and $H + V$ share an eigenbasis, and $H + V$ has reordered eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$. All associated choices of $V$ can be calculated as follows.

$$\lambda_1, \lambda_2, \ldots, \lambda_n \rightarrow \mu_1, \mu_2, \ldots, \mu_n$$

$$V_{[\mu_1, \mu_2, \ldots, \mu_n]} = (H + V)_{[\mu_1, \mu_2, \ldots, \mu_n]} - H$$

$$V_{[\mu_1, \mu_2, \ldots, \mu_n]} = \sum_{i=1}^{n} \mu_i S E_{ii} S^* - \sum_{i=1}^{n} \lambda_i S E_{ii} S^* = \sum_{i=1}^{n} (\mu_i - \lambda_i) S E_{ii} S^* \quad (5.2)$$

$$V_{[\mu_1, \mu_2, \ldots, \mu_n]} = S \begin{pmatrix} \mu_1 - \lambda_1 & 0 & \ldots & 0 \\ 0 & \mu_2 - \lambda_2 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_n - \lambda_n \end{pmatrix} S^*.$$

\[ \square \]

**Example 5.4.** Calculation of a nonzero $V$ such that $\eta_1 \equiv 0$ and $H$ and $H + V$ share an eigenbasis, for $H \in M_5(\mathbb{C})_{sa}$.

Let $H = S \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} S^*$, where $S$ is some appropriate unitary matrix in $M_5(\mathbb{C})$ which results in $H$ being Hermitian, and use Equation (5.1) to calculate $V_{[3, 1, 5, 4, 2]}$ so that $\eta_1(t) \equiv 0$ with a shared eigenbasis.

$$V_{[3, 1, 5, 4, 2]} = S \begin{pmatrix} 3 - 1 & 0 & 0 & 0 & 0 \\ 0 & 1 - 2 & 0 & 0 & 0 \\ 0 & 0 & 5 - 3 & 0 & 0 \\ 0 & 0 & 4 - 4 & 0 & 0 \\ 0 & 0 & 0 & 2 - 5 & 0 \end{pmatrix} S^* = S \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} S^*$$

$\sigma(H) = \sigma(H + V)_{[3, 1, 5, 4, 2]}$

In the simplest case of $S$ being the $5 \times 5$ identity matrix $I$, our original pairing of eigenvalues to eigenvectors of $H$ would be $\{ \lambda_i, (f_i) \}$ equal to:
After the perturbation, our pairing of eigenvalues to eigenvectors of $(H + V)_{[3,1,5,4,2]}$ would be \( \{\lambda_i, (f_i)\} \) equal to:

\[
\left\{ \begin{array}{c}
3, \\
1, \\
5, \\
4, \\
2,
\end{array} \right. \quad \left\{ \begin{array}{c}
\left( \begin{array}{cccc}
1 \\
0 \\
0 \\
0
\end{array} \right), \\
\left( \begin{array}{cccc}
0 \\
1 \\
0 \\
0
\end{array} \right), \\
\left( \begin{array}{cccc}
0 \\
0 \\
1 \\
0
\end{array} \right), \\
\left( \begin{array}{cccc}
0 \\
0 \\
0 \\
1
\end{array} \right), \\
\left( \begin{array}{cccc}
0 \\
0 \\
0 \\
1
\end{array} \right) \right. \right\}.
\]

Proposition 5.5 (Well-known from combinatorics). Let \( A = a_1, a_2, \ldots, a_n \) be any list of objects, and let \( p \leq n \) of these elements be distinct. Let \( B = b_1, b_2, \ldots, b_p \) be a list of those \( p \) distinct objects, and let \( m_i \) denote the multiplicity of \( b_i \) with respect to list \( A \). Then the number of distinct permutations of the list \( A \) is:

\[
\frac{n!}{(m_1!)(m_2!)(m_3!)}.
\]

(5.3)

Example 5.6. Calculation of the number of distinct permutations of a string of letters.

Determine the number of distinct permutations of the letters contained in the string spectralshiftfunctions.

There are a total of 22 letters in this string. There are 4 letters with multiplicity 2 (i.e. c, i, f, n). There are 2 letters with multiplicity 3 (i.e. s, t). All other letters have multiplicity 1 (i.e. p, e, r, a, l, h, u, o). Therefore by Equation (5.3), the number of distinct permutations is:

\[
\frac{22!}{2!2!2!3!3!} = \frac{(2)(3)(4)(5)(6)\ldots(22)}{(2^6)(3^2)} \approx 1.95139 \times 10^{18}.
\]

Thus there are nearly two quintillion distinct permutations.

Corollary 5.7. Let \( H \in M_n(\mathbb{C})_{sa} \) with \( \sigma(H) = \lambda_1, \lambda_2, \ldots, \lambda_n \), and let \( \sigma_0(H) = \mu_1, \mu_2, \ldots, \mu_p \) represent the list of all distinct elements of \( \sigma_H \). Further suppose each \( \mu_i \in \sigma_0(H) \) has a corresponding multiplicity of \( m_i \) with respect to \( \sigma_H \). Then the
number of distinct choices of $V$ such that $\eta_1(t) \equiv 0$ and $H$ and $H + V$ share an eigenbasis is:

$$n! \frac{1}{(m_1!)(m_2!)(m_3!)}.$$ 

**Sketch of Proof.** Apply Equation (5.3) to the result in the proof of Proposition 5.1 that each permissible choice of $V$ corresponds to a specific permutation of the list of eigenvalues of $H$. 

**Conjecture 5.8.** For $H \in M_n(\mathbb{C})_{sa}$ there are infinitely many choices of $V$ for which $\eta_1(t) \equiv 0$ such that $H$ and $H + V$ do not share an eigenbasis.

We have already shown via Proposition 4.2 that Conjecture 5.8 holds when working in $M_2(\mathbb{C})_{sa}$. For $n = 3$, it is likely possible (though much more tedious) to use a similar method along with Cardano’s formula for cubics to derive explicit formulas describing infinitely many appropriate choices of $V \in M_3(\mathbb{C})_{sa}$. We could also theoretically use the same process to solve this problem in $M_4(\mathbb{C})_{sa}$, though of course this would require the general formulas regarding quartic polynomials and thus be even more tedious than when $n = 3$.

Explicit algebraic formulas determining all entries of these $V$ may not exist when $n \geq 5$ as deriving such formulas may be equivalent to the mathematically forbidden (per the famous Abel-Ruffini Theorem) feat of solving quintic and higher degree polynomials. This is due to the general connection of such $V$ to the characteristic polynomial for $H + V$ in a similar manner to that used earlier in the proof of Proposition 4.2.

In order to prove that these formulas cannot exist, we would have to verify that any method of deriving such formulas would actually be equivalent to solving quintic or higher order characteristic polynomials of Hermitian matrices, and that this is ultimately equivalent to solving general quintic polynomials outside the context of Hermitian matrices.

6. **Zero Third Order Spectral Shift for $H, V \in M_2(\mathbb{C})_{sa}$**

**Proposition 6.1.** Let $H, V \in M_n(\mathbb{C})_{sa}$, and let $E$ be the spectral measure of $H$ as in Definition 2.14. Recall also Definition 2.17, noting that:

$$\omega^{(2)}(\lambda_1, \lambda_2) = \text{Tr} (E(\lambda_1)VE(\lambda_2)V).$$

Further let $f_1, f_2$ be the orthonormal eigenvectors of $H$ associated with the eigenvalues $\lambda_1, \lambda_2$, respectively, i.e. $\|f_1\| = \|f_2\| = 1$ and $H\lambda_1 = \lambda_1 f_1, H\lambda_2 = \lambda_2 f_2$. Then:

$$\omega^{(2)}(\lambda_1, \lambda_2) = |\langle Vf_1, f_2 \rangle|^2.$$  (6.1)
Proof. This fact was conveyed to the author by A. Skripka. The proof relies on properties of complex inner products and on the fact that for any matrix $M \in M_2(\mathbb{C})$ we can write $\text{Tr}(M) = \langle Mf_1, f_1 \rangle + \langle Mf_2, f_2 \rangle$.

\[ \square \]

Notations 6.2. Let $p_1$, $p_2$ denote normalized eigenvectors paired, respectively, to the eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$ of $P$ (with $P$ as in Notations 4.1). That is, let:

\[
p_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.
\]

Given $H \in M_2(\mathbb{C})_{sa}$, the we began investigating the case of $\eta_3(t) \equiv 0$ by first restricting conditions to $\eta_1(t) \equiv 0$ and setting $H = P$ as in Notations 4.1. It was suspected that the facts that $\eta_2(t)$ is piecewise constant (see Proposition 3.5) in this situation and $\eta_3(t)$ depends on $\eta_2(t)$ per Equation (2.11) might limit the choices of $V$. Indeed, this turns out to be true.

Proposition 6.3. Let $H = P$, and let $V$ be such that $\eta_1(t) \equiv 0$. Moreover, suppose $\eta_3(t) \equiv 0$. Then $V = 0$ or $V = \begin{pmatrix} -1 & v_{12} \\ v_{12} & 1 \end{pmatrix}$ with $|v_{12}| = 1$.

Proof. Let:

\[
\mathcal{Y}(t) = \sum_{\lambda_1, \lambda_2 \in \{-1, 1\}} ((\lambda - t)^+)^{[1]}(\lambda_1, \lambda_2)\omega^{(2)}(\lambda_1, \lambda_2)
\]

(6.2)

Then, referring back to Equation (2.11), we see that $\eta_3(t) \equiv 0$ implies:

\[
\mathcal{Y}(t) = \text{Tr}(V^2) - 2 \int_{-\infty}^{t} \eta_2(s) \, ds \quad \forall \ t \in (-\infty, \infty).
\]

(6.3)

We now investigate the consequences of this statement by evaluating the right-hand side of Equation (6.2) as follows.

\[
\mathcal{Y}(t) = ((\lambda - t)^+)^{[1]}(-1, 1)\omega^{(2)}(-1, 1) + ((\lambda - t)^+)^{[1]}(1, -1)\omega^{(2)}(1, -1)
\]

\[
+ ((\lambda - t)^+)^{[1]}(-1, -1)\omega^{(2)}(-1, -1) + ((\lambda - t)^+)^{[1]}(1, 1)\omega^{(2)}(1, 1)
\]

Then, according to Proposition 6.1 and the calculations of $((\lambda - t)^+)^{[1]}(\lambda_1, \lambda_2)$ from Example 2.20, for the interval $t \in (-1, 1)$ we have:

\[
\mathcal{Y}(t) = \frac{1 - t}{2} \left[ \omega^{(2)}(-1, 1) \right] + \frac{1 - t}{2} \left[ \omega^{(2)}(1, -1) \right] + 0 + \omega^{(2)}(1, 1)
\]

\[
\mathcal{Y}(t) = \frac{1 - t}{2} \left( |\langle VP_1, p_2 \rangle|^2 + |\langle VP_2, p_1 \rangle|^2 \right) + |\langle VP_1, p_1 \rangle|^2
\]

(6.4)
For conciseness, let:

\[ Z = |\langle Vp_1, p_2 \rangle|^2 + |\langle Vp_2, p_1 \rangle|^2 \]  

(6.5)

Now, setting Equation (6.4) equal to Equation (6.3):

\[
\left(1 - \frac{t^2}{2}\right) Z + |\langle Vp_1, p_1 \rangle|^2 = \text{Tr} (V^2) - 2 \int_{-\infty}^{t} \eta_2(s) \, ds,
\]

\[
\frac{1}{2} Zt = \frac{1}{2} Z + |\langle Vp_1, p_1 \rangle|^2 + 2 \int_{-\infty}^{t} \eta_2(s) \, ds - \text{Tr} (V^2).
\]  

(6.6)

By Proposition 3.5, since we have restricted ourselves to the case of \( \eta_1(t) \equiv 0 \), we know that \( \eta_2(t) = \alpha \) is constant in this interval \( t \in (-1, 1) \) and \( \eta_2(t) = 0 \) for \( t \in (-\infty, -1) \). Thus:

\[
\int_{-\infty}^{t} \eta_2(s) \, ds = \int_{-\infty}^{-1} 0 \, ds + \int_{-1}^{t} \alpha \, ds,
\]

\[
\int_{-\infty}^{t} \eta_2(s) \, ds = \alpha (t + 1).
\]  

(6.7)

Combining Equations (6.6) and (6.7) yields:

\[
\frac{1}{2} (Z - 2\alpha) t = 2\alpha - \text{Tr} (V^2) + \frac{1}{2} Z + |\langle Vp_1, p_1 \rangle|^2.
\]  

(6.8)

This is the point in the investigation of these formulas at which it was suspected it might be possible to derive restrictions on \( V \). Algebraically, Equation (6.8) can hold for all \( t \in (-1, 1) \) only if the coefficient of \( t \) on the left-hand side and the expression on the right-hand side are both equal to zero. Noting this, we obtain (with \( V \) as in Proposition 4.2):

\[ Z = 2\alpha, \]

\[ \text{Tr} (V^2) = 3\alpha + |\langle Vp_1, p_1 \rangle|^2 = 3\alpha + |v_{11}|^2. \]  

(6.9)

It is unknown at this step whether this is possible for finitely or infinitely many choices of \( V \). Of course, it holds at least for \( V = 0 \). Then, combining Equations (6.7), (6.9) and (3.1):
\[
3\alpha + |v_{11}|^2 = \text{Tr} (V^2) = 2 \int_{\mathbb{R}} \eta_2(t) \, dt
\]
\[
= 2 \left( \int_{-\infty}^{-1} \eta_2(t) \, dt + \int_{1}^{1} \eta_2(t) \, dt + \int_{1}^{\infty} \eta_2(t) \, dt \right)
\]
\[
= 2 \int_{-1}^{1} \eta_2(t) \, dt = 2 \int_{-1}^{1} \alpha \, dt = 2\alpha(1 - (-1)) = 4\alpha .
\]

Thus we have:

\[
\text{Tr} (V^2) = 4\alpha , 
\tag{6.10}
\]

\[
|v_{11}|^2 = \alpha .
\tag{6.11}
\]

Since \(v_{11} \in \mathbb{R}, |v_{11}|^2 = v_{11}^2\) so Equation (6.11) yields:

\[
v_{11} = \pm \sqrt{\alpha}
\tag{6.12}
\]

Recall from Proposition 4.2 that since \(\eta_1(t) \equiv 0\), \(V\) is of the form:

\[
V = \begin{pmatrix}
\frac{v_{11}}{v_{12}} & \frac{v_{12}}{-v_{11}}
\end{pmatrix}
\text{ with } v_{11} = -1 \pm \sqrt{1 - |v_{12}|^2}.
\]

Then:

\[
V^2 = \begin{pmatrix}
v_{11}^2 + |v_{12}|^2 & 0 \\
0 & v_{11}^2 + |v_{12}|^2
\end{pmatrix}
\]

\[
\text{Tr} (V^2) = 2(v_{11}^2 + |v_{12}|^2).
\tag{6.13}
\]

Combining Equations (6.9), (6.13), and (6.11), we find:

\[
|v_{12}|^2 = v_{11}^2.
\tag{6.14}
\]

Thus we have \(v_{11} = 0\) or \(v_{11} = -1\) as follows:

\[
v_{11} = -1 \pm \sqrt{1 - v_{11}^2} \Rightarrow (v_{11} + 1)^2 = 1 - v_{11}^2 \Rightarrow 2v_{11}(v_{11} + 1) = 0
\]

If \(v_{11} = 0\) then \(|v_{12}| = 0\) as a result of Equation (6.14), so \(V = 0\) and we are done. If, on the other hand, \(v_{11} = -1\), then \(|v_{12}|^2 = 1\) by Equation (6.14). Then:

\[
V = \begin{pmatrix}
\frac{-1}{v_{12}} & \frac{v_{12}}{1}
\end{pmatrix}, \quad |v_{12}| = 1.
\tag{6.15}
\]

\[
\square
\]
**Conjecture 6.4.** Let $H, V \in M_2(\mathbb{C})_{sa}$, $\eta_1(t) \equiv 0$, and $\eta_3(t) \equiv 0$. Then $V = 0$.

We suspect that the nonzero form of $V$ presented in Proposition 6.3 for the case of special $H$ is insufficient to yield $\eta_3(t) \equiv 0$ for reasons not yet fully explored. This will be investigated in preparation of a future paper.

As discussed in Remark 4.3, results for the special case of $H = P$ should be sufficient to treat the more general case of $H \in M_2(\mathbb{C})_{sa}$ via formulas in development by A. Skripka relating $P$ to all other such $H$. If there are indeed no nonzero choices of $V$ in the special case, it is expected we would also have $V = 0$ in the general case.

We plan to further explore this and other methods of determining necessary and sufficient conditions on $V \in M_2(\mathbb{C})_{sa}$ to force $\eta_3(t) \equiv 0$ following submission of this undergraduate thesis.
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References


