ALGEBRA QUALIFYING EXAM
Spring 1994

01/94

Directions: 1. Solve all eight problems.

2. Start each question on a new sheet of paper. Write only on one side of the page.

Note: Assume that all rings are commutative with identity and all modules are unitary.

Problems

1. Prove that every finite group of order n is isomorphic to a subgroup of S_n.

2. Let G be a finite group of order $p^n q$, where p and q are primes with $p > q$, and n is a positive integer. Prove that G contains a unique normal subgroup of index q.

3. Let P be a prime ideal in a ring R. Let $P[x] = \{f(x) \in R[x] |$ all the coefficients of $f(x)$ belong to $P\}$. Prove that $P[x]$ is a prime ideal in the ring of polynomials $R[x]$.

4. Let I be an ideal in an integral domain R. Let $T(I) = \{x \in I | \exists r \in R, r \neq 0$ such that $rx = 0\}$.

   Prove that $T(I) = \ker \phi$, where $\phi : I \longrightarrow K \otimes_R I$, $\phi(x) = 1 \otimes x$ for all $x \in I$, and K is the quotient field of R.

5. Let

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

   be a commutative diagram of R-modules and R-module homomorphisms, with exact rows. Prove that if $\alpha_1, \alpha_2, \alpha_4, \alpha_5$ are isomorphisms, then α_3 is an isomorphism.

6. Let $f(x) = x^3 + 3x^2 + 2x - 1$. Find the smallest subfield F of $\mathbb{C}$ such that the Galois group of $f(x)$ over F is isomorphic to A_3.

7. Let F be an extension field of a field K and let D be a subring of F that contains K. Prove that if F is an algebraic extension of K, then D is a field.

8. Let I be the ideal generated by $y - x^2$ in the polynomial ring $\mathbb{C}[x, y]$. Prove that the quotient ring $\mathbb{C}[x, y]/I$ is isomorphic to $\mathbb{C}[x]$.

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