MASTERS AND PH.D QUALIFYING EXAMINATION
ALGEBRA
AUGUST 1995

Instructions

A. There are 9 problems divided into four sections. The number of points for each problem is indicated in parenthesis.

B. Write your code number and problem number on each sheet of paper.

1) Prove that a group of order 24 cannot be simple. (10 points)

2) Prove that if G is a group containing a normal subgroup H such that both H and G/H are solvable then G itself is solvable. (10 points)

3) Prove that the group $\mathbf{Q}/\mathbf{Z}$ is not finitely generated. (10 points)

4) Let R be a commutative ring with multiplicative identity 1_R. Show that an ideal $I \subset R$ is maximal if and only if R/I is a field. (10 points)

5) Prove that the ring $\mathbf{Z}[\sqrt{-5}] = \{a + b\sqrt{-5}; a, b \in \mathbf{Z}\}$ is not factorial. (10 points)

6) Consider the polynomial $x^3 - 2 \in \mathbf{Q}[x]$. Find a splitting field F of this polynomial over $\mathbf{Q}$ and determine the Galois group $Aut_{\mathbf{Q}}F$. (15 points)

7) Let $\zeta_n := \exp(2\pi i/n) \in \mathbf{C}$. Prove that the cyclotomic field $\mathbf{Q}(\zeta_n)$ is a Galois extension of $\mathbf{Q}$ with abelian Galois group. (15 points)

MASTERS AND PH.D QUALIFYING EXAMINATION
ALGEBRA
AUGUST 1995

9) Determine the Jordan canonical form of a 3×3 matrix A with complex entries knowing that $A^3 = 0$ and $A^2 \neq 0$. (10 points)

(15 points)

8) Let E be a vector space of finite dimension $d \geq 2$ over the complex numbers and let $n \geq 2$ be an integer. Prove that there exist infinitely many linear transformations $\phi : E \longrightarrow E$ such that $\phi^n = Id_E$. (10 points)

(pages combined)