MASTERS AND PH.D QUALIFYING EXAMINATION
ALGEBRA
AUGUST 1995

Instructions

A. There are 9 problems divided into four sections. The number of points for each problem is indicated in parenthesis.

B. Write your code number and problem number on each sheet of paper.

I. Group Theory.

1) Let A be a finite Abelian group and let p be a prime. Consider the p-th power map $\psi : A \longrightarrow A$ given by $\psi(a) = a^p$. Show that there is an isomorphism of Abelian groups $\ker \psi \simeq A/\text{im } \psi$. (10 points)

2) Let G be a group of order p^n, the power of a prime p. Show that the center of G has order at least p. (10 points)

3) Show that a group of order 36 cannot be simple. (15 points)

II. Rings and Modules.

4) Let R be a commutative ring with identity $\mathbb{I}_R \neq 0$. Prove that an ideal $M \subset R$ is maximal if and only if the quotient ring R/M is a field. (10 points)

5) Let P be a projective module over a ring R and let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of R-modules. Prove from first principals (i.e do not quote a theorem) that the Hom sequence

$$0 \longrightarrow \hom_R(P, A) \longrightarrow \hom_R(P, B) \longrightarrow \hom_R(P, C) \longrightarrow 0$$

is exact. (15 points)

III. Galois Theory.

6) Show that $\mathbb{Q}(\sqrt[6]{3}, i)$ is a splitting field of the polynomial $x^6 - 3$ over the field of rationals $\mathbb{Q}$. (10 points)

7) Determine the Galois group of the polynomial $x^6 - 3$. (15 points)

IV. Linear Algebra.

8) Let $\mathbb{K}$ be a field, and $GL(n, \mathbb{K})$ the group of invertible $n \times n$ matrices over $\mathbb{K}$. If $A, B \in GL(n, \mathbb{K})$ prove that $A + aB$ is invertible for all but a finite number of $a \in \mathbb{K}$. (5 points)

9) Let E be a finite dimensional vector space over a field $\mathbb{K}$, and let $\phi : E \longrightarrow E$ be a linear transformation. Prove that there exists a unique monic polynomial of positive degree $q_\phi \in \mathbb{K}[x]$ such that $q_\phi(\phi) = 0$ and if $f \in \mathbb{K}[x]$ satisfies $f(\phi) = 0$, then q_ϕ divides f. (10 points)