QUALIFYING - ALGEBRA 96

A. There are 8 problems divided into four sections.
B. Write your code number and problem number on each sheet of paper.

Group theory

1. Let N be a normal subgroup of a group G. Assume that N has prime order p and assume that any prime divisor of the order of G is greater or equal to p. Prove that N is contained in the center of G.

2. Prove that if p is an odd prime and m is any positive integer then the group $(\mathbf{Z}/p^m\mathbf{Z})^*$ of invertible elements in the ring $\mathbf{Z}/p^m\mathbf{Z}$ is cyclic.

Linear algebra

3. Let A, B be two $n \times n$ matrices with entries in $\mathbf{C}$ such that $AB = BA$. Prove that there exists an invertible matrix P such that both PAP^{-1} and PBP^{-1} are upper triangular.

4. Find the Jordan canonical form (over the complex numbers) of the 5×5 matrix A knowing that $(A - I)^4 = 0$ and $(A - I)^3 \neq 0$, where I is the identity matrix.

Module theory

5. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of finite abelian groups such that the orders of A and C are coprime. Prove that the sequence is split.

6. Let R be a commutative unitary ring and let I, J be two ideals. Prove that $(R/I) \otimes_R (R/J) \simeq R/(I + J)$.

Field theory

7. Prove that any finite group is the Galois group of a suitable Galois field extension.

8. Prove that if p and q are two prime numbers such that $\mathbf{Q}(\sqrt{p}) = \mathbf{Q}(\sqrt{q})$ then $p = q$. Deduce that there exists an infinite sequence of primes $p_1, p_2, \ldots$ such that $\sqrt{p_1}, \sqrt{p_2}, \ldots$ are linearly independent over $\mathbf{Q}$.