QUALIFYING - ALGEBRA January 1997

There are 10 problems. Each problem counts 10 points. Write your code number and problem number on each sheet of paper.

1. If H is cyclic normal subgroup of a group G, show that every subgroup of H is normal in G.

2. a) Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} \mathbf{Z} \to 0$ is an exact sequence of abelian groups. Show that there is a homomorphism $h : \mathbf{Z} \to B$ such that $g(h(n)) = n$ for all $n \in \mathbf{Z}$ ($\mathbf{Z}$ =integers).

b) Give an example of an exact sequence of abelian groups $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ such that there does not exist a homomorphism $h : C \to B$ with $g(h(x)) = x$ for all $x \in C$.

3. a) Let A_n =subgroup of all even permutations in S_n(=group of all permutations of n letters.) Show that A_4 is generated by 3−cycles.

b) Show that A_4 contains no subgroup of order 6.

4. Let A be an $n \times n$ matrix with n distinct nonzero eigenvalues. Let B be the $2n \times 2n$ matrix given by

$$B = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$$

where I is the $n \times n$ identity matrix. Show that the eigenvalues of B are the square roots of the eigenvalues of A.

5. Find the eigenvalues of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and evaluate A^{301}.

6. Let R be a commutative ring with identity and let S be a subset of R which is closed under multiplication. Let I be an ideal of R such that $I \cap S = \emptyset$ and assume I is maximal with respect to this property (i.e. if J is an ideal with $I \subset J$ and $J \cap S = \emptyset$ then $J = I$.) Show I is a prime ideal.

7. Show that every group of order 56 has a non-trivial normal subgroup.

8. Let $f \in \mathbf{C}(X)$ be a rational function with complex coefficients such that

$$f(X) = f\left(\frac{1}{X}\right)$$

Prove that there exists a rational function $g \in \mathbf{C}(X)$ such that

$$f(X) = g\left(X + \frac{1}{X}\right)$$

9. Let $\alpha \in \mathbf{C}$ be a complex number, $\alpha \notin \mathbf{Q}$ ($\mathbf{Q}$ =rationals). Let S be the set of all subfields K of $\mathbf{C}$ that are algebraic over $\mathbf{Q}$ and do not contain α.

1) Prove that S has a maximal element.

2) Let M be a maximal element of S; prove that any finite Galois extension of M is cyclic.

10. Let $\bar{\mathbf{Q}}$ be the algebraic closure of $\mathbf{Q}$. Prove that the group of all automorphisms of $\bar{\mathbf{Q}}$ is infinite.