QUALIFYING ALGEBRA - AUGUST 1997

Each problem is worth 10 points.

1. Prove that any finitely generated subgroup of the additive group $\mathbf{Q}$ is generated by one element.

2. Let G be a finite group and let $C(G)$ be its center. Assume $G/C(G)$ is cyclic. Prove that G is commutative.

3. Let G be a finite group of order $p^n q$ with p, q primes, $p > q$. Prove that G is not simple.

4. Let G be an additive subgroup of $\mathbf{R}$. ($\mathbf{R}$=the field of real numbers.) Assume there exists an interval $I = (a, b) \subset \mathbf{R}$ such that such that $G \cap I = \{0\}$. Prove that G is generated by one element.

5. Let A be a commutative ring with unit element. Assume $a \in A$ is contained in all prime ideals of A. Prove that a is nilpotent (i.e. that there exists an integer $n \geq 1$ such that $a^n = 0$.)

6. Prove that the ring of Gauss integers $\mathbf{Z}[i] := \{a + bi | a, b \in \mathbf{Z}\}$ is principal.

7. Let $a_1, ..., a_n$ be integers with greatest common divisor 1. Prove that there exists a matrix A with integer coefficients, whose first row is $[a_1, ..., a_n]$, such that $det(A) = 1$. (Hint: consider the $\mathbf{Z}$−module $\mathbf{Z}^n$ and the submodule generated by the vector $[a_1, ..., a_n]$.)

8. Determine the Galois group over $\mathbf{Q}$ of the polynomial $x^6 - 5$.

9. Prove the fundamental theorem of algebra (that is show that the field of complex numbers $\mathbf{C}$ is algebraically closed.)

10. Let A be a $n \times n$ matrix with complex coefficients. Prove that $A^n = 0$ if and only if $tr(A) = tr(A^2) = ... = tr(A^n) = 0$.